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H-CLOSED EXTENSIONS AND ABSOLUTE OF INVERSE LIMIT SPACE

The main purpose of this paper is the application of the Katetov extension kX to an inverse system and its limit.
By the method of the extension theory the theorems concerning continuity of the Katetov functor, H-closedness and nearly-compactness of an inverse limit space are given.

H-closed extension, inverse system

1. KATETOV EXTENSION OF A LIMIT SPACE

If X is a topological space, then the closure and the interior of a subset $A \subseteq X$ is denoted by $Cl_X A$ and $Int_X A$ or by ClA and IntA.

A Hausdorff space X is *H-closed* if for everey open cover U of X there exists a finite subfamily $\{U_1, \ldots, U_k\}$ of U such that $X = ClU_1 \cup \ldots \cup ClU_k$ ([17]).

A continuous mapping $f: X \longrightarrow Y$ is said to be *proper* [17]. if for each $y \in Y$ and each $V \ni y$ open in Y there exists a $V' \ni y$ which is open in Y and such that $Intf^{-1}(ClV') \subseteq Clf^{-1}(V)$ [17].

An inclusion $A \subseteq Y$ is *proper* if for each $y \in Y$ and each $V \in y$ open in Y there exists a $V' \in y$ open in Y and such that $Int_A(A \cap Cl_V V') \subseteq Cl_A(A \cap V)$.

1.1.LEMMA. [17]. Let $f: X \longrightarrow$ by a continuous mapping. Then:

(i) f is proper, if Y is regular;

(ii) f is proper if X is H-closed and if Y is a Hausdorff space;
(iii) a closed subspace A of H-closed X is H-closed iff the inclusion $A \subset X$ is proper;
(iv) each open and dense embedding is proper.

Let *F* be a family of all open free ultrafilters on a Hausdorff space X. The Katetov extension kX of X [17] is the set $X \cup F$ with topology consisting of all open subsets of X and all sets of the form $\{x\} \cup U$, where $x \in F$ and $U \in x$.

1.2.LEMMA. ([16]), ([6]). Let X be a Hausdorf space. Then:
(i) kX is H-closed;
(ii) X is open and dense (i.e. proper) embedded in kX;
(iii) kX-X is discrete in the topology induced by the topology on kX;
(iv) a mappinf $f: X \longrightarrow Y$ into H-closed space Y has a unique continuous extension $kf: kX \longrightarrow Y$ if and only if f is proper;

(v) if U and V are dijoint open subsets of X then $Cl_{kx}U \cap Cl_{kxV} \subseteq X$.

We say that an extension Y of X is *majorizable* if there exists an extension Z of X and a map F:Z ---> Y which is an extensin of the identity i:X ---> X.

An extension will be called *r.o.-free* if for each regularly open subset U of X the boundary $Bd_X U$ in X is the same as the bondary $Bd_V V$ of V in Y, where V is an arbitrary open subset of Y such that U = V ∩ X [17 : I.3.].

1.3.LEMMA. [17:I.3.1]. If an H-closed extension X ⊂ Y is such that:

a) X is open in Y,

b) The remainder Y-X is discrete in the topology inducede from Y,

c) X ⊂ Y is r.o. - free

then X ⊂ Y is non-majorizable.

1.4.LEMMA. An extension X ⊂ Y which satisfies a) and b) of Lemma 1.3. is r.o.-free iff the folowing condition (K) is satisfied: (K) If U, V is a pair of disjoint open subsets of X then ClvU'∩ClvV' ⊆ X, where U', V' are arbitrary open subsets of Y such that U=U' ∩ X and V=V' ∩ X.

A *p-cover* of X is an open cover of X possessing a finite subfamily which is dense in X [26]. A map f:X ---> is p-map if for each p-cover *U* of Y a cover $f^{-1}(U) = (f^{-1}(U):U \in U)$ is a p-cover of X [26].

A continuous mapping f:X ---> Y is a p-mapping iff f can be continuosly extended to kf:kX ---> kY [26].

1.5.LEMMA. ([11], [22]). Let X_α be non-empty spaces for each α∈A. Then $k(P_\alpha X_\alpha) = P_\alpha kX$ iff at least one of the following two conditions is satisfied.

(a) X_α is H-closed for each α ∈ A.

(b) There exists X_{α_0} which is not H-closed. X_α is finite for all $\alpha \neq \alpha_0$. Moreover, all but finitely many X_α 's have only one point.

In contrast of the above Lemma we show that the functor k is continuous in some non-trivial cases i.e. that k (lim*X*) = limkX.

Now we start with the key lemma of this Section.

1.6.LEMMA. Let X = $\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α, β∈A. Then:

(i) if the mappings $f_{\alpha\beta}$ are p-map then there exists inverse system kX = $\{kX_\alpha, kf_{\alpha b}, A\}$;

(ii) if lim*X* is non-empty and if the projections f_α: lim*X* ---> X_α, α∈A, are p-map, then there exists a continuous mapping *K*:k (limX) ---> limkX which is an extension if the identity i:limX ---> limkX;

(iii) if the projections f_α are p-map and onto, then *K* is onto and limkX is an H-closed extension of limX such that limX is open in limkX.

Proof. (i) Apply Lemma 1.2. (iv).

(ii) Now we have the p-map mappings f_α : limX ---> kX_α, $\alpha \in A$. By virtue of Lemma 1.2. (iv) there exist a continuous mappings kf_α :k (limX) ---> kX_α, $\alpha \in A$. A family $\{kf_\alpha : \alpha \in A\}$ induces a continous mapping K:k (limX) ---> limkX ⌊2: 138⌋. The proof is completed.
(iii) Let us prove that K is onto. For each $x \in$ limkX we consider a points $x_\alpha = f_\alpha'$ (x), $\alpha \in A$, where f' : limkX ---> kX_α, $\alpha \in A$, are the projections. For each x_α we have $\{x_\alpha\} = \{ClU_\alpha : U_\alpha$ is the open neighborgood of $x_\alpha\}$. A family $\{(kf_\alpha)^{-1}(U_\alpha) : \alpha \in A\}$ is a centred family of open subsets in H-closed space k(limX). This means that there exists a point $y \in \cap \{Cl(kf_\alpha)^{-1}(U_\alpha) : \alpha \in A\}$. Clearly kf_α (y)= x_α for each $\alpha \in A$. Thus, K (y) = x. This means that K is onto and that limkX is H-closed as a continuous image of H-closed space k (limX). In order to complete the proof it suffices to prove that limX is dense in X. This is an immediate consequence of the definition of a base of the inverse limit space and the assumption that f_α are onto.

1.7.LEMMA. Let $X=\{X_\alpha, f_{\alpha b}, A\}$ be an inverse system with projections f_α which are onto p-map. For each $x \in$ k(limX)-limX there exists a $\alpha \in A$ such that kf_α (x) $\in kX_\alpha - X_\alpha$.

Proff. An immediate consequence of the fact that x is free ultrafilter and the definition of a base on inverse limit space. From Lemmas 1.3. and 11.7. we obtain the following.

1.8.LEMMA. Let X be an inverse system with projections f_α which are p-map onto. Then limkX = k(limX) if and only if the following conditions are satisfied:

a) limkX - limX is discrete in the topology induced by the topology on limkX,

b) each open subset U ⊆ limX is r.o.-free in limkX.

A mapping f:X ---> Y is said to be *p-perfect* if f is a p-map and f(kX - X) = kY - Y [26].

1.9.LEMMA. Let X be an inverse system with p-perfect onto mappings $f_{\alpha\beta}$ such that f_α are p-perfect and onto. Then f_α (limkX - limX) ⊆ $kX_\alpha - X_\alpha$, $\alpha \in A$.

1.10.LEMMA. Let X be an inverse system as in Lemma 1.9. A subspace limkX - limX is discrete iff the following condition is satisfied: (D) For each point $x_\alpha \in kX_\alpha - X_\alpha$ there exists a $\beta \in A$, $\beta \geq \alpha$, such that for each $\gamma \in A$, $\alpha \leq \beta \leq \gamma$, the fiber $(kf_{\beta\gamma})^{-1}$ (x_β)contains a single point for each $x_\beta \in (kf_{\alpha b})^{-1}$ (x_α).

Proof. The "only if" part. Now the subspace limkX - lim X = Y of the space limkX is the limit of inverse subsystem $Y = \{kX_\alpha - X_\alpha, kf_{\alpha\beta} / (kX_\beta - X_\beta), A\}$. Each point $y \in Y$ is an open subset of Y. This means that {y} contains the fiber $(kf_\alpha)^{-1}$ (U_α) for some open subset U_α of $kX_\alpha - X_\alpha$. Thus (D) is satisfied.

The proof of the "if" part is similar.

1.11. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that $f_{\alpha\beta}$ are p-perfect mappings. If the projections $f_\alpha : \lim X \to X_\alpha$, $\alpha \in A$, are onto p-map, then $\lim kX$ and $k(\lim X)$ are homeomorphic iff X satisfies the condition (D) and $\lim kX$ satisfies the condition (K).

Proof. The "if" part. By virtue of Lemmas 1.8. and 1.10. it follows that $\lim kX$ satisfies the conditions of Lemma 1.3. Thus, the mapping *K* is a homeomorphism.

The "only if" part follows from the fact that kX satisfies the conditions of Lemma 1.3.

1.12. DEFINITION. A mapping $f:X \to Y$ is said to be θ-continuous if for each $x \in X$ and each open $V \ni f(x)$ there is an open $U \ni x$ such that $f(ClU) \subseteq ClV$.

If Y is regular, then each θ-continuous mapping $f:X \to Y$ is continuous.

1.13. DEFINITION. A mapping $f:X \to Y$ is a *θ-homeomorphism* if f is 1 - 1 onto such that f and f^{-1} are both θ-continuous. We say that two extensions Y and Z of a space X are *θ-equivalent* if there exists a θ-homeomorphism $H:Z \to Y$ which is the extension of identity $i:X \to X$.

1.14. LEMMA. Let *X* be an inverse system with p-perfect bonding mappings and proper onto projections. The space $\lim kX$ is θ-equivalent to the space $k(\lim X)$ iff the condition (K) is satisfied.

Proof. The "if" part. Apply the Fomin modification $(\lim kX)$, [9:46] which ih homeomorhic to $k(\lim X)$. Moreover, $(\lim kX)$, is O-homeomorphic to $\lim kX$ [2:46m Lemma 7.] since *K* is 1-1.

The "only if" part is obvious since *K* is θ-homeomorphism.

For an inverse system of a regular spaces we have the following corollary of Theorem 1.11.

1.15. COROLLARY. Let X be an inverse system of a regular spaces and perfect onto bonding mappings. The spaces $k(\lim X)$ and $\lim kX$ are equivalent iff the conditions (D) and (K) are satisfied.

Now we define some special kinds of the proper mappings.

A mapping $f:X \to$ is said to be *skeletal (HJ)* if for each open (regularly open) $U \subseteq X$ we have $\mathrm{Int} f^{-1}(ClU) \subseteq Cl f^{-1}(U)$ [17].

1.16. LEMMA. [17]. Each HJ-mapping is a proper mapping.

A mapping $f:X \to Y$ is *semi-open* if $\mathrm{Int} f(U)$ is non-empty for each non-empty open subset $U \subseteq X$.

Each semi-open mapping is HJ and proper. Each open mapping is semi-open.

We say that a mapping $f:X \to Y$ is *irreducible* if the set $f^{\#}(U) = \{y: f^{-1}(y) \subseteq U\}$ is non-empty for any non-empty open subset $U \subseteq X$.

Every closed irreducible mapping is a semi-open mapping.

A mapping $f:X \to Y$ has *the inverse property* if $f^{-1}(ClV) = Cl f^{-1}(V)$ for any open set $V \subseteq Y$.

Every open mapping has the inverse property and every mapping with the inverse property is HJ-mapping.

In the paper [15] it was proved the following theorem.

1.17. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with

HJ-mapping $f_{\alpha\beta}$. If the projections f_α: L limX ---> X_α, $\alpha \in A$, are onto, then the projections f_α are HJ-mapping.

A mapping f:X ---> Y is *absolutely closed* if there do not exists a proper extension T of X and an extension f:T --> Y of f [26:211].

1.18.LEMMA. [26]. Let f:X ---> Y be a continuous mapping. The following are equivalent:
(1) f is absolutely closed.
(2) (a) If $A \subseteq X$ is regularly closed, then f (A) is closed,
(b) If $x \in kX - X$ and $y \in Y$, then there exists $U \in x$ such that $f^{-1}(y) \cap ClU = \emptyset$

1.19.LEMMA. [26:211]. A p-mapping f:X ---> Y is p-perfect iff f is absolutely closed.

Now we have the following corollary of Theorem 1.11.

1.20.COROLLARY. Let X = $\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with absolutely closed HJ-mapping $f_{\alpha\beta}$ and onto projections f_α : limX ---> X_α, $\alpha \in A$. The space k (limX) is equivalent to the space limkX iff the conditions (K) and (D) are satisfied.

A special role play a closed irreducible mapping since we have the following

1.21.LEMMA. If f:X ---> Y is p-perfect closed irreducible mapping, then the restriction kf/(kX - X) is one-to-one i.e. kf/(kX - X) is a homeomorphism the space kX - X onto kY - Y.

Proff. If $x = (U_\alpha : \alpha \in A)$ is a free ultrafilter, then (f# (U) : U $\in x$) is a free ultrafilter. It is easy to prove that for y = $\{V_\mu : \mu \in M\}$ $y \neq x$ it follows that $\{f\#(U_\alpha) : \alpha \in A\} \neq \{f''(V_\mu) : \mu \in M\}$. This means that kf/(kX - X) is one-to-one. The proof is completed.

1.22.THEOREM. Let X = $\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with perfect irreducible onto mapping $f_{\alpha\beta}$. Then limkX = k (limX).

Proof. The pprojections f_α : limX ---> X_α, $\alpha \in A$, are perfect (=closed with compact fiber $f_\alpha^{-1}(x_\alpha)$). It is easy to prove that f_α, $\alpha \in A$, are irreducible. We infer that limkX - limX is homeomorphic to each $kX_\alpha - X_\alpha$, α A. Thus the condition (D) is satisfied. Let us prove that the condition (K) is satisfied. Let U, V be a pair of disjoint open subsets of limX. A sets f"(U) and $f^\#(V)$ are disjoint open subsets of X_α, $\alpha \in A$, since f_α, $\alpha \in A$. are perfect and irreducible. Since X_α satisfies the condition (K) we have the following relation in kX_α:$Clf^\#_\alpha(U) \cap Clf''_\alpha(V) \subseteq X_\alpha$. By virtue of the irreducibility of f_α it follows that in limkX we have the relation ClU ∩ ClV limX. The condition (K) is satisfied. By 1.11. the proof is completed.

We close this Section with theorems concernig the inverse systems of H-closed spaces. The "only if" part of the following

theorem is new.
1.23. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of H-closed spaces X_α. A space limX is H-closed iff the projections f_α are proper.
Proof. The "only if" part. If limX is H-closed, then by Lemma 1.1. (ii) the projections f_α are proper.
The "if" part. Now $kX = \{kX_\alpha, kf_{\alpha\beta}, A\} = \{X_\alpha, f_{\alpha\beta}, A\} = X$ since the mapping definied in the proof of Lemma 1.6. We have $\lim X \subseteq K[k(\lim X)] \subseteq \lim kX$. Since $\lim kX = \lim X$ we infer that $\lim X = K[k(\lim X)] = \lim kX$. As a continuous image of H-closed space k(limX) the space $K[k(\lim X)]$ is H-closed. The proof is completed.
1.24. REMARK: The "if" part of Theorem 1.23. has been proved in the paper [3].

A space X is said to be *nearly-compact*[7] if for each open cover $U = \{U_\alpha : \alpha \in A\}$ of X there exists a finite subfamily $\{U_1, \ldots, U_k\}$ of such that $X = \mathrm{IntCl}U_1 \cup \ldots \cup \mathrm{IntCl}U$.

It is known that X is neaarly-compact iff X is H-closed and completely Hausdorff [7]. Let us recall that a space X is *completely Hausdorff* if each two distinct point of X have a neighborhoods with disjoint elosures.
1.25. LEMMA. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a completely Hausdorff spaces X_α. A limit limX is completely Hausdorff.
Proof. Trivial.
1.26. THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system. A space limX is nearly-compact iff the spaces X_α, $\alpha \in A$, are nearly-compact and if the projections f_α, $\alpha \in A$, are proper.
Proof. Apply Theorem 1.23. and Lemma 1.25.

2. FOMIN EXTENSION ∂X

Let X be a Hausdorff sppace. We now define a topology on the set $X \cup F$ as follows. For each U open in X let O_U be the union of U and all ultrafilters of *F* which contain U. It is easy to prove that

$$O_{U} \cap v = O_U \cap O_v \quad (1)$$

This means that a family $\{O_U : U \text{ is open in } X\}$ is a base for topology on $X \cup F$. We denote the set $X \cup F$ equpped with this topology by ∂X. The space ∂X is called the *Fomin extension* of a space X [9]
2.1. LEMMA. [9]. The space ∂X is H-closed extension of a Haus-dorff space X. If Y is any H-closed extension of X, then there exists a θ-continuous extension of $f:X \to Y$ of the identity $i:X \to X \subset Y$.

Let $f:X \to Y$ be a continuous mapping. For each ultrafilter $x = \{U_\alpha : U_\alpha \text{ is open in } X\} \in \partial X - X$ we consider a filter-base $\partial f(x) = \{V: V$ is open in Y such that there exists a $U_\alpha \in x$ with $f(U_\alpha) \subseteq V\}$. It is easy to prove that if f is a p-map, then $\partial f(x)$ is an open ultrafilter in Y. By virtue of H-closednes of ∂Y the intersection $Z = \cap \{ClV : V \in \partial f(x)\}$ is non-empty. As in the case od the Katetov extension kX it is easy to prove the folloving Lemma.

2.2.LEMMA. (a) If $f:X \longrightarrow Y$ is a p-mapping, then $\partial f(x)$ contains a single point of Y,

(b) The mapping $\partial f: \partial X \longrightarrow \partial Y$ is θ-continuous,

(c) If f is p-perfect then ∂f is continuous,

(d) If Y is regular and if f is p-mapping then ∂f is continuous.

By the proof similar to proof of Lemma 1.6. we obtain

2.3.LEMMA. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Haus-dorf spaces X_α and a p-mapping $f_{\alpha\beta}$. Then:

(i) There exists inverse system $\partial X = \{\partial X_\alpha, \partial f_{\alpha\beta}, A\}$ with θ-continuous mappings $\partial f_{\alpha\beta}$.

(ii) If the mappings $f_{\alpha\beta}$ are p-perfect or if X_α, $\alpha \in A$, are regular, then the mappings $\partial f_{\alpha\beta}$ are continuous. Moreover, there exists a continuous mapping $S : \partial(\lim X) \longrightarrow \lim \partial X$.

(iii) If in (ii) $f_{\alpha\beta}$ are onto then S is onto.

2.4.PROBLEM. Under what conditions the mapping S is a homeomorphism?

If the bonding mapping are perfect and irreducible then we have

2.5.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Haus-dorff spaces X_α and perfect closed irreducible onto mapping $f_{\alpha\beta}$. Then the mapping $S : \partial(\lim X) \longrightarrow \lim \partial X$ is a homeomorphism.

Proof. By virtue od Lemma 1.21. the mapping S is onto and 1-1. It remains to prove that S is an open mapping. A subspace $\partial(\lim X) - \lim X$ is homoemorphic to each $\partial X_\alpha - X_\alpha$, $\alpha \in A$, since the projections $f_{\alpha\beta}$ are perfect onto and irreducible. This means that the subspace $\partial(\lim X) - \lim X$ is homeomorphic to the subspace $\lim \partial X$. The proof is completed.

3. ABSOLUTE OF AN INVERSE LIMIT APACE

The space θX. Let θX denotes a family of all open (fixed or free) ultrafilters on a Hausdorff space X. We introduse a topology into θX in the following way. Let O_U be the set of all ultra-filters that contain U, where U is open in X [9]; O_U is to be a base on θX. That this definition is correct it follows from the relation

$$O_{U \cup V} = O_U \cup O_V \qquad (2)$$

It is easy to prove that

$$O_U = \Theta X - O_{X-ClU} \qquad (3)$$

This means that O_U is open and closed subset of ΘX.

3.1.LEMMA. If X is a Hausdorff space then ΘX is zero-dimensional and compact.

Proof. See [9].

A space X is called *extreally disconnected* if for each disjoint open sets $U, V \subseteq X$ we have $ClU \cap ClV = \emptyset$

If X is extremally disconnected and Y is dense in X, then Y is extremally disconnected [9].

3.2.LEMMA. [9:41]. If X ia a Hausdorff space, then ΘX is extremally disconnected zero-dimensional compact space. The equation $X = \Theta X$ holds iff X is a compact extremally disconnected Haudsorff space.

The absolute wX of a space x. A subspace wX of ΘX containing all fixed open ultrafilters on X is called the *absolute* (in the sense od Iliadis) of the space X or the extremally disconnected *resolution* of the space X.

3.3.LEMMA. The absolute wH is dense in ΘX and, consequently, wx is extremally disconnected.

Proof. See [9:41]

3.4.LEMMA. [9:44]. The absolute wX is Θ-homeomorphic to X iff X is extremally disconnected. If X is regular extremally disconnected, then wX is homeomorphic to X.

For each $x \in wX$ we define a point $p_x(x)$ such that $p_x(x) = \cap \{Cl_x U : U \in x\}$.

3.5.LEMMA. [9:55]. The natural projection $p_x : wx \to X$ is Θ-continuous, irreducible and perfect. It is continuous iff X is regular.

3.6.THEOREM. [9:56]. Let $f: X \to Y$ be a Θ-continuous irreducible perfect mapping of a Hausdorff space X onto a Hausdorff space Y. Then there exists a homeomorphism $wf: wX \to wY$ onto wY such that $fp_x = p_y wf$.

The absolute wX and the extensions of a space X.

3.7.LEMMA. [9:60] Let γX be an arbitrary extension of a Hausdorff space X. Then there exists a homeomorphism $h: \Theta(\gamma X) \to \Theta X$ such that $h(p^{-1}_x(x)) = p_x^{-1}(x)$ for each $x \in X$.

3.8.COROLLARY. [9]. $\Theta(\beta X) = \Theta(kX) = \Theta X$.

3.9.COROLLARY. [9]. If bX is an arbitray extension of X, then $w(bX) = \beta(wX)$. In particular: $w(\beta X) = \beta(wX)$.

The absolute in the sense of Mioduszewski. Now we enlarge the Iliadis topology definied at the begin of this Section by adding sets of the forma $p_x^{-1}(U)$, U being an open subset of X. It is easy to verify that the sets of the form $O_U \cap p_x^{-1}(V)$ may be taken as a members of a topology on the set wX. We denote this space by aX.

3.10.LEMMA. The space aX is extremally disconnected and the mapping $p_x : aX \to X$ is continuous, ireducible and perfect.

The space aX is minimal in the following sense:

3.11.LEMMA.[17:33] For any extremally disconnected space E and any HJ-mapping h:E ---> X there exists a *unique* mapping ah:E ---> aX such that h = p_x(ah).

The following theorem plays a spacial role in our investigation of the absolute of na inverse limit space.

3.12.THEOREM. Let f:X ---> Y be a continuos mapping. A mapping f has *a unique absolute af* such that $p_v$af = fp_x iff the mapping f is HJ.

3.13.REMARK. A) The "if" part of Theorem 3.12. has been proved in the paper [17:24] and the "only if" part in the paper: Shapiro L.B., Ob absoljutah topologiceskih prostranstv i nepreryvnyh otobrazenijah, DAN SSSR 226:3(1976), 523-526.

B) Let us note that the absolute of a continuous mapping always exists but need not be unique.

C) From the proof of the "if" part of Theorem 3.12. it follows that the "if" part holds for the absolute wX in the sense of Iliadis.

D) Another construction of the absolute for regular spaces can be found from [1:363-370]

3.14.LEMMA. ([18:124] or [1:363-370]). Let f:X ---> Y be a continuous mapping. Then there exists the absolute af:aX ---> aY. Moreover:

a) If f is bicompact, then af is bicompact;

b) If f is irreducible and perfect (into, onto) Y, then af is a homemorphism (into, onto) aY.

3.15.LEMMA.{18}. If f>X ---> Y is an open onto mapping, then af:aX ---> aY is onto.

The absolute of the inverse limit space

Now we apply this expository material to the inverse systems and their limits.

3.16.THEOREM. Let X = {X_α, $f_{\alpha\beta}$, A} be an inverse system of a Hausdorff spaces X_α. If the mappings $f_{\alpha\beta}$ are HJ-mapping then there exists an inverse system wX = {wX_α, $wf_{\alpha\beta}$, A} and a mapping W:w (limX) ---> limwX.

Proof. Apply Remark 3.13. C) and the fact that the projections f_α are HJ-mappings. Then modify the proof of Theorem 1.6.

3.17.REMARK. A) Similarly from Theorem 3.12. it follows that there exists an inverse system aX = {aX_α, $af_{\alpha\beta}$, A} for an inverse system as in Theorem 3.16.

B) There exists inverse system θX = {$θX_\alpha$, $θf_{\alpha\beta}$, A} if X is the inverse system of Hausdorff spaces and HJ bonding mappings.

C) If X is an inverse sequence then by total induction on can construct the inverse systems *wX (aX, θX)* without the assumption that the absolute wf (af, θf) are unique.

3.18.THEOREM. Let X = {X_α, $f_{\alpha\beta}$, A} be an inverse system of a Hausdorff spaces X_α with irreducible perfect mappings $f_{\alpha\beta}$ such that the projections f_α are onto. Then the mapping $W: w(\lim X) \longrightarrow \lim wX$ is a homeomorphism.

Proof. From Theorem 3.6. it follows that $wf_{\alpha\beta}$ are homemorphisms. Similarly we infer that wf_α are homeomorphisms. This means that the spaces $w(\lim X)$ and $\lim wX$ are homeomorphic to wX_α, $\alpha \in A$. The proof is completed.

3.19.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of Hausdorf spaces X_α and HJ bonding mappings $f_{\alpha\beta}$ such that the projections f_α are onto. Then the spaces $\theta(\lim X)$ and $\lim\theta X$ are homemorphic iff the following concition (S) is satisfied:

(S) For each two disjoint open subsets U and V of $\lim X$ there is a $\alpha \in A$ such that $f_\alpha(U)$ and $f_\alpha(V)$ have a disjointneighborhoods.

Proof. The "if" part. Let x and y be two distinct points in the space $\theta(\lim X)$. This means that there exists a pair U, V of disjoint open subsets of $\lim X$ such that $U \in x$, $V \in y$. From the condition (S) it follows that $\theta f_\alpha(x) = \{W: W$ open in X_α and there exists $U' \in x$ such that $f_\alpha(U') \subseteq W\}$ is not equal to $\theta f_\alpha(y) = \{W: W$ is open in X and there exists $V' \in y$ such that $f_\alpha(V') \subseteq W\}$. This means that the mapping $\theta: \theta(\lim X) \longrightarrow \lim \theta X$ is 1-1. Since θ is onto and $\theta(\lim X)$ is compact we infer that θ is a homeomorphism. The proof of the"if" part is completed. The proof of the "only if" part is similar.

3.20.LEMMA. The condition (S) is satisfied : (a) if the projectins f_α are closed irreducible or (b) if for each open subset $U \subseteq \lim X$ there exists a $\alpha \in A$ and an open subset U_α of X such that $f_\alpha^{-1}(U_\alpha) = U$.

Proof. Obvious.

3.21.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α such that $f_{\alpha\beta}$ are closed irreducible and the projections f_α are closed onto (or the condition (b) of 3.20. is satisfied), then $\theta(\lim X) = \lim\theta X$.

Proof. Apply Lemma 3.20. and Theorem 3.19.

3.22.COROLLARY. Let $X = \{X_{fn}, \ldots, f_{fn}, \ldots, N\}$ be an inverse sequence of a Hausdorff spaces X_n with close irreduciblem onto mappings f_{mn}. Then the spaces $\theta(\lim X)$ and $\lim\theta X$ are homeomorphic.

Proof. It is well known that the projections f_{mn} are closed and irreducible onto mappings. Now apply Theorem 3.21.

If the mappings $f_{\alpha\beta}$ and the projections f_α in Theorem 3.21. are p-perfect then a restictions $\theta f_{\alpha\beta}/wX_\beta$, $\beta \in A$, are identical with $wf_{\alpha\beta}$. Similarly a restriction $\theta f_\alpha/w(\lim X)$ is identical with Wf_α. Thus we have

3.23.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system as in 3.21. If the mappings $f_{\alpha\beta}$ an the projections f_α are p-perfect, then the spaces w(limX) and limwX are homeomorphic.

If the bonding mappings $f_{\alpha\beta}$ are perfect irreducible then from 3.23. holds Theorem 3.18.

For the absolute aX in the sense of Mioduszewski we now prove

3.24.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α. If the spaces w (limX) and limwX are homeomorphic, then the spaces a(limX) and limaX are homeomorphic.

Proof. Let G be any open neighborhods of $x \in$ a(limX). By the definition of a base in a(limX) there exist a neighoborhood of *x* of the form $O_U \cap p_{lim}^{-1}$ (V) contained in G, where V is open in limX and O_U is open in w(limX). From the relations w(limX) = limwX and $x \in O_U$ it follows there exists an open $U_\alpha \subset X_\alpha$ such that a set $(wf_\alpha)^{-1}(O_{U\alpha})$ is a neighborhod of x contained in O_U. Similarly there exists an open $V_\alpha \subset X$ such that $f_\alpha^{-1}(V_\alpha) \leq V$ is a neighborhood of x. This means that a set $p_{lim}^{-1} f_\alpha^{-1}(V) \cap (wf_\alpha)^{-1}$ $(O_{U\alpha})$ is a neighborhood of x which is contained in $O_U \cap p_{lim}^{-1}$ (V). From the relation $p_{limx}^{-1} f_\alpha^{-1} (V_\alpha) = (wf_\alpha)^{-1} p_{x\alpha}^{-1} (V_\alpha)$we infer that there exists a neighborhood $p_{x\alpha}^{-1} (V_\alpha) \cap O_{U\alpha} = G_\alpha \subset aX_\alpha$ such that $(wf_\alpha)^{-1}$ (G) = (af_α) is contained in G. This means that G is open in limaX. Thus the mapping A:a(limX) ---> limaX is 1-1 continuous and open mapping onto limaX i.e. A is a hemeomorphism. The proof is completed.

We closed this Section with some theorems concerning the non-emptiness of the inverse limit space.

3.25.LEMMA. A Hausdorff space is H-closed iff θX = wX.

Proof. If X is H-closed, then each open ultrafilter on X is fixed. Thus θX = wX. Conversely, if θX = wX, then X is H-closed since the mapping p_x:wX = θX ---> X is θ-continuous and θX is compact. The proof is completed.

3.26.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of H-closed spaces X_α and HJ-mapping $f_{\alpha\beta}$. The space limX is non-empty iff the spaces X_α, $\alpha \in A$, are non-emppty. Moreover, if the mappings $f_{\alpha\beta}$ are onto, then the projections f_α are onto.

Proof. By Theorem 3.16. we obtain the inverse system wX = {wXα, $wf_{\alpha\beta}$, A} which is the inverse system of compact spaces $wX_\alpha = \theta X_\alpha$. It is well known that limwX is non-empty. This means that limX is non-empty since there is a mapping p: wX ---> X, $p = \{p_{x\alpha} : \alpha \in A\}$

3.27.COROLLARY. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α and p-maps $f_{\alpha\beta}$ such that $kf_{\alpha\beta}: kX_\beta$ ---> kX_α are

HJ onto mappings. Then limX is non-emppty.

3.28.COROLLARY. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α such that for each $x_\alpha \in X_\alpha$ and each $\beta \geq \alpha$ $f_{\alpha\beta}^{-1}(x_\alpha) = Y_\beta$ is non-empty H-closed subspace of X_β. If the restrictions $f_{\beta\gamma} / Y_\gamma$ are HJ, them limX is non-empty.

If the mappings $f_{\alpha\beta}$ are open, then the restrictions $f_{\beta\gamma} / Y_\gamma$ are open [2:95]. Thus we have

3.29.COROLLARY. If X is an inverse system of a Hausdorff spaces X_α and open onto mappings $f_{\alpha\beta}$ such that each $f_{\alpha\beta}^{-1}(x_\alpha)$ is H-closed, then limX is non-empty.

4. *ALMOST REALCOMPACTIFICATION rX*

A class of almost realcompact spaces was introduced by Frolik (see [26]).

We say that an open ultrafilter $U = \{U_\mu : \mu \in M, U_\mu \subseteq X\}$ is *countably almost centred* if each countable subfamily $\{U_1, \ldots, U_n, \ldots\}$ of U has the property that $\cap \{Cl_x U_i : i \in N\}$ is non-empty.

4.1.DEFINITION. A Hausdorff space X is *almost realcompact* if each countably almost centred open ultrafilter on X is fixed.

Frolik has been proved the followings theorems.

4.2.THEOREM. The Cartesian product af almost realcompact spaces is almost realcompact.

4.3.THEOREM. Each closed subset of a regular almost realcompact space X is almost realcompact.

It is well-known that an inverse limit of a Hausdorff spaces is closed in the Cartesian product [2]. Thus we have the following theorem.

4.4.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a regular almost realcompact spaces X_α, then limX is a regular almost realcompact space.

4.5.THEOREM. [26]. For each completely regular space X there exists an almost realcompact space rX with the following properties:

a) $X \subseteq rX \subseteq \beta X$, where βX is the Stone-Cech's compactification of X;

b) If $f: X \longrightarrow Y$ is a mapping into any almost realcompact completely regular space, then there exist $rf: rX \longrightarrow Y$ such that $f = rf/X$.

Let us note that rf is the restriction of βf onto rX.

4.6.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a completely regular spaces X_α such that the projections f_α are onto. If $\beta(\lim X) = \lim \beta X$ then $r(\lim X) = \lim rX$. _

Proof. From the properties of the Stone-Cech's compactification and from Theorem 4.5. b) it follows that there exist inverse systems $\beta X = \{\beta X_\alpha, \beta f_{\alpha\beta}, A\}$ and $rX = \{rX_\alpha, rf_{\alpha\beta}, A\}$. The inverse system rX is the subsystem of the system βX. By virtue of the surjectivity of the mappings $f_{\alpha\beta}$ we infer that limX is densely embedded in limrX and limβX. On can also construct a mappings R:r(limX) ---> limrX and B:β(limX) ---> limβX. Moreover, R is the restriction of B onto limX. It is clear that if B is the homeomorphism, then R is the homeomorphism. The proof is completed.

4.7.REMARK. The notion of the almost realcompactification is a generalization of the Hewitt realcompactification νX of a completely regular space X [2:277]. The space νX is the subspace of βX such that each real-valued functin f:X ---> R has an extension on X. It is evident that Theorem 4.7. holds also for the spaces ν(limX) and limνX.

4.8.THEOREM. Let X be an inverse system as in 4.6. The spaces limrX and r(limX) are homeomorphic if the following condition (CS) is satisfied:

(CS) For every pair F_1, F_2 of completely separated subsets of limX there exists a $\alpha \in A$ such that $f_\alpha(F_1)$ and $f_\alpha(F_2)$ are completely separated subsets of X_α.

Proof. Aplly theorem 4.6. and Lemma 1.1. of the paper [7].

If the spaces limX and X_α, $\alpha \in A$, are normal then each pair of a closed subsets of these spaces are completely separated. Thus the condition (CS) can be replaced by the following condition:

(S) For each pair F_1, F_2 of disjoint closed subsets of limX there exists a $\alpha \in A$ such that $Cl_x\, f_\alpha(F_1) \cap Cl_x\, f_\alpha(F_2) = \emptyset$.

There condition (CS) is satisfied if the inverse system X is a *factorizable* or *f-system* [17]. This means that for each real-valued function f:limX ---> R there exists a $\alpha \in A$ and a real-valued function $g_\alpha:X_\alpha$ ---> R such that $f = g_\alpha f_\alpha$.

4.9.THEOREM. If X is an f-system with onto projections f_α:limX ---> X_α, $\alpha \in A$, then r(limX) = limrX.

Proof. Each f-system satisfies the condition (CS). Apply Theorem 4.8.

4.10.THEOREM. Let X be an ∂-directed inverse system with onto projections f_α such that a space limX is a Lindelof space. Then r(limX) = limrX.

Proof. From [17:Theorem 1.10] it foollows that β(limX) = limβX. Apply Theorem 4.8.

4.11.THEOREM. Let X = {X_n, f_{nm}, N} be an inverse sequence of a normal spaces X_n and onto bonding mappings f_{nm}. If a space limX is countably compact, then r(limX) = limrX.

Proof. By virtue of Theorem 1.3. of the paper [17] it follows that β(limX) = limβX. Apply Theorem 4.8. replacing the condition (CS) by the condition (S).

If the spaces X_n are countably compact and if the mappings f_{nm} are closed, then limX is countably compact [13]. Thus we have

4.12.THEOREM. Let $X = \{X_n, f_{nm}, N\}$ be an inverse sequence of a normal countably compact spaces X_n and a closed onto mappings f_{nm}. Then r(limX) = limrX.

By the same method of proof on can prove for a sequentially compact (strongly countably compact, D-compact) spaces the following

4.13.THEOREM. Let X be an inverse sequence of a normal suquentially compact (strongly countably compact, D-compact) spaces. Then r(limX) = limrX.

4.14.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with a perfect fully closed onto mappings $f_{\alpha\beta}$. If the spaces X_α, $\alpha \in A$, are normal countably compact, then r(limX) = lim rX.

Proof. Let us recall that a mapping f:X ---> Y [17] is *fully closed* if for each point y ∈ Y and each finite open cover $\{U_i : i = 1, \ldots, s\}$ of $f^{-1}(y)$ by open sets U_i, i = 1,..., s, the set {y} ∪ {f#(U_1) ∪ ... ∪ f"(U_)} is an open set in Y. Now from Theorem 1.16. of [17] it follows that β(limX) = lim βX. Theorem 4.8. completes the proof.

We say that a Hausdorff space X is *m-compact*, $m \geq s\backslash^{s}_{o}$, if each open cover U of X has a subcover W of the cardinality |W| < m.

Each countably compact space X is an $s\backslash^{s}_{o}$-compact space.

4.15.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an well-ordered inverse system of $s\backslash^{s}_{m}$-compact normal spaces X_α such that $f_{\alpha\beta}$ are closed onto mappings and cf(A) < $s\backslash^{s}_{m}$. Then r(limX) = limrX.

Proof. Let us recall that cf(A) is the smallest ordinal number which is cofinal in A. Now the condition (S) is satisfied [13]. Theorem 4.8. completes the proof.

4.16.REMARK. By the same method of proof on can see that Theorems 4.6. - 4.15. holds for the realcompactification ν(limX).

We close this Sectin by the consideration of the almost realcompactification r(limX) of an inverse system of a Hausdorff spaces.

If X is a Hausdorff space then an almost realcompactification rX has been definied by Liu and Strecker [12] as follows. Let rX be a subspace of the Katetov extension kX containing a points of X and all countably almost centred open ultrafilters on X. The topology on rX is the subspace topology.

Liu and Strecker was proved the following lemma.

4.17.LEMMA.[12]. a) The space rX is the almost realcompact Hausdorff space in wich X is densely embedded.

b) If Y is any almost realcompactification of X then there exists an extension f:rX ---> Y of the identity i:X ---> X ⊆ Y.

4.18.THEOREM. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a Hausdorff spaces X_α and p-perfect onto $f_{\alpha\beta}$ such that the mapping rf_α are onto. If k(limX) = limkX then r(limX) = limrX.

Proof. There exists the inverse system kX since $f_{\alpha\beta}$ are perfect. The inverse system rX is the subsystem of kX. Clearly, if the spaces k(limX) and limkX are homeomorphic, then the spaces r(limX) and limrX are homeomorphic. The proof is completed.

4.19.REMARK. Now on cannot be proved that the inverse limit of any almost realcompact spaces is almost realcompact since a closed subset of any nonregular almost realcompact space need not be almost realcompact.

4.20.THEOREM. Let X be an inverse system as in theorem 4.18. If the spaces X_α, $\alpha \in A$, are almost realcompact, then limX is almost realcompact.

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Lončar I. H-zatvorena proširenja i apsolut inverznog limesa

S A D R Ž A J

U radu su istraživana H-zatvorena proširenja inverznog limesa. Pri tome je posebna pažnja posvećenja nužnim i dovoljnim uvjetima koje mora ispunjavati inverzni sistem da bi Katetovljevo proširenje k(limX) bilo ekvivalentno limesu inverznog sistema kX (Theorem 1.11.). pomoću ovog teorema dobiveni su neki teoremi za H-zatvorenost i blisku kompaktnost inverznog limesa (Teoremi 1.23. - 1.26.). Za Fominovo proširenje ∂(limX) dobiven je Teorem 2.5. Teorem 3.19. daje nužne i dovoljne uvjete da bi apsolut inverznog limesa bio ekvivalentan inverznom limesu apsoluta. Pomoću pridruženog inverznog sistema aX moguće je dobiti neke teoreme za nepraznost inverznog limesa (Teoremi 3.25. - 3.29.)