Prof. dr. Ivan Lončar
Fakultet organizacije i informatike
V a r a ž d i n

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INVERSE LIMITS OF THE COUNTABLE-DIMENSIONAL SPACES

In this paper we investigate the countable-dimensionality of the inverse limit spaces. Section One contains theorems concerning the countable-dimensionality of a limit space of an inverse system with closed bonding mappings. In Section Two we give some theorems on countable-dimensionality of a limit of an inverse system with open mapping. Section Three is devoted to the inverse system with d-mappings or with α-reducible mappings.

Countable-dimensional; strongly countable-dimensional; inverse system; inverse limit space

0. INTRODUCTION

0.1. A mapping $f : X \to Y$ is open (closed) if $f(A)$ is open (closed) for each open (closed) $A \subseteq X$.

0.2. If $f : X \to Y$ is a mapping, then we define $f^{\#}(A)$ as the set $\{ y : f^{-1}(y) \subseteq A\}$.

0.3. The cardinality of a set A is denoted by $|A|$.

0.4. cf(A) means a smalest ordinal number which is cofinal in a well-ordered set A.

0.5. The closure of a set A we denote by Cl(A).

0.6. We use the notions of inverse system and of inverse limit space as in [6].

0.7. The inverse system $\underline{X} = \{ X_\alpha, f_{\alpha\beta}, A\}$ is σ-directed if for each sequence $\alpha_i \in A$, $i \in N$, there is a $\alpha \in A$ such that $\alpha > \alpha_i$ for each $i \in N$.

0.8. Let $f : X \to Y$ be a mapping of locally connected spaces and let $Cl(V) \subseteq U \subseteq Y$, where U and V are open. The splitting number $s(f, Cl(V), U)$ is the nubmer of components of $f^{-1}(U)$ which meet $Cl(f^{-1}(V))$.

The inverse system $\underline{X} = \{ X_\alpha, f_{\alpha\beta}, A\}$ is locally connected (briefly, lc-system) if the spaces X_α are locally connected and for each $\alpha \in A$ and each pair $Cl(V_\alpha) \subseteq U_\alpha$ there is $\alpha' \geq \alpha$ such that $s(f_{\alpha\beta}, Cl(V_\alpha), U_\alpha)$, $\beta \geq \alpha'$, are finite and $s(f_{\alpha\alpha'}, Cl(V_\alpha), U_\alpha) = s(f_{\alpha\beta}, Cl(V_\alpha), U_\alpha)$.

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The notion of lc-system was introduced in [10]. See also [17].

0.9. By FrA a boundary of a set A is denoted.

1. COUNTABLE-DIMENSIONAL SPACES AND CLOSED MAPPINGS

A space X is called (ind-, Ind-) **countable-dimensional** [22:162] if $X=\cup\{X_i : i\in N\}$ for some subspaces X_i of dimension (ind, Ind) dim $X_i \leq n$.

A space X is called (ind-, Ind-, dim-) countable-dimensional **in the strong sense** [24:162] (or strongly countable-dimensional) if X is the countable sum of (ind, Ind, dim(finite-dimensional closed sets. Clearly, every finite-dimensional spaces is countable-dimensional.

In the class of (separable) metric spaces the notions of (ind-) Ind- and dim- -countable-dimensionality is equivalent.

We say that a metric space X is countable-dimensional if X is Ind- ord dim-countable-dimensional.

In the sequel we use the following theorem (see [2:5-3-517] and [20:163-177]).

1.1. **Theorem.** If X is a metric space, then the following conditions are equivalent:

(i) X is countable-dimensional,
(ii) there exists a closed onto mapping $f:X^o\to X$ of a metric space X^o with dim $X^o\leq 0$ such that $f^{-1}(x)$ < for each $x\in X$,
(iii) there exists a closed onto mapping $f : X^o\to X$ of a metric space X^o with dim $X^o\leq 0$ such that $f^{-1}(x)<\aleph_0$
(iv) for all sequences $\{U_i : i\in N\}$ of open sets and $\{F_i : i\in N\}$ of closed sets satisfying $F_i\subset U_i$, i=1,2,..., there exists a sequence $\{V_i : i\ N\}$ of open sets such that $F_i\subset V_i\subset U_i$ and $\mathrm{ord}_x\ \{\mathrm{Fr}\ V_i : i\in N\}<\aleph_0$ for each $x\in X$.

We say that $f:Y\to X$ has a **weak local order** iff for each $x\in X$ there exists a point $a_x\in f^{-1}(x)$, a neighborhood U_x of a_x, and positive integer n_x such that ord $f|U_x\leq n_x$.

In the paper [31: Theorem 2.] it is proved

1.1'. **Theorem.** A metric space X is strongly countable-dimensional iff there is a 0-dimensional metric spaces B and closed, finite-to-one mapping from B onto X with weak local order.

By a straightforward proof we have

1.1''. **Lemma.** If a mappings $f:X\to Y$ and $g : Y\to Z$ have a weak local order, then $gf : X\to Z$ has a weak local order.

Every local homeomorphism has a weak local order. If $f^{-1}(y)\leq k\in N$ for each $y\in Y$, then $f:X\to Y$ has a weak local order.

1.1'''. **Problem.** Let $\underline{X}=\{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that the mappings $f_{\alpha\beta}$ have a finite local order. Is it true that the projections $f_\alpha : \lim \underline{X}\to X_\alpha$, $\alpha\in A$, have a weak local order?

From (iii) of Theorem 1.1. it follows

1.2. **Theorem.** Let $f : X \to Y$ be a closed mapping of a metric countable-dimensional space X onto a metric space Y. If $f^{-1}(y) \leq \aleph_0$ for each $y \in Y$, then Y is countable-dimensional.

1.3. **Theorem.** Let $f: X \to Y$ be a closed mapping of a metric space X onto a metric spce Y. If $f^{-1}(y) = k < \aleph_0$ for every $y \in Y$, then X is countable-dimensional iff Y is countable-dimensional.

Proof. If X is countable-dimensional then so is Y (1.2. Theorem). If Y is countable-dimensional, then $Y = \cup \{Y_i : i \in N, \dim y_i \leq 0\}$. The mapping $f_{y_i} : f^{-1}(Y_i) \to Y_i$ is closed and $f_{y_i}(y)_{-1} = k$ for each $y \in Y_i$. By Suzuki theorem [7:286] it follows that $\dim Y_i = \dim f^{-1}(Y_i)$, $i \in N$. The proof is completed since $X = \{f^{-1}(y_i) : i \in N\}$.

Now we apply Theorem on dimension-lowering mappings [7].

1.4. **Theorem.** Let $f : X \; Y$ be a closed mapping onto metric countable-dimensional space Y. If $\dim f^{-1}(y) \leq k$ for each $y \in Y$, then X is countable-dimensional.

Proof. Let $Y = \cup \{Y_i : i \in N\}$ with $\dim Y_i < \aleph_0$. The mappings $f_{y_i} : f^{-1}(Y_i) \to Y_i$, $i \in N$, satisfy Theorem on dimension-lowering mappings. Hence, $\dim f^{-1}(y_i) \leq \dim y_i + k < \aleph_0$. The proof is completed, since $X = \cup \{ f^{-1}(Y_i) : i \in N\}$.

From Theorem 1.2. and 1.4. it follows

1.5. **Theorem.** Let $f: X \to Y$ be a closed mapping of a metric space X onto a metric space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$. The space X is countable-dimensional iff the space Y is countable-dimensional.

1.6. **Remark.** A) Theorem 1.3. is a corollary of Theorem 1.5. B) By the well known method of proof it follows that in Theorem 1.2. one can assume that $\mathrm{Fr}\, f^{-1}(y) \leq \aleph_0$. C) The alternative proof of Theorem 1.5. in the case when X has a countable network is the following. From [4:354, Ex.97.] it follows that if $f: X \to Y$ is a closed onto mapping with $f^{-1}(y) \leq \aleph_0$ for each $y \in Y$, there exist the subsets X_i, $i \in N$, of X such that: a) $X = \cup \{X_i : i \in N\}$ b) $Y = \cup \{f(X_i) : i \in N\}$ and $f_i = f/X_i : X_i \to f(X_i)$ is a homeomorphism. This means that X_i is countable-dimensional iff $Y_i = f(X_i)$ is countable-dimensional. From the relations a) and b) it follows that X is countable dimensional iff Y is countable-dimensional.

Now we study the countable-dimensionality of the inverse limit space

1.7. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a metric spaces X_α and closed mappings with $|f_{\alpha\beta}^{-1}(x_\alpha)| \leq k \in N$. If the space $X = \lim \underline{X}$ is a metric space, then X is countable dimensional iff the spaces X_α are countable-dimensional.

Proof. It is readily seen that $|f_\alpha^{-1}(x_\alpha)| \leq k$. On the other hand, the projections f_α are perfec [2:148]. Now apply Theorem 1.5. Q.E.D.

Similarly, one can prove

1.8. **Theorem.** Let a metric space X be a limit of a σ-directed inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$, where X_α are metric spaces and $f_{\alpha\beta}$ a closed mappings with

$|f_{\alpha\beta}^{-1}(x_\alpha)| < \aleph_0$. The space X is countable-dimensional iff the spaces X_α are countable-dimensional.

For compact metric spaces we have

1.9. **Theorem.** Let a metric space X be a limit of a σ-directed inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$, where X_α are compact metric spaces. If X_α are countable-dimensional, then X is countable-dimensional.

Proof. Suppose $\mathcal{U} = \{U_i : i\varepsilon N\}$ and $\mathcal{F} = \{F_i : i\varepsilon N\}$ are sequences as in Theorem 1.1. For every U_i and $F_i \subset U_i$ there exist $\alpha_i \varepsilon A$ and open set W_{α_i} such that $f_{\alpha_i}(F_i) \subseteq W_{\alpha_i}$ and $F_i \subseteq f_{\alpha_i}^{-1}(W_{\alpha_i}) \subseteq U_i$. Since A is σ-directed, there exist $\alpha > \alpha_1, \alpha_2, \ldots$. Let $W_i = f_{\alpha_i}^{-1}(W_{\alpha_i})$, $i\varepsilon N$. From the assumption that X_α is countable-dimensional it follows that there exists a family $\mathcal{V}_\alpha = \{V_i : i\varepsilon N\}$ such that $f_\alpha(F_i) \subseteq V_i \subseteq W_i$, $i\varepsilon N$, and $\mathrm{ord}_x \{F_r(V_i) : i\varepsilon N\} < \infty$ at every point $x\varepsilon X$. Let $\mathcal{V} = \{f_\alpha^{-1}(V_i) : V_i \varepsilon \alpha\}$ Clearly, $F_i \subseteq f^{-1}(V_i) \subseteq U_i$, $i\varepsilon N$, and $\mathrm{ord}_x \{F_r(V) : V \varepsilon \mathcal{V}\} < +\infty$ at every point $x\varepsilon X$. Q.E.D.

1.10. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\Omega\}$ be an well ordered inverse system of metric spaces X with the property $hl(X_\alpha) < \aleph_m$, $\alpha\varepsilon\Omega$, and $cf(\Omega) > \omega_m$. If the spaces X_α are countable-dimensional and $X = \lim\limits_{\leftarrow} \underline{X}$ is a metric space, then X is countable-dimensional.

Proof. From [16] and [28] it follows that every open (closed) $U_i \subset X$ is the form $U_i = f_{\alpha_i}^{-1}(U_{\alpha_i})$ for some open (closed) $U_{\alpha_i} \subset X_{\alpha_i}$. Thus, for every $F_i \subset U_i$ there exists a_i such that $F_i \subseteq f_{\alpha_i}^{-1} \subseteq (W_{\alpha_i}) \subset U_i$ for some open set $W_{\alpha_i} \subseteq X_{\alpha_i}$. The remaining part of the proof coincides with the corresponding part of the proof of Theorem 1.9.

If X_α are separable metric spaces and a) $f_{\alpha\beta}$ are perfect or b) $f_{\alpha\beta}$ is open or c) $\underline{X}$ is cointinuous then w (lim $\underline{X}$) [26] i.e. $X = \lim\limits_{\leftarrow} \underline{X}$ is a metric space. Hence, we have

1.11. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well ordered inverse system of separable metric spaces X_α such that $cf(A) > \omega_1$. If $\underline{X}$ is continuous (or $f_{\alpha\beta}$ are perfect, or $f_{\alpha\beta}$ are open) and X_α are countable-dimensional, then $X = \lim\limits_{\leftarrow} \underline{X}$ is countable-dimensional.

1.12. **Theorem.** Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of metric (strongly) countable-dimensional spaces X_n. If f_{nm} are closed mappings such that $\dim f_{nm}^{-1}(x_n) \leq k$ for some $k\varepsilon N$, then $X = \lim\limits_{\leftarrow} \underline{X}$ is (strongly) countable-dimensional.

Proof. From [7:261] it follows that $\dim f_n^{-1}(x_n) \leq k$ i.e. $\dim f_n \leq k$, $n\varepsilon N$. Let $X_n = \cup \{X_{ni} : i \varepsilon N\}$, where X_{ni}, $i\varepsilon N$, is finite-dimensional. Let $Y_i = f_n^{-1}(X_{ni})$, $i\varepsilon N$. Clearly, $X = \{Y_i : i\varepsilon N\}$. The restriction f_n/y_i, $i\varepsilon N$, are closed since f_n is a closed mapping. From the inequality $\dim Y_i \leq \dim f_n + \dim X_{ni}$ [2,pp.452] it follow that Y_i is finite-dimensional. The proof is completed.

1.13. **Theorem.** Let X be a limit space of an inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ of compact metric (strongly) countable-dimensional spaces X_α. If $\dim f_{\alpha\beta} \leq k$ for some $k\varepsilon N$, then X is (strongly) countable-dimensional.

Proof. X is a compact space and the projections f_α, $\alpha\in A$, are perfect mappings with dim $f_\alpha \leq k$. Let $X_\alpha = \{X_{\alpha i} : i\in N\}$ where $X_{\alpha i}$, $i\in N$, are finite-dimensional subspaces of X_α. The restrictions $f_{\alpha i} = f_\alpha / Y_i$, $i\in N$, are closed [6:pp.50.] and perfect. It follows that $Y_i = f_\alpha^{-1}(X_{\alpha i})$, $i\in N$, are paracompact subspaces of X. From the inequality dim $Y_i \leq$ dim $f_{\alpha i}$ + dim $X_{\alpha i}$ [2:pp.452.] it follows that Y_i, $i\in N$, are finite-dimensional. The proof is completed since X = $\{Y_i : i\in N\}$.

The proof of the next theorem is similar to the proof of Theorem 1.12.and 1.13.

1.14. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of hereditary paracompact spaces X_α and the perfect mappings $f_{\alpha\beta}$. If the spaces X_α are (strongly) countable-dimensional and if dim $f_{\alpha\beta} \leq k$, $k\in N$, then X = lim $\underline{X}$ is a hereditary paracompact (strongly) countable-dimensional space.

1.15. **Problem.** It is true that X in Theorems 1.12.-1.14. (strongly) countable-dimensional if $f_{\sigma\beta}^{-1}(x_\alpha)$ is (strongly) countable-dimensional for every α, x and β?

1.16. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of the completely regular (strongly) countable-dimensional spaces X_α. If $f_{\alpha\beta}$ are open-closed $\leq$ k-to-one mappings, then X = $\varprojlim$ $\underline{X}$ is (strongly) countable-dimensional.

Proof. The projections $f_\alpha : X \to X_\alpha$, $\alpha\in A$, are open-and-closed. It is readily seen that f_α, $\alpha\in A$, are k-to-one mappings. Let X = $\{X_{\alpha i}$, $i\in N$, where $X_{\alpha i}$, $i\in N$, are finite-dimensional subspaces of X_α. From [13]it follows that dim X_{α_i} = dim $f_\alpha^{-1}(X_{\alpha i})$, $i\in N$. This finishes the proof since X $= \bigcup\{f_\alpha^{-1}(X_{\alpha i}) : i\in N\}$.

A mapping $f:X\to Y$ is fully closed [9] it for each $y\in Y$ and each open cover $\{U_1,\dots,U_k\}$ of the set $f^{-1}(y)$ the set $\{y\} \cup (f^{\#}U_k)$ is open in Y.

A mapping $f:X\to Y$ is fully closed iff f is closed and if $f(F_1)\cap f(F_2)$ is a discrete subspace of Y for each pair F_1, F_2 of a closed disjoint subsets of X.

1.17. **Lemma.** [9: Lemma 3.]. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with perfect mappings $f_{\alpha\beta}$. The projections f_α: $\varprojlim$ $X\in X$, $\alpha\in A$, are fully closed iff $f_{\alpha\beta}$ are fully closed.

On can prove

1.18. **Lemma.** Let X = $\{X_n, f_{nm}, N\}$ be an inverse sequence with pseudoperfect mappings f_{nm}. The projections $f_n : \varprojlim \underline{X} \to X_n$, $n\in N$, are fully closed iff f_{nm} are fully closed.

1.19. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of normal spaces X_α and fully closed perfect mappings $f_{\alpha\beta}$ such that dim $f_{\alpha\beta}^{-1}(x_\alpha) \leq k$. If a space X = $\varprojlim$ $\underline{X}$ is normal, then X is strongly countable-dimensional iff spaces X_α, $\alpha\in A$, are strongly countable-dimensional.

Proof. The projections $f_\alpha : X\to X_\alpha$, $\alpha\in A$, are perfect fully closed such that dim $f_\alpha^{-1}(x_\alpha) \leq k$. Let $X_\alpha = \cup\{X_{\alpha,i} : i\in N\}$for some $\alpha\in X_\alpha$. A mapping $f_i = f_\alpha | f_\alpha^{-1}(X_{\alpha,i}) \to X_{\alpha,i}$

is perfect fully closed. From [9:Theorem 3.] it follows $\dim f_\alpha^{-1}(X_{\alpha,i}) \le \max\{\dim f_i, \dim f_{\alpha,i}\} < \aleph_0$. The proof is completed since $X = \bigcup\{f_\alpha^{-1}(X_{\alpha,i}) : i \in N\}$. Conversely, if $X = \bigcup\{X_i : i \in N\}$, where X_i are closed finite-dimensional subsets of X, then for every $\alpha \in A$ we have $X_\alpha = \bigcup\{f_\alpha(X_i) : i \in N\}$ and $\dim f_\alpha(X_i) \le \dim X_i + 1$ [9: Theorem 4.]. The proof is completed.

1.20. **Remark.** Is it true that Theorem 1.19. holds for countable-dimensionality if we assume that X_α, $\alpha \in A$, and X are hereditarily normal? Clearly, the "if" part holds.

1.21. **Corollary.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of a normal countably compact spaces X_α and fully closed perfect mappings $f_{\alpha\beta}$ such that $\dim f_{\alpha\beta}^{-1}(X_\alpha) \le k$. A space $X = \lim \underline{X}$ is strongly countable-dimensional iff all X_α are stronlgy countable-dimensional.

Proof. From Lemma 1.17. and Theorem 1.19. it follows that it suffices to prove that $X = \varprojlim \underline{X}$ is normal. If F_1, F_2 are disjoint closed subsets of X, then $Y_\alpha = f_\alpha(F_1) \cap f_\alpha(F_2)$, $\alpha \in A$, are finite since Y_α is discrete (see definition of the fully closed mappings). This means that $\underline{Y} = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ is the inverse system of compact spaces Y_α. If we assume that $Y = \varprojlim \underline{Y} \neq 0$ then we obtain the contradiction $\emptyset \neq Y \subset F_1 \cap F_2 = \emptyset$. It follows that there exist $\alpha \in A$ such that $Y_\alpha = \emptyset$. Since X_α is normal, there exist disjoint open sets $U_\alpha \supset f_\alpha(F_1)$, $V_\alpha \supset f_\alpha(F_1)$. The space X is normal since $f_\alpha^{-1}(U_\alpha) \supset F_1$ and $f_\alpha^{-1}(V_\alpha) \supset F_2$. The proof is completed.

1.22. **Remarks.** A) From [9:115] it follows that if $\dim X_\alpha \le n$, then $\dim X \le n$ and $\dim X \ge n-1$.

B) From the proof of the normality of X in Theorem 1.21. it follows that if X_α are connected then X is connected.

C) If $\underline{X}$ is locally connected [10] then a local connectedness of X_α implies local connectedness of X.

D) Similarly it follows that $\beta(\varprojlim \underline{X})$ is homeomorphic with $\varprojlim \beta\underline{X}$, where $\beta\underline{X} = \{\beta X_\alpha, f_{\alpha\beta}, A\}$.

From Lemma 1.18. one can deduce

1.23. **Theorem.** Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of a normal countably compact spaces X_n and fully closed mappings f_{nm} such that $\dim f_{nm}^{-1}(x_n) \le k$. A space $X = \varprojlim \underline{X}$ is strongly countable-dimensional iff spaces X_n are strongly countable-dimensional.

Proof. The projections $f_n : X \to X_n$, $n \in N$, are fully closed by Lemma 1.18. Furthemore, $\dim f_n^{-1}(X_n) \le k$ [15:2.1. Theorem]. The remaining part of the proof is as in the proof of Theorem 1.19.

We close this Section with the inverse systems of finite-dimensional spaces which have infinite-dimensional (= not finite-dimensional) limit.

1.24. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of finite-dimensional spaces X_α and closed mappings $f_{\alpha\beta}$ such that $1 \leq f_{\alpha\beta}^{-1}(x_\alpha) \leq k+1$ for some fixed $k\in N$. If $\underleftarrow{\lim}\ \underline{X} = X$ is normal and for every $n\in N$ there is a $m\in N$ sch that $\dim X_m \geq n$, then X is infinite-dimensional.

Proof. The mappings $f_\alpha : \lim X \to X_\alpha$, $\alpha\in A$, are closed and $|f_\alpha^{-1}(x_\alpha)| \leq k+1$ [2]. From [2:Zamečanie] it follows that $\dim X_n \leq \dim X+k$ for each X_n. This is impossible since $\dim X_n \to \infty$ if $n\to\infty$. The proof is completed.

1.25. **Remark.** On can prove that Theorem 1.24. is valid if we asume that $|Fr\ f_{\alpha\beta}^{-1}(X_\alpha)| \leq k+1$.

1.26. **Question.** Is it true that X in 1.24. is countable-dimensional? The answer is positive if $\underline{X}$ is an inverse sequence of metric spaces (or $\lim \underline{X}$ is metric). (See 1.7. Theorem, 1.8. Theorem and 1.10. Theorem).

1.27. **Remark.** From [2:449. Theorem 1] it follows that $X = \lim \underline{X}$ in Theorem 1.24. is not Ind-finite-dimensional.

2. OPEN MAPPINGS AND COUNTABLE-DIMENSIONALITY

We start with the following theorem.

2.1. **Theorem.** Let $f:X\to Y$ be an open onto mapping between a separable metric space such that each $f^{-1}(y)$ has an isolated point. If X is countable-dimensional, then so is Y.

Proof. Let A_i be a set defined in the proof of Lemma 1.12.5. of [7]. It follows that A_i is countable-dimensional. Since f/A_i is a homeomorphism, the set $f(A_i)$ is countable-dimensional. From the relation $Y = \cup\{f(A_i):i\in N\}$ it follows that Y is countable-dimensional.

If X and Y are locally compact separable metric spaces, then from $|f^{-1}(y)| \leq \aleph_0$ it follows that $f^{-1}(y)$ has an isolated point [7:139]. Thus, we have

2.2. **Theorem.** Let $f:X\to Y$ be an open surjection between separable locally compact metric spaces. If $|f^{-1}(y)| \leq \aleph_0$ for every $y\in Y$, then Y is countable-dimensional if so is X.

If each $f^{-1}(y)$ is a discrete subspace of X, then one can assume that A_i, $i\in N$, are closed [3:194] and $X = \cup\{A_i:i\in N\}$. Thus we have

2.4. **Theorem.** Let $f:X\to Y$ be an open onto mapping between separable metric spaces. The space X is countable-dimensional iff Y is countable-dimensional.

In [3:3.2. Theorem] it is proved

2.5. **Theorem.** If $f:X\to Y$ is an open surjection between metric spaces such that for each $y\in Y$ $|f^{-1}(y)| < \aleph_0$ then X is countable-dimensional iff Y is countable-dimensional.

We say that a mapping $f : X \to Y$ is inductively open [3:209] if there exists a subspace $X_1 \subseteq X$ such that $f/X_1 : X_1 \to Y$ is an open mapping and $f(X_1) = Y$.

2.6. **Theorem.** Let $f : X \to Y$ be an inductively-open closed surjection between metric spaces such that for each $y \in Y$ $|f^{-1}(y)| \leq \aleph_0$. If X is countable-dimensional (countable-dimensional in the strong sense), then Y is countable-dimensional (countable-dimensional in the strong sense).

2.7. **Remark.** If X is a complete metric space (of $f^{-1}(y)$ are complete), then the assumption of the closedness of f can be amitted [3:219.]

If the fibers $f^{-1}(y)$ are finite, then we have

2.8. **Theorem.** [3:9.5. Theorem]. Let $f:X \to Y$ be an inductively-open onto mapping with finite fibers. If X is countable- dimensional (in the strong sense), then Y is countable-dimensional (in the strong sense).

Now, we prove the following theorems.

2.9. **Theorem.** Let $f:X \to Y$ be open onto mapping betwen metric spaces such that $f^{-1}(y)$ is discrete. If Y is (strongly) countable-dimensional, then X is countable-dimensional.

Proof. Let $Y = \cup\{Y_i : i \in N\}$, where Y_i is finite-dimensional (closed) subspace. The mappings $f_i : f^{-1}(Y_i) \to Y$, $i \in N$, are open with discrete fibers. By Hodel's theorem [7:288] it follows that $\dim f^{-1}(y_1) \leq \dim Y_i$. The proof is completed since $X = \cup \{ f^{-1}(Y_i) : i \in N \}$.

2.10. **Theorem.** If $f:X \to Y$ is an open onto mapping between locally compact metric spaces such that $|f^{-1}(y)| \leq \aleph_0$ for each $y \in Y$ and Y is (strongly) countable-dimensional, then X is (strongly) countable-dimensional.

Proof. In the proof of Theorem 2.9. apply [7:288, 4.3.E(d)] instead of [7:288, 4.3E(a)].

For inverse system of a metric spaces we have

2.11. **Theorem.** Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse system of metric spaces X_n and open mappings f_{nm} such that $f_{nm}^{-1}(x_n)$ are discrete. If X_n are (strongly) countable-dimensional, then $X = \lim\limits_{\leftarrow} \underline{X}$ is (strongly) countable-dimensional.

Proof. Let $X_n = \cup\{X_{n,i} : i \in N\}$. For every $m>n$ a mapping $f_{nm,i} = f_{nm} / f_{nm}^{-1}(X_i) : f_{nm}^{-1}(X_i) \to X_i$ is open with discrete fibers. This means that $\dim f_{nm,i}^{-1}(X_i) \leq \dim X_i$ [7:288.]From Nagami's theorem [7:261] it follows that inverse system $\underline{X}_i = \{ f_{nm,i}^{-1}(X_i), m>n\}$ has a finite-dimensional limit X_i.

Since $X = \cup \{X_i : i \in N\}$, we conclude that X is countable-dimensional.

Similarly, by [7:288, 4.3.E8D)] and [7:261.4.1.22.Theorem] we have

2.12. **Theorem**. Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse system of a locally compact metric spaces X_n and open mappings f_{nm} such that $|f^{-1}_{nm}(X_n)| \leq \aleph_0$. If X_n are (strongly) countable-dimensional, then $X = \lim \underline{X}$ is (strongly) countable-dimensional).

Generaly we have the following

2.13. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of the spaces X_α and mappings $f_{\alpha\beta}$ such that:

i) $X_\alpha = \cup\{X_{\alpha,i} : i \in N\}$, where $X_{\alpha,i} \in \mathcal{K}$

ii) $\dim f^{-1}_{\alpha\beta}(X_{,i}) \leq \dim X_{\alpha,i}$

iii) Every system $\underline{X_i} = \{X_{\alpha,i}, f_{\alpha\beta}, N\}$

has the limit X_i with $\dim X_i \leq \dim X_{n,i}$ then, if X_α are countable-dimensional so is the $\lim \underline{X}$.

From this Theorem we have

2.14. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a σ-directed inverse system of compact spaces X_α and open mappings $f_{\alpha\beta}$ such that $|f^{-1}_{\alpha\beta}(X| \leq \aleph_0$. If the spaces X_α are strongly countable-dimensional, then $X = \lim \underline{X}$ is countable-dimensional.

Proof. Let $X = \cup\{X_{\alpha,i} : i \in N\}$, where $X_{\alpha,i}$ is a closed finite-dimensional subspace of X_α. For every $\beta > \alpha$ we consider $X_{\beta,i} = f^{-1}_{\alpha\beta}(X_{\alpha,i})$. The restriction $f'_{\beta\alpha} : X_{\beta,i}\ X_{\alpha,i}$ is open with $|f'^{-1}_{\beta\alpha}(X_\alpha)| \leq \aleph_0$.

From [2:459] it follows $\dim X_{\beta,i} = \dim X_{\alpha,i} < \aleph_0$. The condition ii) of Theorem 2.13. is satisfied. Inverse system $\underline{X_i} = \{X_{\beta,i}, f'_{\beta\gamma}, \alpha \leq \beta \leq \gamma\}$ is an inverse system of compact spaces such that iii) is satisfied. The proof is completed.

We close this Section with theorems of infinite-dimensionality of a limit of an inverse system of finite-dimensional spaces.

2.15. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a σ-directed inverse system of compact spaces X_α, and open mappings $f_{\alpha\beta}$ such that $1 \leq |f^{-1}_{\alpha\beta}(X_\alpha) < \aleph_0$. If $\dim X_n \to \infty$ when $n \to \infty$, then $X = \lim \underline{X}$ is not finite-dimensional (= infinite-dimensional).

Proof. Before the proof, let us observe that X is countable-dimensinal (see Theorem 2.14.). Now, suppose that $\dim X$ is finite. From [2:459] it follows that $\dim X = \dim X_n$ for each X_n. This is a contradiction since $\dim X_n \to \infty$. Q.E.D.

2.16. **Theorem.** Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of separable metric spaces with open mappings f_{nm} such that $1 \leq |f^{-1}_{nm}(X_n)| \leq k$ for some natural number $k \in N$. If $\dim X_n \to \infty$ when $n \to \infty$, then $X = \lim \underline{X}$ is not a finite-dimensional countable-dimensional space.

Proof. The countable-dimensionality of X it follows from 2.11. Theorem. If we suppose that $\dim X$ is finite, we obtain a contradiction $\dim X = \dim X_n$ (see [7:1.12.7. Theorem.]) since $\dim X_n \to \infty$. The proof is completed.

3. REMAINING CLASSES OF MAPPINGS

We say that a mapping $f : X \to Y$ is dissipative (briefly, d-mapping) [25] if for each point $x \in X$ and each open set $U_x \ni x$ there is an open set $U_y \ni y = f(x)$ such that $f^{-1}(U_y)$ is the union of two disjoint open sets V_1, V_2, and $x \in V_1 \subseteq U_x$.

3.1. Remarks. A) Each closed mapping $f : X \to Y$ defined on a normal space X with ind $f = 0$ (i.e. ind $f^{-1}(y) = 0$ for each $y \in Y$) is a d-mapping [25:143].

B) If $f : X \to Y$ is d-mapping, then ind $X \leq$ ind Y [25].

C) Let X be a T_1-space, Y a metric space. If $f: X \to Y$ is a d-mapping, then, if X is compact (Lindelöf, strongly or totally paracompact), we have dim X = ind X = Ind X [25:145]

D) [25:149, Theorem 1.2]. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system. If $f_{\alpha\beta}$ are d-mappings, then the projections $f_\alpha : \lim \underline{X} \to X_\alpha$, $a \in A$, are d-mappings.

E) Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that X_α, $a \in A$, are perfectly normal compact spaces and $f_{\alpha\beta}$ are ind-zero-dimensional mappings. If ind $X_\alpha \leq k$, then Ind $(\lim \underline{X}) \leq r$.

Now we prove

3.2. Theorem. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with d-mappings $f_{\alpha\beta}$. If the spaces X_α are (strongly) ind-countable-dimensional, then $X = \lim \underline{X}$ is an(strongly) ind-countable-dimensional space.

Proof. Let $X_\alpha = \cup\{X_{\alpha,i} : i \in N\}$, where $X_{\alpha,i}$ is a subset of X_α for each $i \in N$ (closed subset in the case of a strongly ind-countable-dimensionality). For each $\beta \geq \alpha$ let $X_{\beta,i} = f_{\alpha\beta}^{-1}(X_{\alpha,i})$. A mappings $f_{\alpha\beta}/X_{\beta,i}$ ($x \in N$) are d-mappings [25:144(5)]. From 3.1.D) it follows that $f_\alpha / f_\alpha^{-1}(X_{\alpha,i})$ is d-mapping since $f_\alpha^{-1}(X_{\alpha,i})$ is a limit of a system $\{X_{\beta,i}, f_{\beta\gamma}/X_{\gamma,i}, \alpha \leq \beta \leq \gamma\}$. Now we have ind $f_{\alpha\beta}^{-1}(X_{\alpha,i}) \leq$ ind $X_{\alpha,i}$ (see 3.1. B)). Since $\lim \underline{X} = \cup\{f_\alpha^{-1}(X_{\alpha,i}) : i \in N\}$, we infer that $X = \lim \underline{X}$ is (strongly) ind-countable-dimensional Q.E.D.

From this Theorem and 3.1.C) it follows

3.3. Theorem. Let X be a limit of an inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ such that X_α, $\alpha \in A$, are metric and $f_{\alpha\beta}$ d-mappings. If X_α are (strongly) countable-dimensional and X is compact (Lindelöf, strongly or totally paracompact), then X is (strongly) countable-dimensional.

3.4. Theorem. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of compact metric spaces X_α and d-mappings $f_{\alpha\beta}$. If X_α are strongly countable-dimensional, then X is also strongly countable-dimensional.

Proof. The spaces X_α are separable. This means that ind X_α = Ind X_α = dim X_α [7]. Apply Theorem 2.13., 3.1.C. and the well known theorem on the dimension dim for a limit of an inverse system of compact spaces.

3.5. **Theorem.** Let $X = \{X_n, f_{nm}, N\}$ be an inverse sequence of separable metric (strongly) dim-countable-dimensional space X_n. If f_{nm} are d-mappings, then $X = \lim\limits_{\leftarrow} \underline{X}$ is (strongly) dim-countable-dimensional space.

Proof. Apply Theorem 2.13. and Nagata's theorem [7:261, 4.1.22. Theorem]

3.6. **Remark.** The hypothesis that f_{nm} are d-mappings is essential. Namely, the space I^{ω} is a limit of an inverse sequence of the finite-dimensional cubes I^n, but I^{ω} is not countable-dimensional [7:81,1.8.20.Theorem.].

Let $f:X \to Y$ be a mapping. We say that a point $x \in X$ is a point f-perfectness if there is a neighborhood $U \ni x$ such that the restriction f/ClU is a perfect mapping onto a closed subset of Y [21].

If $A \subseteq X$, then for a mapping $F_A = f/A$ we define the residue $A^{\alpha}_{f_A}$ in the following way. Let $A^0_{f_A} = A$. Suppose that $A^{\beta}_{f_A}$ is defined for each $\beta<\alpha$. If there exists $\alpha-1$, we define $A^{\alpha}_{f_A}$ as the set of all points $x \in A^{\alpha-1}_{fA}$ which are not the points $f_{A^{\alpha-1}_{f_A}}$ - perfectness. If α is a limit ordinal, let $A^{\alpha}_{f_A} = \bigcap\{A_{f_a}{}^{\beta} : \beta<\alpha\}$.

A mapping f_A is α-**reducible** if there exists an ordinal α sucht that $A^{\alpha}_{f_A} = \emptyset$. A mapping f is 1-reducible iff it is locally perfect.

We say that $I(f) < \infty$ if for $f:X \to Y$ there is a natural number $n \in N$ such that $A^n_f = \emptyset$ [21:119].

In the paper [21:Sledstvie 7.] the following theorem is proved.

3.7. **Theorem.** Let $f:X \to Y$ be a mapping such that $I(f)<\infty$. If $W(X)$, $W(Y) \leq$, then $\dim X \leq \dim f + \dim Y$.

Now, we prove

3.8. **Theorem.** Let $\underline{X} = \{X_{\alpha}, f_{\alpha\beta}, A\}$ be an inverse system of separable metric spaces X_{α} and perfect mapping $f_{\alpha\beta}$ such that $\dim f_{\alpha\beta} \leq \emptyset$. If $\dim X_{\alpha} \leq n$ and $W(\lim \underline{X}) \leq \aleph_0$, then $\dim (\lim\limits_{\leftarrow} \underline{X}) \leq n$.

Proof. The projections $f_{\alpha}: \lim \underline{X} \to X_{\alpha}$, $\alpha \in A$, are pefect [2:148]. This means that $\dim f_{\alpha} \leq 0$ [7:247.]. From Theorem 3.7. it follows that $\dim (\lim\limits_{\leftarrow} \underline{X}) \leq \dim X_{\alpha} \leq n$.Q.E.D.

3.9. **Corollary.** Let $\underline{X} = \{X_{\alpha}, f_{\alpha\beta}, A\}$ be an inverse system of separable metric spaces X_{α} and perfect mappings $f_{\alpha\beta}$ such that $\dim f_{\alpha\beta} \leq 0$. If X_{α}, $\alpha \in A$, are (strongly) dim-countable-dimensional and $W(\lim\limits_{\leftarrow} \underline{X}) \leq \aleph_0$ then a limit $\lim \underline{X}$ a (strongly) dim--countable-dimensional space.

3.10. **Remark.** By the total induction one can prove that in Theorem 3.8. $\mathrm{Ind}(\lim X) \leq n$ and $\mathrm{ind}(\lim \underline{X}) \leq n$. thus, a limit $\lim \underline{X}$ in Corollary 3.9. is a ind-(Ind-) countable-dimensional space.

From [28] it follows that $W(\lim \underline{X}) \leq \aleph_0$ if $\underline{X}$ is a well-ordered inverse system such that $cf(A) > \omega_1$ and:

a) $f_{\alpha\beta}$ are perfect or b) f_α are open of c) $\underline{X}$ is continuous.

Usin this assertion we obtain the following theorems.

3.11. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a well-ordered inverse system of separable metric spaces X_α with $cf(A) > \omega_1$. If X_α, $\alpha\varepsilon A$, are (strongly) countable-dimensional and $f_{\alpha\beta}$ are perfect such that $\dim_{\alpha\beta} \leq n$, the $X = \lim \underline{X}$ is (strongly) countable-dimensional.

Proof. Let $X = \cup \{X_{\alpha,i} : i\varepsilon N\}$, $\dim X_{\beta,i} \leq n$.
From 3.7. it follows that $\dim X_{\beta,i} = \dim f_{\alpha\beta}^{-1}(X_{\alpha,i}) \leq n$.

Let $\underline{X}_i = \{X_{\beta,i}, f_{\beta\gamma}/X_{\beta,i}, \alpha \leq \beta \leq \gamma\}$. The system $\underline{X}_i$ has the dimension $\dim \leq n$ [15]. The proof is completed since $X = \cup\{\lim \underline{X}_i : i\varepsilon N\}$.

3.12. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a continuous inverse system of separable metric spaces X_α and perfect mapping $f_{\alpha\beta}$ such that $\dim f_{\alpha\beta} \leq k$. If $\dim X_\alpha \leq n$, then $\dim (\lim \underline{X}) \leq n+k$.

3.13. **Corollary.** If X_α, $\alpha\varepsilon A$, in Theorem 3.12. are (strongly) dim-countable-dimensional, then $\lim \underline{X}$ is (stronlgy) dim-countable-dimensional.

3.14. **Remark.** By the total induction one can prove that in Theorem 3.12. Ind $(\lim \underline{X}) \leq n+k$ and ind $(\lim \underline{X}) \leq n+k$. Hence, a limit $\lim \underline{X}$ in Corollary 3.13. is ind- and Ind-countable-dimensional.

We close the application of Theorem 3.7. with the following theorem.

3.15. **Theorem.** Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be a locally connected inverse system of locally compact separable metric spaces X_α and perfect dim-zero-dimensional mappings $f_{\alpha\beta}$. If $\dim X_\alpha \leq n$ then $\dim (\lim \underline{X}) \leq n$.

Proof. The projections $f_\alpha : \lim \underline{X} \to X_\alpha$ are perfect dim-zero-dimensional. The space $\lim \underline{X}$ is locally connected [17]. From [4] or [18] it follows that $W(\lim \underline{X}) \leq \aleph_0$. Applying Theorem 3.7. we complete the proof.

3.16. **Remark.** If $\underline{X}$ is σ-directed and X_α locally connected, then $\underline{X}$ is locally connected. Hence, in this case we have also $\dim (\lim \underline{X}) \leq n$.

3.17. **Corollary.** If in Theorem 3.15. the spaces X_α are (strongly) countable-dimensional, then $\lim \underline{X}$ is (stronlgy) countable-dimensional.

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Lončar I. Inverzni limesi prebrojivo dimenzionalnih prostora

SAŽETAK

U radu je istraživana prebrojiva dimenzionalnost inverznog limesa sistema prebrojivo dimenzionalnih prostora.

Prvi odjeljak posvećen je prebrojivoj dimenzionalnosti limesa inverznog sistema prebrojivo dimenzionalnih prostora i zatvorenih veznih preslikavanja.

U drugom odjeljku izučavani su analogni inverzni sistemi uz otvorena vezna preslikavanja.

Treći odjeljak sadrži teoreme o prebrojivoj dimenzionalnosti limesa inverznih sistema uz još neke vrste preslikavanja, kao što su d-preslikavanja i α-reducibilna preslikavanja.