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UDK. 513
Znanstveni rad

APPLICATIONS OF Θ-CLOSED AND U-CLOSED SETS

For every Hausdorff space X the spaces X_Θ *and* X_u *are introduced. If X is H-closed (Urysohn-closed) then* X_Θ (X_u) *is compact* T_1*-space.*

If $f : X \to Y$ *is a mapping, then there exist the mappings* $f_\Theta : X_\Theta \to Y_\Theta$ *and* $f_u : X_u \to Y_u$. *We say that* $f : X \to Y$ *is a Θ-closed (u-closed) mapping if* f_Θ (f_u) *is a closed mapping. If X and Y are H-closed (Urysohn-closed) and* $f : X \to Y$ *is the HJ-mapping, then* f_Θ (f_u) *is Θ-closed (u-closed).*

Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ *be an inverse system of the H-closed (Urysohn-closed) spaces* X_α *and the Θ-closed (u-closed) mappings* $f_{\alpha\beta}$. *If* X_α *are non-empty spaces, then* $X = \varprojlim \underline{X} \neq \emptyset$.

0. INTRODUCTION

Throughout this paper a space X always denotes a topological space. No separation axioms are assumed unless otherwise specified. A mapping $f : X \to Y$ means a continuous mapping.

The conventions and elementary results on inverse limits of topological spaces are those given in Engelking |3|.

A number of other tehnical or specialized definitions are given in the text.

1. SPACES X_Θ AND X_u

A Θ-closed sets were introduced by Veličko |16|.

1.1. Definition. A point $x \in X$ is in the Θ-closure of a set $A \subseteq X$, $x \in |A|_\Theta$, if $\bar{V} \cap A \neq \emptyset$ for any V open about x.

A subset $A \subseteq X$ is Θ-closed if $A = |A|_\Theta$. A subset $B \subseteq X$ is Θ-open if $X \setminus B$ is Θ-closed.

Veličko |16| proved the following properties of Θ-closed sets.

1.2. THEOREM (See Herrington |9; Theorem 2|). In any topological space

(a) the empty set and the whole space are Θ-closed,
(b) arbitrary intersections and finite unions of Θ-closed sets are Θ-closed,
(c) $\bar{K} \subseteq |K|_\Theta$ for each subset K,
(d) a Θ-closed subset is closed.

From this Theorem it follows that the family of all Θ-open subsets of X is a topology t_Θ on X.

1.3. Definition Let (X,t) be a topological spaces.

The Θ-space of X is the space (X, t_Θ).

In the sequel we use the denotations X and X_Θ.

It is easily to prove that in any Hausdorff space X every point $x \in X$ is Θ-closed. This fact implies the following lemma.

1.4 LEMMA. If X is a Hausdorff space, then X_Θ is a T_1-space.

1.5. LEMMA. The identity mapping $id_\Theta : X \to X_\Theta$ is a continuous mapping.

1.6. LEMMA. Let X be a Hausdorff space. If for every open set $U \subseteq X$ is $\bar{U} = |U|_\Theta$, then X_Θ is a Hausdorff space.

We say that a space X is an Urysohn space (|6|,|10|) if for $x,y \in X$ with $x \neq y$, there exist open sets V and W about x and y, respectively, satisfying $\bar{V} \cap \bar{X} = \emptyset$.

A topological space X is an extremally disconnected space or e.d. space |3| if for every pair U, V of disjoint open subsets of X we have $\bar{U} \cap \bar{V} = \emptyset$.

Every Hausdorff e.d space in an Urysohn space.

1 7. LEMMA. Let X be an e.d. spaces. For every open subset $U \subseteq X$ we have $\bar{U} = |\bar{U}|_\Theta$.

P r o o f. For $x \notin \bar{U}$ we have an open set $V \ni x$ with $U \cap V = 0$. It follows that $\bar{U} \cap \bar{V} = \emptyset$ i.e. $x \notin |U|_\Theta$. This implies that $|\bar{U}|_\Theta \subseteq \bar{U}$. Since $\bar{U} \subseteq |\bar{U}|_\Theta$, the proof is complete.

1.8. COROLLARY. It X is an e.d. space, then X_Θ is an Hausdorff space.

P r o o f. Apply Lemmas 1.6 and 1.7.

A space X is H-closed if X is a Hausdorff space and every centred family $\{U_\alpha : U_\alpha \text{ open in } X\} = \mathcal{U}$ has a property $\cap\{\bar{U}_\alpha : U_\alpha \varepsilon\ \} \neq \emptyset$. A Hausdorff space X is H-closed |8| iff for every centred family $\{A_\alpha : A_\alpha \subset X\}$ there exists the point $x \varepsilon X$ with property that $\bar{V} \cap A_\alpha \neq 0$ for every open $V \ni x$ and every A_α.

The point x is called Θ-accumulation point of $\{A_\alpha\}$

1.9. LEMMA. If X is H-closed, then every centred family $\{A_\alpha : \alpha \varepsilon A\}$ of Θ-closed sets $A_\alpha \subset X$ has none-empty intersection $\cap\{A_\alpha : \alpha\varepsilon A\}$.

A Hausdorf space X is nearly-compact |5| if every open cover $\{U_\mu : \mu\varepsilon M\}$ of X has a finite subcollection $\{U_{\mu 1},\dots,U_{\mu n}\}$ such that $\operatorname{Int}\bar{U}_{\mu 1}\cup\dots\cup\operatorname{Int}\bar{U}_{\mu n} = X$.

Every nearly-compact spaces is H-closed. A space X is nearly-compact iff X is H-closed and Urysohn |5|.

1.10. LEMMA Let X be an Urysohn space. Every H-closed subspace $A \subset X$ is Θ-closed.

1.11. LEMMA. If X is H-closed and Urysohn, then X_Θ a Hausdorff space.

1.12. THEOREM. If X is H-closed, then X_Θ is a compact T_1-space.

P r o o f. Let $\{F_\mu : \mu\varepsilon M\}$ be a centred family of closed sets in X_Θ. By virtue of the Definition 1.3. it follows that $F_\mu = \cap\{F_{\mu,\alpha} : \alpha\varepsilon A, F_{\mu,\alpha}$ is Θ-cloesd in X$\}$. Lemma 1.9. implies that there exists $x \varepsilon X$ with property $x \varepsilon \cap\{F_{\mu,\alpha} : \mu\varepsilon M, \alpha\varepsilon A\}$. CLearly, $x\varepsilon \cap\{F_\mu : \mu\varepsilon M\}$. The proof is complete.

1.13. PROBLEM. Is it true that X is H-closed if X_Θ is the compact T_1-space?

Let us prove the following theorem.

1.14. THEOREM. If the space X_Θ is Hausdorff, then X is Urysohn.

1.15. THEOREM. If X is nearly-compact, then X_Θ is the compact Hausdorff space.

P r o o f. Apply Lemmas 1.1. and 1.12.

1.16. THEOREM. If X is an H-closed e.d. space, then X_Θ is the Hausdorff compact space.

P r o o f. Apply Lemmas 1.7. and 1.12.

1.17. PROBLEM. Is it true that X is nearly-compact if X_Θ is a compact Hausdorff space?

In |16|, Veličko also introduced the notion of δ-closed sets.

1.18. Definition. A point $x \in X$ is in the δ-closure of a set $A \subset X$, $x \in |A|_\delta$, if $\operatorname{Int} \bar{V} \cap A \neq \emptyset$ for every open V about x.

A subset $A \subset X$ is δ-closed if $A = |A|_\delta$. A subset $B \subset X$ is δ-open if $X \setminus B$ is δ-closed.

1.19. LEMMA. The set $A \subset X$ is δ-open iff A is the union of the sets $\operatorname{Int} \bar{V}$.

The following Lemma is an immediate consequence of the definition of δ-closedness. (See Veličko |16|, Lemma 3.).

1.20. LEMMA. In any topological space

(a) the empty set and the whole space are δ-closed,
(b) arbitrary intersections and finite unions of δ-closed sets are δ-closed,
(c) $K \subset \bar{K} \subset |K|_\delta \subset |K|_\Theta$ for each subset K,
(d) a δ-closed set is closed.

1.21. Definition. Let (X,t) be a topological space. The space (X, t_s) - or X_s - is the set X with topology generated by family t_s of all δ-open set in (X,t).

From Lemma 1.19. it follows that the family of all regularly open sets (i.e. the sets of the form $V = \operatorname{Int} \bar{V}$) is a base for topology t_s. This means that X_s is well known semiregularization X_s of X.

1.22. LEMMA. The identity mapping $id_{s\Theta} : X_s \to X_\Theta$ is continuous.

1.23. LEMMA. A space X is nearly-compact iff X_s is a compact Hausdorf space.

1.24. LEMMA. A space X is nearly-compact iff the spaces X_Θ and X_s are compact homeomorphic spaces.

P r o o f. Apply Lemmas 1.15, 1.22. and 1.23.

We conclude this Section with an discussion of u-closed sets.

We say that (G,H) is an ordered pair of open sets about $x \in X$ if G and H are open subsets of X and $x \in G \subset \bar{G} \subset H$ |6|.

1.25. Definition. (|10|, Definition 2.1). A point $x \in X$ is in the u-closure of a subset $K \subset X$ ($x \in |K|_u$) if each ordered pair (G,H) of open sets about $x \in X$ satisfies $K \cap \bar{H} \neq 0$.

A subset K of a space is u-closed if $K = |K|_u$. A subset K is u-open if $X \setminus K$ is u-closed.

The next Lemma is proved in |10|.

1.26. LEMMA. In any topological space

(a) the empty set and whole space are u-closed,
(b) arbitrary intersections and finite unions of u-closed sets are u-closed,
(c) $K \subset \bar{K} \subset |K|_\Theta \subset |K|_u$ for each subset K,
(d) u-closed set is Θ-closed (closed).

From (a) and (b) of Lemma 1.26. it follows that a family t_u of all u-open sets of the space (X,t) is a topology on X.

1.27. Definition. Let (X,t) be a topological space. The space X_u is the space (X,t_u).

1.28. LEMMA. The identity mapping $id_u : X \to X_u$ is continuous.

P r o o f. Let F be an closed subset in X_u. This means that $(id_u)^{-1}(F) = F$ is an u-closed subset of X. By virtue of Lemma 1.26. (c) it follows that $(id_u)^{-1}(F)$ is closed in X.

1.29. LEMMA. The identity mapping $id_{\Theta u} : X_\Theta \to X_u$ is the continuous mapping.

P r o o f. Apply 1.26. (d).

Every u-closed set is δ-closed since $K \subset \bar{K} \subset |K|_\delta \subset |K|_\Theta \subset |K|_u, K \subset X$ (see Lemmas 1.20. and 1.26). An immediate consequence is the following

1.30. LEMMA. The identity mapping $id_{su} : X_s \to X_u$ is the continuous mapping.

1.30. LEMMA. (|10|) The following statements are equivalent for a space X.

(a) X is Urysohn,
(b) $\{x\} = \bigcap\{|V|_u : V$ open set containing $x\}$ for each $x \in X$,
(c) Each point in X is u-closed.

1.31. LEMMA. If X is an Urysohn space, then X_u is T_1-space.

P r o o f. Lema 1.30. (c) implies that $\{x\}$ is u-closed in X. This means that $\{x\}$ is closed in X_u.

A Urysohn space X is Urysohn-closed |6| if X is a closed set in every Urysohn space in which it can be embedded.

We say that a point $x \in X$ is in the u-adherence of a filterbase $\mathcal{F}$ ($x \in u\text{-ad}\,\mathcal{F}$) if each $F \in \mathcal{F}$ and ordered pair (G H) of open sets about x satisfy $F \cap \bar{H} \neq \emptyset$.

We use the following characterization of the Urysohn-closed spaces (|6|, Theorem 3.2.).

1.32. LEMMA. A Urysohn space is Urysohn-closed if each filterbase on the space has non-empty u-adherence.

1.33. LEMMA. If X is a Urysohn-closed space, then every centred family $\mathcal{F} = \{F_\mu : F_\mu$ is u-closed in X, $\mu \in M\}$ has non-empty intersection $\bigcap\{F_\mu : \mu \in M\}$.

P r o o f. By virtue of Lemma 1.3. it follows that there exists a point $x \in U\text{-ad}\,\mathcal{F}$. Since each $F_\mu \in \mathcal{F}$ is u-closed, we have $x \in F_\mu$ i.e. $x \in \bigcap\{F_\mu : \mu \in M\}$. The proof is complete.

1.34. LEMMA. If X is Urysohn-closed, then X_u is compact T_1-space.

P r o o f. Apply Lemmas 1.31. and 1.33.

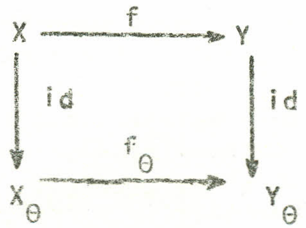
We say that X is a strongly Urysohn space if every distinct points $x, y \in X$ have disjoint u-open neighbourhoods.

1.35. LEMMA. A space X is a strongly Urysohn space iff X_u is a Hausdorff space.

1.36. LEMMA. If X is a strongly Urysohn and Urysohn-closed space, then X_u is a compact Hausdorff space.

2. MAPPINGS F_Θ NAD F_u

Let $f : X \to Y$ be a mapping. We define a mapping $f_\Theta : X_\Theta \to Y_\Theta$ such that $f_\Theta (x) = f (x)$ for every $x \in X_\Theta$ i.e. such that the commutativity holds in the diagram



2.1. LEMMA. If $f : X \to Y$ is a continuous mapping, then $f_\Theta : X_\Theta \to Y_\Theta$ is a continuous mapping.

P r o o f. It suffices to prove that $f^{-1}(A)$ is Θ-closed in X if A is Θ-closed in Y. Let us assume that $x \in |f^{-1}(A)|_\Theta \setminus f^{-1}(A)$. This means that $f(x) \notin A$ and that $\bar{V}_x \cap f^{-1}(A) \neq 0$ for each open set $V_x \ni x$. Since A is Θ-closed, there exists open set $U \ni f(x)$ such that $\bar{U} \cap A = \emptyset$. The set $f^{-1}(U)$ is a neighbourhood of x. It follows that $f^{-1}(U) \cap f^{-1}(A) \neq \emptyset$. The contradiction ($\bar{U} \cap A = \emptyset$ and $f^{-1}(U) \cap f^{-1}(A) \neq \emptyset$) complete the proof.

2.2. Definition. A mapping $f : X \to Y$ is called Θ-closed if $f(A)$ is Θ-closed for each Θ-closed subset $A \subseteq X$.

2.3. LEMMA. Let $f : X \to Y$ be a continuous mapping. The following conditions are equivalent:

(a) f is Θ-closed,
(b) for every $B \subset Y$ and each Θ-open set $U \supseteq f^{-1}(B)$ there exists Θ-open set $V \supseteq B$ such that $f^{-1}(V) \subseteq U$,
(c) f_Θ is a closed mapping.

Remark. The proof is similar to the proof of the corresponding theorem for closed mappings (|3|, pp.52.).

From 1.15. and 2.3. we obtain

2.4. THEOREM. If X and Y are nearly-compact spaces, then every continuous mapping $f : X \to Y$ is Θ-closed.

Lemmas 1.16. and 2.3. imply the following theorem.

2.5. THEOREM. If $f : X \to Y$ is a continuous mapping betven H-closed e.d. spaces X and Y, then f is Θ-closed.

An open (closed) subset A of X is called regularly open (regularly closed) if $A = \mathrm{Int}\,\bar{A}$ ($A = \overline{\mathrm{Int}\,A}$).

2.6. Definition |12|. A mapping $f : X \to Y$ is said to be skeletal (HJ) if for each open (regularly open) $U \subset X$ we have $\mathrm{Int}\, f^{-1}(\bar{U}) \subseteq \overline{f^{-1}(U)}$.

Now we prove the following important theorem.

2.7. THEOREM. If X and Y are H-closed, then every HJ-mapping $f : X \to Y$ is Θ-closed.

P r o o f. Let A be a Θ-closed subset of X. This means that

$$A = \bigcap \{\bar{V}_\lambda : V_\lambda \text{ open in } X,\ V_\lambda \supseteq A\} \qquad (1)$$

where $\mathcal{V} = \{V_\lambda : \lambda \in \Delta\}$ is the maximal family of open subsets containing A. Let $\mathcal{U} = \{U_\mu : \mu \in M\}$ be a family of all open subsets $U_\mu \subset Y$ such that there exists $V_\lambda \in \mathcal{V}$ with property $f(V_\lambda) \subseteq U_\mu$.

Clearly $f(A) \subset U_\mu$ for each $U_\mu \in \mathcal{U}$. Let us prove that

$$f(A) = \bigcap \{\bar{U}_\mu : U_\mu \in \mathcal{U}\} \qquad (2)$$

We prove only that $f(A) \supset \bigcap \{\bar{U}_\mu : U_\mu \in \mathcal{U}\}$ since $f(A) \subset \bigcap \{\bar{U}_\mu : U_\mu \in \mathcal{U}\}$. Suppose that $y \in \bigcap \{\bar{U}_\mu : \mu \in M\}$. For every open $W \ni y$ we have $\bar{W} \cap f(V_\lambda) \neq \emptyset$ since $\bar{W} \cap f(V_\lambda) = \emptyset$ implies that $Y \setminus \bar{W} \supset f(V_\lambda)$, $Y \setminus \bar{W} \in \mathcal{U}$ and $y \in Y \setminus \bar{W}$. Now, the set $W' = \operatorname{Int} \bar{W}$ is regularly open and, by virtue of 2.6., we have

$$\operatorname{Int} f^{-1}(\overline{W'}) \subseteq \overline{f^{-1}(W')} \qquad (3)$$

From (3) and $f^{-1}(\overline{W'}) \cap V_\lambda \neq \emptyset$ it follows that $f^{-1}(W') \cap V_\lambda \neq \emptyset$ for each $V_\lambda \in \mathcal{V}$. The family $\mathcal{V}' = \{V_\lambda' : V_\lambda' = f^{-1}(W') \cap V_\lambda\}$ is centred family in H-closed space X. Hence, there exist $x \in X$ with property $x \in \bigcap \{\overline{V_\lambda'} : V_\lambda' \in \mathcal{V}'\}$ It is easily to prove that $x \in A$ and $f(x) \in \bigcap \{\bar{W} : W$ is open set about $y\}$. This means that $y = f(x)$ since Y is a Hausdorff space. The proof is complete.

2.8. EXAMPLE. We shall now show that there exists a Θ-closed mapping which is not a HJ-mapping. Let $X = [0;1]$ with the following topology: the neighbourhoods of every point $x \neq 0$ are the same as those in the usual topology of $[0,1]$, but the neigbourhoods of $x_1 = 0$ are the sets of the form $[0,\varepsilon) \setminus D$, where $D = \{1, \frac{1}{2}, \ldots, \frac{1}{n} \ldots\}$; $0<\varepsilon<1$. The space X is H-closed and Uryshon i.e. X is nearly-compact. Let us define $f : X \to X = Y$ so that $f(x) = x$ for $x < 0{,}6$; $f(x) = 0{,}6$ if $0{,}6 \leq x \leq 0{,}8$ and $f(x) = 2x - 1$. The mapping $f : X \to X$ is continuous. By virtue of Theorem 2.4. f is Θ-closed. Let us prove that f is not an HJ-mapping. Let $V = (0{,}6;\ 1]$ be regularly open subset of Y. Now, $f^{-1}(V) = (0{,}8;\ 1] \subset X$ and $\overline{f^{-1}(V)} = [0{,}8;\ 1]$. In the other hand we have $\bar{V} = [0{,}6;\ 1]$, $f^{-1}(\bar{V}) = [0{,}6;\ 1]$ and $\operatorname{Int} f^{-1}(\bar{V}) = (0{,}6;\ 1)$. It follows that $\operatorname{Int} f^{-1}(\bar{V}) \supset \overline{f^{-1}(V)}$ an, therefore, f is not HJ.

A mapping $f : X \to Y$ is semi-open if $\operatorname{Int} f(\bar{U}) \neq \emptyset$ for any open subset $U \subset X$.

Every semi-open mapping is HJ. An open mapping $f : X \to Y$ is semi-open.

We say that the mapping $f : X \to Y$ is irreducible if the set $f\#(U) = \{y : f^{-1}(y) \quad U\}$ is non-empty for any non-empty open set $U \quad X$.

Every closed irreducible mapping is a semi-open mapping.

A mapping $f : X \to Y$ has the inverse property if, for each open $V \subseteq Y$, we have $f^{-1}(\bar{V}) = \overline{f^{-1}(V)}$.

Every open mapping has the inverse property, and every mapping with inverse property is an HJ-mapping.

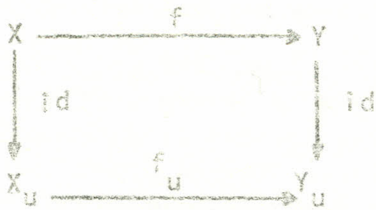
2.9. LEMMA. Let X be an H-closed space. If the mapping $f : X \to Y$ has the inverse property or is semi-open (open, closed irreducible), then $f : X \to Y$ is Θ-closed.

P r o o f. Lemma follows from Theorem 2.7.

2.10. REMARK. The notion of the δ-closed mapping were introduced by Noiri |13|.

Let us go to the discusion on the u-closed mappings.

For each mapping $f : X \to Y$ we define the mapping $f_u : X_u \to Y_u$ such that the commutativity holds in the diagram



2.11. LEMMA. If $f : X \to Y$ is continuous, then $f_u : X_u \to Y_u$ is continuous.

P r o o f is parabell to the proof of Lemma 2.1.

2.12. Definition. A continuous mapping $f : X \to Y$ is called u-closed if $f(A)$ is u-closed for each u-closed subset $A \subseteq X$.

2.13. LEMMA. For each continuous mapping $f : X \to Y$ the following conditions are equivalent:

(a) f is u-closed,
(b) for each $B \subseteq Y$ and each u-open set $U \supseteq f^{-1}(B)$ there exists an u-open set $V \supseteq B$ with property $f^{-1}(V) \subseteq U$,
(c) $f_u : X_u \to Y_u$ is closed.

P r o o f. See the remark on the proof of Lemma 2.3.

2.14. LEMMA. Every continuous mapping $f : X \to Y$ betwen the strongly Urysohn and Urysohn-closed spaces X and Y is u-closed.

P r o o f. Apply Lemmas 1.36. and 2.13.

We close this Section with Lemma which characterize the u-closed subsets and with Theorem on the u-closedness of the HJ-mappings.

2.15. LEMMA. A subset A of X is u-closed if and only if $A = \cap\{\bar{U}_\mu : U_\mu$ is open in X such that $A \subseteq V \subseteq \bar{V} \subseteq U_\mu$ for some open $V\}$.

P r o o f. Necessity. Let A be u-closed. If $x \in \cap\{\bar{U}_\xi\}$ is not in A, then there exists an ordered pair (G,H) about x such that $\bar{H} \cap A = \emptyset$. The sets $V = X \setminus \bar{H}$ and $U = X \setminus \bar{G}$ have the property $A \subset V \subset \bar{V} \subset U$. It follows that $x \notin \cap\{\bar{U}_\mu\}$ which contradicts with $x \in \cap\{\bar{U}_\mu\}$.

Sufficiency. If $A = \cap\{\bar{V}_\mu : \mu \in M\}$, then for $x \notin A$ there exist open sets U and V such that $A \subset V \subset \bar{V} \subset U$ and $x \notin \bar{U}$. The sets $G = X \setminus \bar{U}$ and $H = X \setminus \bar{V}$ have the property $x \in G \subset \bar{G} \subset H$. It follows that (G,H) is an ordered pair of open sets about x and that $\bar{H} \cap A = \emptyset$. This means that $x \notin |A|_u$. Therefore, A is u-closed. The proof is complete.

2.16. THEOREM. Every HJ-mapping $f : X \to Y$ betwen Urischn-closed spaces X and Y is an u-closed mapping.

P r o o f. Let A be an u-closed subsets of X. By wirtue of Lemma 2.15. we have $A = \cap\{\bar{U} : U$ is open X such that $A \subset V \subset \bar{V} \subset U$ for some open $V \subset X\}$. It follows that

$$A = \cap\{\bar{U}_\mu : U_\mu \supset A,\ U_\mu \text{ is open},\ \mu \in M\} \tag{4}$$

We dennote the family $\{U_\mu : U_\mu \supset A,\ U_\mu$ is open, $\mu\in M\}$ by $\mathcal{U}$. Let $\mathcal{V} = \{V_\alpha : V_\alpha$ is open in Y such that $f(A)\subset W\subset \bar{W}\subset V_\alpha$ for some open W and $f(U_\mu)\subset V_\alpha$ for some $U_\mu\in\mathcal{U}\}$. We shall show that

$$f(A) = \cap\{\bar{V}_\alpha : V_\alpha \in \mathcal{V}\} \qquad (5)$$

Let y be a point of $\cap\{\bar{V}_\alpha : V_\alpha\in\mathcal{V}\}$. It follows that $W\cap V_\alpha \neq 0$ for each $V_\alpha\in\mathcal{V}$ and each open $W\ni x$. Relation $\bar{W}\cap f(U_u) = \emptyset$ implies that there exists a pair (W_α, V_α) such that $f(U_u)\subset V_\alpha$. This means that $W_\alpha^* = W_\alpha\setminus\bar{W} = \overline{V_\alpha}\setminus\bar{W}$ are open sets with property $f(A)\subset W_\alpha^*\subset \overline{W_\alpha^*}\subset V_\alpha^*$. Since $x\notin \overline{V_\alpha^*}$, we have $x\notin \cap\{\bar{V}_\alpha : V_\alpha\in\mathcal{V}\}$; a contradiction. Thus $\bar{W}\cap f(U_u)\neq\emptyset$ for each $U_u\in\mathcal{U}$ and each open $W\ni x$. Clearly, $f^{-1}(\bar{W})\cap U_u\neq\emptyset$. It we assume that $f^{-1}(W)\cap U_u=\emptyset$, then $U^*_\mu = U_u\setminus f^{-1}(\bar{W})$ is an open set since $f^{-1}(\bar{W}\setminus W)$ is nowhere dense (f is an HJ-mapping).

(A) If $U^*_\mu\supset A$ then $f(U^*_\mu)\cap W = \emptyset$; a contradiction. This means that $f^{-1}(W)\cap U_\mu \neq\emptyset$ for each $W\ni x$ and each $U_\mu\in\mathcal{U}$. Moreover, we have that $Z_\mu = f^{-1}(W)\cap U_\mu \neq 0$. The family $\{Z_\mu : \mu\in M\}$ is centred. Thus, since X is Uryšohn-closed, we have u-ad $\{Z_\mu\}\neq\emptyset$. Let x be a point of u-ad $\{Z_\mu\}$. The assumption that $x\notin A$ implies that there exist open sets U and V such that $x\in V\subset\bar{V}\subset U$ and $\bar{U}\cap A = \emptyset$. Thus $U^* = X\setminus\bar{U}$ is in $\{Z_\mu\}$ since $U^*\supseteq A$. It follows that $w\notin$ u-ad $\{Z_\mu\}$; a contradiction. From $x\in$ u-ad $\{Z_\mu\}$ we have that $x\in|f^{-1}(W)|_u$ for every open $W\ni y$. Since f is continuous we conclude that $f(x)\in|W|_u$ for every open $W\ni x$. From Lemma 1.30 (b) it follows that $f(x) = y$.

(B) If $A\setminus U^*_\mu\neq\emptyset$ then $\bar{W}\cap f(A)\neq\emptyset$ for each open W about y. Since $\bar{W}\subset|W|_u$ we have $|W|_u\cap f(a)\neq\emptyset$. This means that $A_\mu = f^{-1}(|W|_u)\cap A\neq\emptyset$. The family $\{A_\mu\}$ is centred family in X. Since X is Urysohn-closed, there exist $x\in X$ such that $x\in$ u-ad $\{A_\mu\}$. It is clear that $x\in A$ since A is u-closed. Moreover, $f(x)\in\{\cap|W|_u : W$ open set about $y\}$. Lemma 1.30. (b) implies that $f(x) = y$.

Finally, we conclude that for every point of $\cap\{\bar{V} : V\in\mathcal{V}\}$ there exist $x\in A$ such that $f(x) = y$ i.e. (5) is proved. From Lemma 2.15. and (5) it follows that f(A) is u-closed. The proof is complete.

2.16. EXAMPLE. If X is the space as in Example 2.8. then $|A|_\Theta = |A|_u$ for each $A \subset X$. Moreower, X is Urysohn-closed. It follows that $f : X \to Y$ definied in 2.8. is u-closed but not an HJ-ma-ping.

3. INVERSE SYSTEM $\underline{X}_\Theta$ AND $\underline{X}_u$

For every inverse system $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ we introduce two inverse system denoted by $\underline{X}_\Theta$ and $\underline{X}_u$. Indeed, for every space X_α there exist the spaces $(X_\alpha)_\Theta$ and $(X_\alpha)_u$ which are definied in Section 1. From the second Section it follows that for each mapping $f_{\alpha\beta} : X_\beta \to X_\alpha$ there exist the mappings $(f_{\alpha\beta})_\Theta$ nad $(f_{\alpha\beta})_u$. From the diagrams of the second Section it follows the transitivity conditions $(f_{\alpha\beta})_\Theta (f_{\beta\gamma})_\Theta = (f_{\alpha\gamma})_\Theta$ and $(f_{\alpha\beta})_u (f_{\alpha\beta})_u = (f_{\alpha\gamma})_u$.

3.1. LEMMA. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system, then there exist the identity mappings $Id_\Theta : \underline{X} \to \underline{X}_\Theta$ and $Id_u : \underline{X} \to \underline{X}_u$.

The following Lemma is an immediate consequence of Lemma 3.1.

3.2. LEMMA. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system, then $\varprojlim \underline{X} \neq \emptyset$ iff $\varprojlim \underline{X}_\Theta \neq \emptyset$ ($\varprojlim \underline{X}_u \neq \emptyset$).

In the sequel we use the following Stone´s theorems.

3.3. THEOREM. (A.H. Stone |14|). Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of compact non-empty T_0 spaces and closed maps. Then $\varprojlim \underline{X} \neq \emptyset$; in fact, there exists $x \in \varprojlim \underline{X}$ such that $x_\alpha = f_\alpha(x)$ is a closed point of X_α for all $\alpha \in A$.

3.4. THEOREM. (A.H.Stone |14|). Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty compact T_0 spaces and continuous maps, and suppose that each $f_{\alpha\beta}$ is a surjection (for all $\alpha, \beta \in A$ with $\beta > \alpha$) and that A is linearly ordered. Then $\lim \underline{X} \neq \emptyset$; in fact, there exist $X \in \varprojlim \underline{X}$ such that (for all $\alpha \in A$) $x_\alpha = f_\alpha(x)$ is a closed point of X_α.

Now we shall show the following theorem.

3.5. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty H-closed spaces X_α and Θ-closed mappings $f_{\alpha\beta}$. Then $X = \varprojlim \underline{X} \neq 0$. Moreover, if $f_{\alpha\beta}$ are onto mappings, then the projections $f_\alpha : X \to X_\alpha$ are onto.

P r o o f. The inverse system $\underline{X}_\Theta = \{(X_\alpha)_\Theta, (f_{\alpha\beta})_\Theta, A\}$ satisfies the conditions of Theorem 3.3. since $(X_\alpha)_\Theta$, $\alpha\in A$, are compact T_1 space (Lemma 1.12) and the mappings $(f_{\alpha\beta})_\Theta$ are closed(Lemma 2.3). It follows that $\lim\limits_{\leftarrow} \underline{X}_\Theta \neq \emptyset$. By virtue of Lemma 3.2. we have $\lim\limits_{\leftarrow} \underline{X} \neq \emptyset$. Further, if $f_{\alpha\beta}$, $\beta>\alpha$, are onto mappings, then for $x_\alpha \in X_\alpha$ the sets $Y_\beta = f^{-1}_{\alpha\beta}(x_\alpha)$ are non-empty Θ-closed sets (lemma 2.1.). This means that the system $\underline{Y}=\{(Y_\beta)_\Theta, (f_{\beta\gamma})_\Theta/Y_\gamma, \alpha<\beta<\gamma\}$ satisfies. Theorem 3.3. and has non-empty limit Y. From Lemma 3.2. it follows that there exist $y\in Y \subset \lim\limits_{\leftarrow} \underline{X}$ such that $f_\alpha(y) = x_\alpha$. The proof is complete.

3.6. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system definied over a linearly ordered set A. If $X_\alpha \neq \emptyset$, $\alpha\in A$, is H-closed and each $f_{\alpha\beta}$, $\alpha<\beta$, is a surjecton, then $X = \lim \underline{X} \neq \emptyset$ and each projection $f_\alpha : X\to X_\alpha$ is a surjection.

P r o o f. Apply Lemma 1.12. and Theorem 3.4. (See also Friedler and Pettey |4|, Theorem 3.5.).

3.7. REMARK. If in the definition of the H-closed space we omit T_2-separation axiom, then we obtain the definition of the quasi-H-closed space.

3.8. THEOREM. Let $\underline{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of non-empty quasi-H-closed spaces X_α and Θ-closed mappings f_{nm}. Then $X = \lim\limits_{\leftarrow} \underline{X} \neq 0$.

P r o o f. The inverse sequence $\underline{X}_\Theta = \{(X_n)_\Theta, (f_{nm})_\Theta, N\}$ is an inverse sequence of non-empty compact (without separation axiom!) spaces $(X_n)_\Theta$ and closed mappings $(f_{nm})_\Theta$. By virtue of |14; Theorem 2.| it follows that $\lim\limits_{\leftarrow} \underline{X} \neq \emptyset$. Lemma 3.2. complete the proof.

3.9. PROBLEM. Is it true that X in Theorem 3.5. is H-closed?

3.10. PROBLEM. Is it true that the projections $f_\alpha : X\to X_\alpha$ in Theorem 3.5. are Θ-closed?

Theorem 3.5., along with 2.7., gives

3.11. THEOREM. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system of non-empty H-closed spaces and HJ-mappings $f_{\alpha\beta}$, then $X=\lim\limits_{\leftarrow} \underline{X}=\emptyset$. Moreover, if $f_{\alpha\beta}$ are onto, then the projections $f_\alpha : X\to X_\alpha$ are onto mappings.

3.12. COROLLARY. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty H-closed spaces X_α and mappings $f_{\alpha\beta}$ with the inverse property or with semi-open (open, closed and irreducible) mappings. Then $X = \varprojlim \underline{X} \neq \emptyset$.

Let us prove that $X = \varprojlim \underline{X}$ in Theorem 3.11. and Corollary 3.12. is H-closed.

3.13. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system with HJ-mappings $f_{\alpha\beta}$. If the projections $f_\alpha : X \to X_\alpha$ are onto, then f_α are HJ-mapping.

P r o o f. Suppose that some projection f_α is not HJ-mapping. This means that there exists regularly open set $V_\alpha \subset X_\alpha$ such that $Y = \operatorname{Int} \overline{f_\alpha^{-1}(\bar{V}_\alpha)} \setminus \overline{f_\alpha^{-1}(V_\alpha)} \neq \emptyset$. Let $y \in Y$. It follows that there exists open set $f_\beta^{-1}(U_\beta)$, $\beta > \alpha$, with property $\underline{f_\beta^{-1}(U_\beta)} \subseteq Y$. It is easily to prove that $f_\beta(x) \in \operatorname{Int} f_{\alpha\beta}^{-1}(\bar{V}_\alpha) \setminus \overline{f_{\alpha\beta}^{-1}(V_\alpha)}$. This is a contradiction since $f_{\alpha\beta}$ is HJ-mapping.

3.14. THEOREM. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system of the H-closed spaces X_α and HJ-mappings $f_{\alpha\beta}$, then $X = \varprojlim \underline{X}$ is H-closed.

P r o o f. If $X = \emptyset$, then Theorem holds. Let $X \neq \emptyset$. Then $X_\alpha \neq \emptyset$, $\alpha \in A$, and the projections $f_\alpha : X \to X_\alpha$, $\alpha \in A$, are onto HJ-mappings (Theorems 3.11. and 3.13). Let us prove that X is H-closed. It suffices to prove that each maximal centred family $\mathcal{U} = \{U_\mu : \mu \in M, U_\mu$ is open subset of $X\}$ has the property $\cap\{\bar{U}_\mu : \mu \in M\} \neq \emptyset$. For each $\alpha \in A$ we define the centred family $\mathcal{U}_\alpha = \{U_{\mu\alpha} : U_{\mu\alpha}$ is open in X_α such that there exists $U_\mu \in \mathcal{U}$ with $f_\alpha(U_\mu) \subseteq U_{\mu\alpha}$, $\mu \in M\}$. Now we shall show that $\mathcal{U}_\alpha$ is maximal. Let $U_\alpha \subset X_\alpha$ be an open set with the property $U_\alpha \cap U_{\mu\alpha} \neq \emptyset$, $U_{\mu\alpha} \in \mathcal{U}_\alpha$. It is readily seen that $\bar{U}_\alpha \cap f_\alpha(U_\mu) \neq \emptyset$ for each $U_\mu \in \mathcal{U}$. Hence, if we denote $\operatorname{Int} \bar{U}_\alpha$ by V_α, we have $\bar{V}_\alpha \cap f_\alpha(U_\mu) \neq \emptyset$ for each $U_\mu \in \mathcal{U}$. From the fact that f_α is HJ we conclude that $\overline{f_\alpha^{-1}(V_\alpha)} \cap U_\mu \neq \emptyset$ since $\overline{f_\alpha^{-1}(V_\alpha)} \cap U_\mu = \emptyset$ implies that $U_\mu^* = X \setminus \overline{f_\alpha^{-1}(\bar{V}_\alpha)} \in \mathcal{U}$; a contradiction. From $\overline{f_\alpha^{-1}(V_\alpha)} \cap U_\mu \neq \emptyset$ we infer that $f_\alpha^{-1}(V_\alpha) \cap U_\mu \neq \emptyset$ for each $U_\mu \in \mathcal{U}$. By virtue of the maximallity of $\mathcal{U}$ it follows $f_\alpha^{-1}(V_\alpha) \in \mathcal{U}$ and, consequently, $V_\alpha \in \mathcal{U}_\alpha$. Thus $U_\alpha \in \mathcal{U}_\alpha$ since $V_\alpha = \operatorname{Int} \bar{U}_\alpha$. This means that $\mathcal{U}_\alpha$ is maximal. In a similar way on can prove that if $U\mu_\alpha \in {}_\alpha$, then $f^{-1}_{\alpha\beta}(U\mu_\alpha) \in \mathcal{U}_\beta$, where $\beta > \alpha$. Since X_α is H-closed and $\mathcal{U}_\alpha$ maximal, there exists $x_\alpha \in X_\alpha$ such that $\{x_\alpha\} = \bigcap\{\bar{U}\mu_\alpha : U\mu_\alpha \in \mathcal{U}_\alpha\}$. Moreover, $f_{\alpha\beta}(x_\beta) = x_\alpha$ since $f^{-1}_{\alpha\beta}(U\mu_\alpha) \in \mathcal{U}_\beta$ for each $U\mu_\alpha \in \mathcal{U}_\alpha$. It is easely to prove that $x = (x_\alpha) \in \bigcap\{\bar{U}_\mu : U_\mu \in \mathcal{U}\}$. This completes the proof.

3.15. PROBLEM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of H-closed spaces X_α and of θ-closed mappings $f_{\alpha\beta}$. Is it true that $(\underleftarrow{\lim}\, \underline{X})_\theta$ is homeomorphic to $\underleftarrow{\lim}\, \underline{X}_\theta$?

We now pass to a discussion of inverse systems of Urysohn-closed spaces.

3.16. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system non-empty Urysohn-closed spaces. If the mappings $f_{\alpha\beta}$ are u-closed, then $X = \underleftarrow{\lim}\, \underline{X} \neq \emptyset$.

P r o o f. Inverse system $\underline{X}_u = \{(X_\alpha)_u, (f_{\alpha\beta})_u, A\}$ is an inverse system of non-empty compact spaces wit closed mappings $(f_{\alpha\beta})_u$ (Lemmas 1.31, 1.34. and 2.13). Theorem 3.3. complete the proof.

3.17. PROBLEM: Is $X = \underleftarrow{\lim}\, \underline{X}$ (in Theorem 3.16) an Urysohn-closed space?

3.18. PROBLEM. Is it true that in the case of Theorem 3.16. the projections $f_\alpha : X \to X_\alpha$ are u-closed?

From Lemma 2.11. and Theorem 3.4. it follows

3.19. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-empty Urysohn-closed spaces definied over a linearly ordered set A. If each mapping $f_{\alpha\beta}$ is onto, then $X = \underleftarrow{\lim}\, \underline{X} \neq \emptyset$ and each projection $f_\alpha : X \to X_\alpha$ is onto.

Similarly, from Lemma 1.36. and Theorem 3.3. we obtain the following theorem.

3.20. THEOREM. If $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ is an inverse system of non-empty strongly Urysohn and Urysohn-closed spaces X_α, then $X = \underleftarrow{\lim}\, \underline{X} \neq \emptyset$.

3.21. THEOREM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of non-emtpy Urysohn-closed spaces X_α and the HJ-mappings $f_{\alpha\beta}$. Then $X = \varprojlim \underline{X} \neq \emptyset$. Moreover, the projections $f_\alpha : X \to X_\alpha$ are onto mappings.

P r o o f. Inverse system $\underline{X}_u = \{(X_\alpha)_u, (f_{\alpha\beta})_u, A\}$ is an inverse system of non-empty compact T_1 spaces $(X)_u$ (Lemma 1.34) with closed mapping $(f_{\alpha\beta})_u$ (Theorem 2.16). From Theorem 3.3, it follows that $\varprojlim \underline{X}_u \neq \emptyset$. Hence, $X = \varprojlim \underline{X} \neq \emptyset$. Q.E.D.

We close this Section with the following problem.

3.22. PROBLEM. Let $\underline{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of the Urysohn-closed spaces X_α and the HJ-mappings (semi-open mappings, open mapppings or closed, irreducible mappings) $f_{\alpha\beta}$. Is $X = \varprojlim \underline{X}$ Urysohn-closed?

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Primljeno: 1984-04-06

Lončar I. Primjene Θ-zatvorenih i u-zatvorenih skupova

SAŽETAK

U radu se izučavaju topologije koje su slabije od zadane topologije t prostora (X,t). Te se topologije definiraju pomoću Θ-zatvorenih i u-zatvorenih skupova prostora (X,t). Dobiveni prostori X_Θ i X_u su najčešće T_o-kompakti. Na temelju toga moguće je svakom inverznom sistemu $\underline{X}$ pridružiti inverzne sisteme $\underline{X}_\Theta$ i $\underline{X}_u$ na koje se mogu primijeniti Stoneovi rezultati za T_o-kompakte. Na taj se način dobivaju neki novi teoremi o nepraznosti limesa inverznih sistema.