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Pregledni znanstveni članak

On Computational Aspects of Data Processing of Geodetic Networks with Large Number of Unknown Parameters

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ABSTRACT. Some special characteristics of linear systems of equations, which arise from geodetic network adjustment, optimization and pre-analysis, are outlined in this paper. Advantages and disadvantages of several methods for solving of the linear systems are discussed in the paper, considering their application to the problems arising from the geodetic network with large number of unknown parameters. Some of the scientific and expert achievements incorporated into the NetExpert software package were the subjects of this paper, including those in mathematics, geodesy and computer science.

Keywords: network, adjustment, systems, equations, parameter.

1. Introduction

The problems of planning, optimization, measurement realization and adjustment, and data preserving in maintenance of geodetic networks, are existing since the foundation of the modern Geodesy. The rapid progress of computer technologies brought up the objective of computer assistance in dealing with those problems. It was the problem of solving the large sparse system of linear equations arising from the adjustment, which was essential to the success in achieving the stated objective.

There are different approaches to the problem of adjustment of geodetic networks with large number of unknown parameters. The most usual one is the dividing the networks into groups, and adjusting separately.

Respecting the outlined consideration, the development of the *NetExpert* system was accomplished through giving the greatest care to the efficient solving of the

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large sparse system of linear equations. Further, the same holds for the inverting and pseudo-inverting of the large sparse matrix. This approach also considerably increases the number of unknown parameters and measurements that can be efficiently handled. The purposes of the *NetExpert* software package are: the numerical processing of data in geodetic networks, data maintenance and graphical presentation of data (Aleksić et al. 1997).

2. Solving Methods

Good source for information on topics mentioned in this chapter are George and Liu (1980) and Rice (1991).

Geodetic network adjustment demands the solution of an overdetermined non-linear system of equations:

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= d_1 \\ f_2(x_1, x_2, \dots, x_n) &= d_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ f_m(x_1, x_2, \dots, x_n) &= d_m \end{aligned} \tag{1}$$

where m is the number of measurements and n is the number of unknown parameters ($m > n$).

Instead of solving this system, the linear system is considered. The latter is obtained through the process of linearization, i.e. by developing functions f_i into first-order Taylor's polynomial. Since the approximate values of the unknown parameters are either known or can be calculated, this step is completely acceptable.

Hence, we obtain the overdetermined system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned} \tag{2}$$

or shortly

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}. \tag{3}$$

Since the geodetic measurements most usually involve only two geodetic points, i.e. a few unknown parameters, each of the equations contains only several non-zero coefficients. Therefore, the system is sparse.

2.1. Orthogonal Factorization

The common property of all orthogonal factorization methods is that the matrix of the system is factored in the form

$$\mathbf{A} = \mathbf{Q} \cdot \mathbf{R} \quad (4)$$

where \mathbf{Q} is an orthogonal and \mathbf{R} is an upper triangular matrix. The main advantage of these methods is in their numerical stability. Also, orthogonal factorization gives the possibility of extracting linearly dependent unknowns from the system. If the matrix \mathbf{A} is badly conditioned, the usual approach of normalization can lead to the solving a system of even worse (squared) condition number. In such cases, the orthogonal factorization is the preferred method.

Nevertheless, the rank of the system that arises from the adjustment is known in advance. If the adjustment is performed with fixed points, than the rank is full. Otherwise, if the adjustment is free, the matrix of the system has a defect which is known even before the solving the system. Moreover, it is known that any $r = \text{defect}(\mathbf{A})$ unknowns can be expressed as a linear combination of the rest of the unknowns.

And finally, the condition number of matrices arising from adjustment is usually sufficiently small (or not too big) so that solving of the normalized system can be successfully realized. The conclusion that orthogonal factorization does not have the precedence over the normalization in most cases, can be safely drawn.

2.2. Normalization

The usual way of obtaining the best solution of the described overdetermined system, is the method of solving the normalized system:

$$\mathbf{A}^T \mathbf{A} \cdot \mathbf{x} = \mathbf{A} \cdot \mathbf{b} \quad (5)$$

or

$$\mathbf{N} \cdot \mathbf{x} = \mathbf{v} \quad (6)$$

where $\mathbf{N} = \mathbf{A}^T \mathbf{A}$ and $\mathbf{v} = \mathbf{A}^T \cdot \mathbf{b}$.

2.3. Factorization

2.3.1. Iterative Methods

The normalized system can be solved by utilization of iterative methods. The principal advantage of the iterative methods is that only non-zero elements of the original matrix \mathbf{N} need be used. There is no fill up, i.e. no newly created non-zero elements. Other methods, on the other hand, produce the increase in non-zeros which overwhelms their initial number by far.

Unfortunately, there are substantial disadvantages to the iterative methods. The greatest of those is the fact that not only the solution of the system is required, but also the diagonal (and some of the non-diagonal) elements of the matrix \mathbf{N}^{-1} . The same goes for the diagonal elements of the product $\mathbf{AN}^{-1}\mathbf{A}^T$. These requirements inevitably lead to the necessity of solving a large number of systems with the same coefficients matrix. This number is equal to the number of the unknowns. If this number is several thousands or ten of thousands, then the application of iterative methods is practically unrealizable.

2.3.2. Cholesky Factorization

Cholesky factorization is contained in the process of calculating the matrix \mathbf{L} such that

$$\mathbf{N} = \mathbf{LL}^T. \quad (7)$$

There are different approaches in application of Cholesky factorization. The key point in choosing the way that Cholesky factorization is implemented is the method of storing non-zero elements of the matrix.

2.3.2.1. The Envelope Method of Storing Non-Zeros

The following figure illustrates the way that non-zeroes in the matrix are stored (Fig. 1).

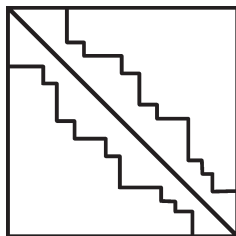


Fig. 1. *Non zero elements grouped about the main diagonal.*

Since the non-zeroes are scattered all over the original matrix, the envelope would include almost all the elements in it. To avoid such a waste, the rearrangement of the unknowns is done, where the aim is to group the non-zeros about the main diagonal of the matrix. Doing so, the envelope is minimized to an acceptable level.

The structure of zeroes in the matrix coming from adjustment is usually suitable for achieving this minimization. This holds true because, in classical geodetic network without GPS measurements, only physically neighbouring points are linked with a measurement. This comes as a consequence of a necessity that the points are mutually visible.

2.3.2.2. Symbolic Factorization

GPS measurements play a very significant role in modern geodetic networks. Since the physical proximity of the points is not meaningful to those measurements, there are no obstacles to include an arbitrary number of GPS measurements into the network. This seriously affects the envelope, which becomes too large, with many zeroes inside it.

Instead of the minimizing the envelope, rearrangement of the unknowns in order of minimization of newly created non-zeroes should be considered. This approach gives better results (faster calculation and less memory requirements) even for networks with terrestrial observations only.

When the reordering of the unknowns is done, the symbolic factorization is to be applied. This is the method of determining the structure of non-zeroes in the matrix \mathbf{L} . After determining this structure, the Cholesky factorization can be performed, using only non-zero elements during the calculation.

It can be shown that the matrix \mathbf{L} has the same structure as the matrix \mathbf{N} , i.e. the same pattern of non-zero elements.

The method of Symbolic factorization was the choice for implementation into the *NetExpert* software.

3. Matrix Inversion

For this problem, the crucial fact is that it can be proved that the calculation of the inverse matrix \mathbf{N}^{-1} can be reduced to the non-zero pattern only. This is essential, because the calculation of the complete inverse is either lengthy or even practically impossible for the case of very large matrix.

Diagonal elements of the matrix \mathbf{N}^{-1} are necessary for obtaining the accuracy of the unknown parameters. Also, calculation of the adjusted measurements accuracy and their reliability requires the diagonal elements of the product $\mathbf{AN}^{-1}\mathbf{A}$. It can be shown that only elements from the non-zero pattern are required for the calculation of $\mathbf{AN}^{-1}\mathbf{A}$.

4. Matrix Pseudo-Inversion

Adjusting geodetic networks without fixed points eventually imposes the necessity of calculating the pseudo-inverse of the matrix \mathbf{N} .

The method implemented in the *NetExpert* is the method of orthogonal augmentation. We augment matrix \mathbf{N} in the following manner

$$\mathbf{N}^* = \begin{pmatrix} \mathbf{N} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix}. \quad (8)$$

Matrix \mathbf{B} has the number of columns equal to the defect of the matrix \mathbf{N} . It is formed as follows

$$\mathbf{B} = (\mathbf{S} \ \mathbf{I}). \quad (9)$$

Here \mathbf{I} is the unity matrix of the appropriate order, and the matrix \mathbf{S} is extracted from the matrix \mathbf{N} in the process of factorization. More precisely, by suitable reordering of rows and columns of the matrix \mathbf{N} , it can be partitioned so that

$$\mathbf{P}\mathbf{N}\mathbf{P}^T = \begin{pmatrix} \mathbf{N}_r & \mathbf{S}^T \\ \mathbf{S} & \mathbf{W} \end{pmatrix}, \quad (10)$$

where $\dim(\mathbf{N}_r) = \text{rank}(\mathbf{N})$, and \mathbf{P} is the permutation matrix. We usually say that \mathbf{N}_r is the regular part of the matrix \mathbf{N} .

It can be verified that matrix \mathbf{B} satisfies the condition $\mathbf{A}\mathbf{B}^T = \mathbf{0}$. Therefore, if we find the inverse of the augmented matrix \mathbf{N}^* (which is regular), then we have

$$\mathbf{N}^+ = \mathbf{Q}_{11} \quad (11)$$

where

$$\mathbf{N}^{*-1} = \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12}^T \\ \mathbf{Q}_{12} & \mathbf{Q}_{22} \end{pmatrix} \quad (12)$$

and $\dim(\mathbf{Q}_{11}) = \dim(\mathbf{N})$, and accordingly $\text{defect}(\mathbf{N}) = \dim(\mathbf{Q}_{22})$ (see Bjerhammar 1973).

This method requires optimization and further development for the case of large, sparse matrices. It was successfully realized and implemented into the *NetExpert* software.

5. Adjustment

Last-Squares estimation process has two equally important components, the functional and the stochastic model (Leick 1989). The most important component with aspect computational data processing of geodetic networks with large number of unknown parameters is adjustment algorithm or Last-Squares method (Fig. 2).

More appropriate approach to this problem is the complete inclusion of all measurements and unknown parameters into the unique functional and stochastic model of adjustment (Pelzer 1985). The evident advancement of PC hardware and also the creation of more powerful mathematical models made

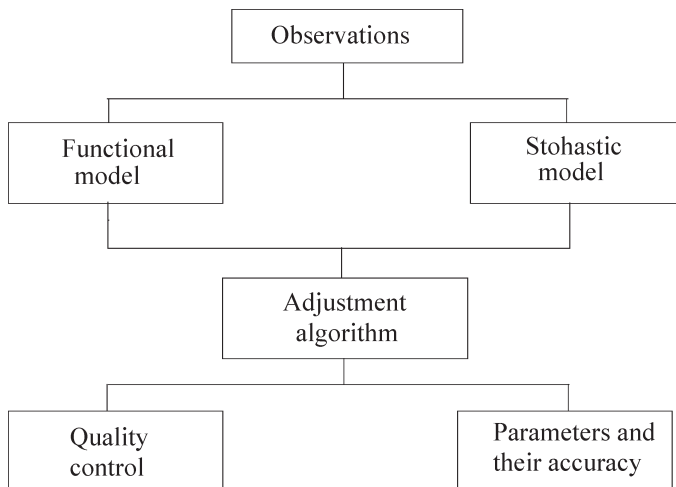


Fig. 2. *Least-Squares estimation process.*

possible the implementation of this approach. It is now possible to adjust a network of hundreds of thousand points in network. Furthermore, as the amount of labour necessary to perform the adjustment accordingly drops, it is now possible to do frequent periodical adjustments in purpose of geodetic network maintenance.

6. Optimization

Last-Squares estimation process can be used in solving problems of optimization of geodetic networks. Sequential method of optimization belongs to second order design and method may be used for obtaining the optimal standard deviation of planned observations.

The problem of determination of optimal standard deviation shall be defined by a general matrix equation (Aleksić 1988, Aleksić 1990)

$$\mathbf{K}_x = (\mathbf{A}^T(\mathbf{K}_1)^{-1}\mathbf{A})^+ \quad (13)$$

or

$$\mathbf{D}_x \mathbf{R}_x \mathbf{D}_x = (\mathbf{A}^T(\mathbf{D}_1^*)^{-1}(\mathbf{R}_1^*)^{-1}(\mathbf{D}_1^*)^{-1}\mathbf{A})^+ \quad (14)$$

where

\mathbf{K}_x – variance-covariance matrix criteria,

\mathbf{K}_1 – variance-covariance matrix planned measurements,

\mathbf{A} – design matrix,

\mathbf{D}_x – diagonal matrix of standard deviation parameters,

\mathbf{R}_x – correlation matrix of parameters,

\mathbf{D}_1^* – diagonal matrix optimal standard deviation of measurements,

\mathbf{R}_1^* – correlation matrix of measurements.

The matrices \mathbf{A} , \mathbf{D}_x and \mathbf{R}_1^* are constant and known in advance. By observing the last matrix equation, we see that it is necessary to determine the unknown diagonal matrix of standard deviations of the values to be measured in the geodetic networks

$$\mathbf{D}_1^* = \text{Diag}(\sigma_{l_1}^* \sigma_{l_2}^* \dots \sigma_{l_n}^*) \quad (15)$$

or

$$\sigma_{l_i}^* = \frac{\sigma_{l_i}}{\sqrt{u_i}} = F(\mathbf{A}, \mathbf{K}_1, \mathbf{D}_x) \quad (i=1, 2, \dots, n) \quad (16)$$

where $\sigma_{l_i}^*$ is optimal standard deviation of measurement. The sequential method is presented in the Fig. 3, with aspect of accuracy and reliability (Mihailović and Aleksić 2008).

7. Software Development

Development of the first software in Geodesy is closely related to the problem of network adjustment. For example, in year 1969, Geographical Survey Office of Sweden had developed the first prototype of TRUT, the software for adjustment of triangulation networks. It was an EDP program for adjusting horizontal control survey networks and it was adapted for CDC 6600 (Ussisoo 1977). It performed a single run adjustment on the plane with a capacity not exceeding 79 stations. In year 1977 new TRUT version had a capacity of 330 stations per single run.

Program OPERA is a computer program for operational adjustment of numerous kinds of geodetic observations in the sense of integrated geodesy (Hein and Landau 1983). It is intended for handling large networks, up to 800 3D points and 1600 observations.

In review ZfV a comparison of 15 different program systems for adjustment of planar geodetic networks was made in year 1985. The selected criteria of comparison also define the quality standard for up-to-date adjustment procedures (Bill 1985).

At the Institute of Geodesy and Geomatics, Faculty of Civil Engineering, University of Belgrade the *NetExpert* software package was developed. Besides the adjustment, it was also designed for the pre-analysis and network optimization, with full quality and gross-error control. It is capable of handling extremely large networks, with hundreds of thousands of points.

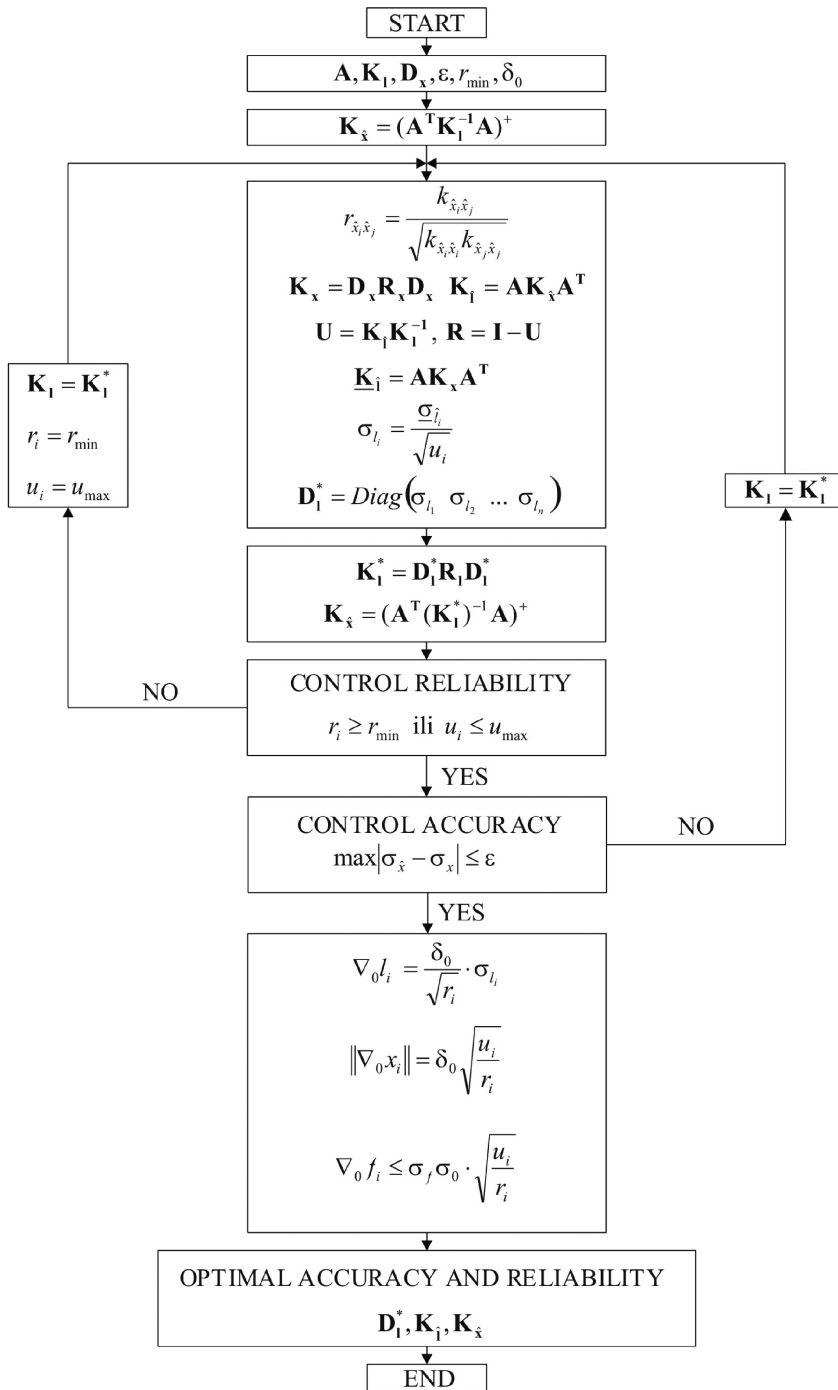


Fig. 3. Sequential method – aspect of accuracy and reliability.

8. The Results of the Tests

Apart from examples in geodetic practice, which are of moderate size, systems arising from simulated geodetic networks were being solved, where the measurements values were generated by the computer. Here are the timings for several such networks (Table 1), on a medium-powered PC computer (AMD Turion64 x2 2.6 GHz, 2GB RAM).

Table 1. *The timings for several geodetic networks.*

Points	Measurements	Unknowns	Time [sec]
1 000	4 000	2 000	0.13
5 000	20 000	10 000	0.89
10 000	40 000	20 000	2.1
20 000	80 000	40 000	5.4
50 000	200 000	100 000	19.6
100 000	400 000	200 000	47.3

These are values for the network with fixed points. Time measured covers all necessary calculations, including Cholesky decomposition, solving, and inverting. For the case of the free network and rank-deficient system of linear equations, times are only slightly greater (10% cca).

9. Conclusion

The use of computers has become a prerequisite for efficient handling of data in any geodetic discipline. This especially holds true for geodetic network adjustment.

Nevertheless, the computer capabilities are limited, and it is absolutely essential that the highest theoretical results in numerical mathematics and computer science be exploited when dealing with data processing in Geodesy.

As a consequence, this inevitably necessitates the existence of the inter-disciplinary teams, consisting of geodetic experts, mathematicians and computer science experts. Without such an approach, the achievable results are undoubtedly very much limited.

The described software allows that very large scale of geodetic networks can be handled on the PC platform. The software capabilities actually transcend the size of any realistic geodetic network.

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O računskim aspektima obrade podataka geodetskih mreža s velikim brojem nepoznatih parametara

SAŽETAK. U radu su u glavnim crtama prikazane neke posebne karakteristike linearnih sustava jednadžbi koje proizlaze iz izjednačenja geodetskih mreža, optimiranja i prethodne analize. Raspravlja se o prednostima i nedostacima nekoliko metoda za rješavanje linearnih sustava uzimajući u obzir njihovu primjenu na probleme koji nastaju iz geodetskih mreža s velikim brojem nepoznatih parametara. Neka od znanstvenih i stručnih postignuća unesena u softverski paket NetExpert bila su predmet ovog rada uključujući ona iz matematike, geodezije i računalne znanosti.

Ključne riječi: mreža, izjednačenje, sustavi, jednadžbe, parametar.

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