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LOCAL COMPUTABILITY OF COMPUTABLE METRIC SPACES AND COMPUTABILITY OF CO-C.E. CONTINUA

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Abstract. We investigate conditions on a computable metric space under which each co-computably enumerable set satisfying certain topological properties must be computable. We examine the notion of local computability and show that the result by which in a computable metric space which has the effective covering property and compact closed balls each co-c.e. circularly chainable continuum which is not chainable must be computable can be generalized to computable metric spaces which have the effective covering property and which are locally compact. We also give examples which show that neither of these two assumptions can be omitted.

1. Introduction

We say that a function $f : \mathbb{N}^k \to \mathbb{Q}$, $k \geq 1$, is *computable* if there exist computable (i.e., recursive) functions $a, b, c : \mathbb{N}^k \to \mathbb{N}$ such that $f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1}$ for each $x \in \mathbb{N}^k$. A function $f : \mathbb{N}^k \to \mathbb{R}$ is said to be *computable* if there exists a computable function $F : \mathbb{N}^{k+1} \to \mathbb{Q}$ such that $|f(x) - F(x,i)| < 2^{-i}$ for all $x \in \mathbb{N}^k$ and $i \in \mathbb{N}$ and a number $x \in \mathbb{R}$ is said to be *computable* if there exists a computable function $g : \mathbb{N} \to \mathbb{Q}$ such that $|x - g(i)| < 2^{-i}$ for each $i \in \mathbb{N}$.

A function $f : \mathbb{N}^k \to \mathbb{R}^n$, $n \geq 1$, will be called computable if the component functions of f are computable as functions $\mathbb{N}^k \to \mathbb{R}$. We say that $x \in \mathbb{R}^n$ is a *computable point* if $x = (x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are computable numbers.

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A tuple (X,d,α) is said to be a *computable metric space* if (X,d) is a metric space and $\alpha:\mathbb{N}\to X$ is a sequence dense in (X,d) such that the function $\mathbb{N}^2\to\mathbb{R}$, $(i,j)\mapsto d(\alpha(i),\alpha(j))$ is computable.

If (X,d,α) is a computable metric space, then a sequence (x_i) in X is said to be *computable* in (X,d,α) if there exists a computable function $F:\mathbb{N}^2\to\mathbb{N}$ such that $d(x_i,\alpha_{F(i,k)})<2^{-k}$ for all $i,k\in\mathbb{N}$ and a point $a\in X$ is said to be *computable* in (X,d,α) if the constant sequence $a,a,\dots$ is computable.

For example, if $\alpha:\mathbb{N}\to\mathbb{R}^n$ is a computable function whose image is dense in $\mathbb{R}^n$, then $(\mathbb{R}^n,d,\alpha)$ is a computable metric space, where d is the Euclidean metric on $\mathbb{R}^n$. A sequence (x_i) is computable in this computable metric space if and only if (x_i) is a computable sequence in $\mathbb{R}^n$ (i.e., a computable function $\mathbb{N}\to\mathbb{R}^n$) and $x\in\mathbb{R}^n$ is a computable point in this space if and only if x is a computable point.

Let $q:\mathbb{N}\to\mathbb{Q}$ be some fixed computable function whose image is $\mathbb{Q}\cap\langle 0,\infty\rangle$, where $\langle a,b\rangle$ denotes the open interval of the reals with given endpoints a and b. Let $\tau_1,\tau_2:\mathbb{N}\to\mathbb{N}$ be some fixed computable functions such that $\{(\tau_1(i),\tau_2(i))\mid i\in\mathbb{N}\}=\mathbb{N}^2$. We are going to use the following notation: $\langle i\rangle_1$ instead of $\tau_1(i)$ and $\langle i\rangle_2$ instead of $\tau_2(i)$.

Let (X,d,α) be a computable metric space. For $i\in\mathbb{N}$ we define

$$I_i=B(\alpha_{\langle i\rangle_1},q_{\langle i\rangle_2}),\ \widehat{I}_i=\widehat{B}(\alpha_{\langle i\rangle_1},q_{\langle i\rangle_2}).$$

Here, for $x\in X$ and $r>0$, we denote by $B(x,r)$ the open ball of radius r centered at x and by $\widehat{B}(x,r)$ the corresponding closed ball, i.e., $B(x,r)=\{y\in X\mid d(x,y)<r\}$, $\widehat{B}(x,r)=\{y\in X\mid d(x,y)\le r\}$. For $A\subseteq X$ we will denote the closure of A by $\overline{A}$.

A closed subset S of (X,d) is said to be *computably enumerable* in (X,d,α) if

$$\{i\in\mathbb{N}\mid S\cap I_i\neq\emptyset\}$$

is a c.e. subset of $\mathbb{N}$. A closed subset S is said to be *co-computably enumerable* in (X,d,α) if there exists a computable function $f:\mathbb{N}\to\mathbb{N}$ such that

$$X\setminus S=\bigcup_{i\in\mathbb{N}}I_{f(i)}.$$

These definitions do not depend on functions τ_1, τ_2 and q. We say that S is a *computable* set in (X,d,α) if S is both computably enumerable and co-computably enumerable ([2, 10]).

Let $\sigma:\mathbb{N}^2\to\mathbb{N}$ and $\eta:\mathbb{N}\to\mathbb{N}$ be some fixed computable functions with the following property: $\{(\sigma(j,0),\dots,\sigma(j,\eta(j)))\mid j\in\mathbb{N}\}$ is the set of all nonempty finite sequences in $\mathbb{N}$, i.e., the set $\{(a_0,\dots,a_n)\mid n\in\mathbb{N},\ a_0,\dots,a_n\in\mathbb{N}\}$. Such functions, for instance, can be defined using the Cantor pairing function. We are going to use the following notation: $(j)_i$ instead of $\sigma(j,i)$

and $\overline{j}$ instead of $\eta(j)$. Hence

$$\{((j)_0, \ldots, (j)_{\overline{j}}) \mid j \in \mathbb{N}\}$$

is the set of all nonempty finite sequences in $\mathbb{N}$. For $j \in \mathbb{N}$ the set $\{(j)_i \mid 0 \leq i \leq \overline{j}\}$ will be denoted by $[j]$.

Let (X, d, α) be a computable metric space. For $j \in \mathbb{N}$ we define

$$J_j = \bigcup_{i \in [j]} I_i, \ \widehat{J}_j = \bigcup_{i \in [j]} \widehat{I}_i.$$

The sets J_j represent finite unions of rational balls and the sets $\widehat{J}_j$ finite unions of closed rational balls.

A computable metric space (X, d, α) has the *effective covering property* if the set

$$\{(i, j) \in \mathbb{N}^2 \mid \widehat{I}_i \subseteq J_j\}$$

is computably enumerable ([2]). It is not hard to check that this definition does not depend on the choice of the functions $q, \tau_1, \tau_2, \sigma, \eta$ which are necessary in the definitions of sets I_w and J_j.

If (X, d, α) is a computable metric space, than a compact set K in (X, d) is said to be *computable compact* in (X, d, α) if K is computably enumerable in (X, d, α) and if the set $\{j \in \mathbb{N} \mid K \subseteq J_j\}$ is c.e. ([1]).

If (X, d) is a complete metric space, then each nonempty computable set S in (X, d, α) contains a computable point, moreover there exists a computable sequence of points dense in S. On the other hand, there exist nonempty co-computably enumerable sets (even in $\mathbb{R}$) which contain no computable points ([9]). So, while each computable set is co-c.e. by definition, the implication

$$S \text{ co-computably enumerable } \Rightarrow S \text{ computable} \tag{1.1}$$

does not hold in general, in fact there are co-c.e. sets which are "far away from being computable". However, under certain assumptions (1.1) holds ([6], [1], [5]). In particular, by [5], we have the following theorem.

THEOREM 1.1. *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls and let $S \subseteq X$.*

(a) *If S is co-c.e. and, as a subspace of (X, d), a circularly chainable, but not a chainable continuum, then S is computable.*
(b) *If S is a co-c.e. continuum chainable from computable points a and b in (X, d, α), then S is computable.*
(c) *If S is a co-c.e. chainable decomposable continuum, then for every $\varepsilon > 0$ there is a computable subcontinuum K of S which is $\varepsilon-$close to S with respect to the Hausdorff metric.*

The definitions of a chainable, circularly chainable and decomposable continuum can be found in [3], [4], [7], as well as the definition of the Hausdorff metric. For example, each topological circle (i.e., metric space homeomorphic to a cirle) is circularly chainable, but not chainable, and each arc is chainable and decomposable.

In this paper we prove that the assumption in Theorem 1.1 that (X, d) has compact closed balls, which means that the set $\widehat{B}(x, r)$ is compact for all $x \in X$, $r > 0$, can be replaced with the assumption that (X, d) is locally compact (if a metric space has compact closed balls, then it is clearly locally compact, converse does not hold in general which shows the example of the metric space (X, d), where d is the discrete metric on an infinite set X).

In fact, we will show that Theorem 1.1 holds under the assumption that a computable metric space (X, d, α) is locally computable. A computable metric space (X, d, α) is *locally computable* ([1]) if for each compact set A in (X, d) there exists a computable compact set K in (X, d, α) such that $A \subseteq K$. As we will see, each computable metric space which has the effective covering property and which is locally compact is locally computable. On the other hand, there exists a computable metric space which is locally computable and locally compact, but which does not have the effective covering property (Example 4.4).

2. Basic facts and techniques

Let $k, n \in \mathbb{N}$, $k, n \geq 1$. By a partial computable function $f : S \to \mathbb{N}^n$, $S \subseteq \mathbb{N}^k$, we mean a function whose component functions $f_1, \dots, f_n : S \to \mathbb{N}$ are partial computable. Of course, such a function will be called computable if $S = \mathbb{N}^k$. In the following proposition we state some elementary facts.

Proposition 2.1. (i) (Projection Theorem) *Let $T \subseteq \mathbb{N}^{k+n}$ be a computably enumerable set. Then the set $S = \{x \in \mathbb{N}^k \mid \exists y \in \mathbb{N}^n$ such that $(x, y) \in T\}$ is computably enumerable.*

(ii) (Single-Valuedness Theorem) *If $S_1 \subseteq \mathbb{N}^k$ and $S_2 \subseteq \mathbb{N}^n$ are c.e. sets such that for each $x \in S_1$ there exists $y \in S_2$ such that $(x, y) \in T$, then there exists a partial computable function $f : S_1 \to \mathbb{N}^n$ such that $f(S_1) \subseteq S_2$ and $(x, f(x)) \in T$ for each $x \in S_1$.*

In the following proposition we state some elementary facts about computable functions $\mathbb{N}^k \to \mathbb{Q}$ and $\mathbb{N}^k \to \mathbb{R}$.

Proposition 2.2. (i) *If $f, g : \mathbb{N}^k \to \mathbb{Q}$ are computable, then $f + g, f - g : \mathbb{N}^k \to \mathbb{Q}$ are computable.*

(ii) *If $f, g : \mathbb{N}^k \to \mathbb{R}$ are computable, then $f + g, f - g : \mathbb{N}^k \to \mathbb{R}$ are computable.*

(iii) *If $f, g : \mathbb{N}^k \to \mathbb{Q}$ are computable functions, then the sets $\{x \in \mathbb{N}^k \mid f(x) < g(x)\}$ and $\{x \in \mathbb{N}^k \mid f(x) \leq g(x)\}$ are computable.*

(iv) *If $f, g : \mathbb{N}^k \to \mathbb{R}$ are computable functions, then the set $\{x \in \mathbb{N}^k \mid f(x) < g(x)\}$ is c.e.*

(v) *If* $f : \mathbb{N}^k \to \mathbb{R}$ *and* $F : \mathbb{N}^{k+1} \to \mathbb{R}$ *are functions such that* F *is computable and* $|f(x) - F(x,i)| < 2^{-i}$ *for all* $x \in \mathbb{N}^k$ *and* $i \in \mathbb{N}$*, then* f *is computable.*

COROLLARY 2.3. *Let* (X, d, α) *be a computable metric space.*

(i) *If* (x_i) *and* (y_j) *are computable sequences in* (X, d, α)*, then the function* $(i, j) \mapsto d(x_i, y_j)$ *is computable.*
(ii) *If* $x \in X$*, then* $\{x\}$ *is a c.e. set in* (X, d, α) *if and only if* x *is a computable point in* (X, d, α)*.*

PROOF. (i) This follows from Proposition 2.2(v) and the fact that $|d(x, y) - d(a, a')| \leq d(x, a) + d(y, a')$ for all $x, y, a, a' \in X$.

(ii) Let $L = \{i \in \mathbb{N} \mid x \in I_i\}$. If $\{x\}$ is c.e., then L is c.e. and by Proposition 2.2(iv) and the Single-Valuedness Theorem we can for each $k \in \mathbb{N}$ effectively find $i \in \mathbb{N}$ such that $i \in L$ and $q_{\langle i \rangle_2} < 2^{-k}$, which implies the computability of x.

On the other hand, if $f : \mathbb{N} \to \mathbb{N}$ is a computable function such that $d(x, \alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbb{N}$, then for each $i \in \mathbb{N}$ we have

$$x \in I_i \iff \exists k \in \mathbb{N} \text{ such that } d(\alpha_{\langle i \rangle_1}, \alpha_{f(k)}) + 2^{-k} < q_{\langle i \rangle_2}.$$

The computable enumerability of $\{x\}$ now follows from Proposition 2.2(iv) and the Projection theorem. ☐

LEMMA 2.4. *Let* (X, d, α) *be a computable metric space. Let* (x_i)*,* (y_j) *be computable sequences in this space and* (r_i)*,* (s_j) *computable sequences in* $\langle 0, \infty \rangle$ *(i.e., sequences of positive numbers which are computable in* $\mathbb{R}$*.) Let*

$$A = \{(j, i) \in \mathbb{N}^2 \mid d(x_i, y_j) + s_j < r_i\}.$$

Then

(i) A *is c.e.;*
(ii) *if* $(j, i) \in A$*, then* $\widehat{B}(y_j, s_j) \subseteq B(x_i, r_i)$*;*
(iii) *if* $a \in X$ *and* $i \in \mathbb{N}$ *are such that* $a \in B(x_i, r_i)$*, then there exists* $\varepsilon > 0$ *such that* $a \in B(y_j, s_j)$ *and* $s_j < \varepsilon$ *imply* $(j, i) \in A$*.*

PROOF. (ii) is obvious and (i) follows from Proposition 2.2 and Corollary 2.3(i). If $a \in X$ and $i \in \mathbb{N}$ are as in (iii), then there exists $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, such that $d(x_i, a) + 2\varepsilon < r_i$. Now, if $j \in \mathbb{N}$ is such that $a \in B(y_j, s_j)$ and $s_j < \varepsilon$, then $d(a, y_j) < s_j < \varepsilon$ and

$$d(y_j, x_i) + s_j < d(y_j, a) + d(a, x_i) + \varepsilon < d(a, x_i) + 2\varepsilon < r_i,$$

hence $(i, j) \in A$. ☐

EXAMPLE 2.5. Let (X, d, α) be a computable metric space. Let A be the set associated to the sequences $(\alpha_{\langle i \rangle_1})$, $(\alpha_{\langle j \rangle_1})$, $(q_{\langle i \rangle_2})$, $(q_{\langle j \rangle_2})$ as in Lemma 2.4.

(i) Suppose S and T are subsets of X and $f, g : \mathbb{N} \to \mathbb{N}$ are computable functions such that $X \setminus S = \bigcup_{j\in\mathbb{N}} I_{f(j)}$, $X \setminus T = \bigcup_{k\in\mathbb{N}} I_{g(k)}$. Let B be the set of all $i \in \mathbb{N}$ for which there exist $j, k \in \mathbb{N}$ such that $(i, f(j)) \in A$, $(i, g(k)) \in A$. The set B is c.e. by the Projection theorem. Let $h : \mathbb{N} \to \mathbb{N}$ be a computable function such that $h(\mathbb{N}) = B$. Using Lemma 2.4, it is easy to conclude that

$$X \setminus (S \cup T) = \bigcup_{i\in\mathbb{N}} I_{h(i)}.$$

Hence the union of two co-c.e. sets is co-c.e (note that the intersection of two co-c.e. sets is also co-c.e).

(ii) Suppose that the metric space (X, d) is complete and let S be a nonempty c.e. set in this space. Let $C = \{(j, i) \in \mathbb{N}^2 \mid q_{\langle j\rangle_2} < \frac{1}{2} q_{\langle i\rangle_2}\}$. This set is c.e. by Proposition 2.2. Since the set B defined by $B = \{i \in \mathbb{N} \mid S \cap I_i \neq \emptyset\}$ is c.e., the set

$$D = A \cap C \cap \{(j, i) \in \mathbb{N}^2 \mid i \in B,\ j \in B\}$$

is also c.e. It follows easily from Lemma 2.4(iii) that for each $i \in B$ there exists $j \in B$ such that $(j, i) \in D$. By the Single-Valuedness Theorem there exists a partial computable function $\varphi : B \to \mathbb{N}$ such that $\varphi(B) \subseteq B$ and $(\varphi(i), i) \in D$ for each $i \in B$. Note that the function $B \times \mathbb{N} \to \mathbb{N}$, $(i, k) \mapsto \varphi^{(k)}(i)$, is partial computable ($\varphi^{(0)}(i) = 0$, $\varphi^{(k+1)}(i) = \varphi(\varphi^{(k)}(i))$).

Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function such that $f(\mathbb{N}) = B$. Using the fact that (X, d) is complete, we easily conclude that for each $n \in \mathbb{N}$ the intersection $\bigcap_{k\in\mathbb{N}} I_{\varphi^{(k)}(f(n))}$ contains a unique point x_n and it is not hard to see that the sequence $(x_n)_{n\in\mathbb{N}}$ is computable and dense in S. Hence for each nonempty c.e. set S in (X, d, α) there exists a computable sequence dense in S (see also [2]).

We say that a function $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ is *computable* if the function $\overline{\Phi} : \mathbb{N}^{k+n} \to \mathbb{N}$ defined by

$$\overline{\Phi}(x, y) = \chi_{\Phi(x)}(y),$$

$x \in \mathbb{N}^k$, $y \in \mathbb{N}^n$ is computable. Here $\mathcal{P}(\mathbb{N}^n)$ denotes the set of all subsets of $\mathbb{N}^n$, and $\chi_S : \mathbb{N}^n \to \{0, 1\}$ denotes the characteristic function of $S \subseteq \mathbb{N}^n$. A function $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ is said to be *computably bounded* if there exists a computable function $\varphi : \mathbb{N}^k \to \mathbb{N}$ such that $\Phi(x) \subseteq \{0, \dots, \varphi(x)\}^n$ for each $x \in \mathbb{N}^k$, where $\{0, \dots, \varphi(x)\}^n$ equals the set of all $(y_1, \dots, y_n) \in \mathbb{N}^n$ such that $\{y_1, \dots, y_n\} \subseteq \{0, \dots, \varphi(x)\}$.

We say that a function $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ is *c.c.b.* if Φ is computable and computably bounded. It is not hard to prove the following proposition.

PROPOSITION 2.6. (i) *If $\Phi, \Psi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ are c.c.b. functions, then the sets $\{x \in \mathbb{N}^k \mid \Phi(x) = \Psi(x)\}$, $\{x \in \mathbb{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$, $\{x \in \mathbb{N}^k \mid \Phi(x) = \emptyset\}$ are computable.*

(ii) *If* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *and* $\Psi : \mathbb{N}^{n+k} \to \mathcal{P}(\mathbb{N}^m)$ *are c.c.b. functions, then* $\Lambda : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^m)$ *defined by*

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z, x),$$

$x \in \mathbb{N}^k$*, is a c.c.b. function.*

(iii) *If* $\Phi : \mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$ *is c.c.b. and* $T \subseteq \mathbb{N}^n$ *is c.e., then the set* $S = \{x \in \mathbb{N}^k \mid \Phi(x) \subseteq T\}$ *is c.e.*

Example 2.7. If $\alpha, \beta : \mathbb{N}^k \to \mathbb{N}$ and $f : \mathbb{N}^{k+1} \to \mathbb{N}^n$ are computable functions, then the function $\mathbb{N}^k \to \mathcal{P}(\mathbb{N}^n)$, $x \mapsto \{f(i, x) \mid \alpha(x) \leq i \leq \beta(x)\}$ is c.c.b. In particular, the function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$, $n \mapsto [n]$, is c.c.b.

Example 2.8. Let (X, d, α) be a computable metric space such that the set $\{j \in \mathbb{N} \mid X = J_j\}$ is c.e. Suppose (X, d) is compact. We claim that there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X = B(\alpha_0, 2^{-k}) \cup \cdots \cup B(\alpha_{f(k)}, 2^{-k}) \tag{2.1}$$

for each $k \in \mathbb{N}$.

Let $S = \{(n, k) \in \mathbb{N}^2 \mid q_{\langle i \rangle_2} < 2^{-k} \text{ for each } i \in [n]\}$. It follows readily from Proposition 2.2(iv) and Proposition 2.6(iii) that S is c.e. Since for each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $X = J_n$ and $(n, k) \in S$, there exists a computable function (the Single-Valuedness Theorem) $\varphi : \mathbb{N} \to \mathbb{N}$ such that $X = J_{\varphi(k)}$ and $(\varphi(k), k) \in S$ for each $k \in \mathbb{N}$. Now, if $f : \mathbb{N} \to \mathbb{N}$ is some computable function such that $[\varphi(k)] \subseteq \{0, \dots, f(k)\}$ for each $k \in \mathbb{N}$, then (2.1) holds.

Conversely, if (X, d, α) is a computable metric space such that (X, d) is compact and such that there exists a computable function f with the property (2.1), then it can be shown ([5, Corollary 22]) that (X, d, α) has the effective covering property. Hence we have the following proposition.

Proposition 2.9. *Let* (X, d, α) *be a computable metric space such that* (X, d) *is compact. Then the following statements are equivalent.*

(i) *The set* $\{j \in \mathbb{N} \mid X = J_j\}$ *is c.e.*
(ii) (X, d, α) *has the effective covering property.*
(iii) *There exists a computable function* $f : \mathbb{N} \to \mathbb{N}$ *such that (2.1) holds.*

3. Local computability

Let (X, d, α) be a computable metric space. A computable metric space (Y, d', β) is said to be a *subspace* of (X, d, α) if $Y \subseteq X$, $d' : Y \times Y \to \mathbb{R}$ is the restriction of $d : X \times X \to \mathbb{R}$ and β is a computable sequence in (X, d, α).

Proposition 3.1. *Let* (Y, d', β) *be a subspace of a computable metric space* (X, d, α) *and let* $S \subseteq Y$.

(i) *If S is co-c.e. in (X,d,α), then S is co-c.e. in (Y,d',β).*
(ii) *If S is c.e. in (X,d,α), then S is c.e. in (Y,d',β). Conversely, if S is closed in (X,d) and c.e. in (Y,d',β), then S is c.e. in (X,d,α).*
(iii) *Let $y \in Y$. Then y is a computable point in (Y,d',β) if and only if y is a computable point in (X,d,α).*

Proof. For $y \in Y$ and $r > 0$ let $B_Y(y,r)$ denote the open ball with respect to metric space (Y,d'). For $i \in \mathbb{N}$ let $K_i = B_Y(\beta_{\langle i\rangle_1}, q_{\langle i\rangle_2})$.

(i) Suppose S is co-c.e. in (X,d,α). We want to prove that $Y \setminus S = \bigcup_{j\in\mathbb{N}} K_{f(j)}$ holds for some computable function $f : \mathbb{N} \to \mathbb{N}$. We have

$$X \setminus S = \bigcup_{i\in\mathbb{N}} B(x_i, r_i),$$

where (x_i) is a computable sequence in (X,d,α) and (r_i) a computable sequence in $\langle 0,\infty\rangle$. Let A be the set associated to the sequences (x_i), $(\beta_{\langle j\rangle_1})$, (r_i), $(q_{\langle j\rangle_2})$ as in Lemma 2.4. Then $(j,i) \in A$ implies $K_j \subseteq B(x_i,r_i)$ and, since A is c.e., there exist computable functions $f,g : \mathbb{N} \to \mathbb{N}$ such that

$$A = \{(f(n), g(n)) \mid n \in \mathbb{N}\}.$$

It follows $K_{f(n)} \subseteq B(x_{g(n)}, r_{g(n)})$ for each $n \in \mathbb{N}$, which implies $\bigcup_{n\in\mathbb{N}} K_{f(n)} \subseteq Y \setminus S$. Now it suffices to prove

$$Y \setminus S \subseteq \bigcup_{n\in\mathbb{N}} K_{f(n)}. \tag{3.1}$$

Let $a \in Y \setminus S$. It follows $a \in X \setminus S$ and $a \in B(x_i,r_i)$ for some $i \in \mathbb{N}$. It follows from Lemma 2.4(iii) that there exists $j \in \mathbb{N}$ such that $a \in K_j$ and $(j,i) \in A$. Therefore $i = f(n)$ for some $n \in \mathbb{N}$ and we have $a \in K_{f(n)}$. This proves (3.1). Hence S is co-c.e. in (Y,d',β).

(ii) Suppose S is c.e. in (X,d,α). Hence $B = \{j \in \mathbb{N} \mid S \cap I_j \neq \emptyset\}$ is a c.e. subset of $\mathbb{N}$. Let A be the set associated to the sequences $(\beta_{\langle i\rangle_1})$, $(\alpha_{\langle j\rangle_1})$, $(q_{\langle i\rangle_2})$, $(q_{\langle j\rangle_2})$ as in Lemma 2.4. Then A is c.e. and $(j,i) \in A$ implies $I_j \cap Y \subseteq K_i$.

Let $i \in \mathbb{N}$. Using Lemma 2.4(ii) we get that for each $s \in S$ the following equivalence holds:

$$d(s, \beta_{\langle i\rangle_1}) < q_{\langle i\rangle_2} \ \Leftrightarrow\ \exists j \in \mathbb{N} \text{ such that } s \in I_j \text{ and } (j,i) \in A.$$

From this we conclude the following:

$$S \cap K_i \neq \emptyset \ \Leftrightarrow\ \exists j \in \mathbb{N} \text{ such that } j \in B \text{ and } (j,i) \in A.$$

Now, by the Projection theorem, the set $\{i \in \mathbb{N} \mid S \cap K_i \neq \emptyset\}$ is c.e., hence S is c.e. in (Y,d',β). In the same way we get that computable enumerability of S in (Y,d',β) implies computable enumerability of S in (X,d,α) (under the assumption that S is closed).

(iii) This follows from (ii) and Corollary 2.3. □

EXAMPLE 3.2. Let $f : \mathbb{N} \to \mathbb{Q}$ be a computable sequence which converges to a noncomputable number $b \in \mathbb{R}$ and such that $f(0) = 0$ and $f(i) < f(i+1)$ for each $i \in \mathbb{N}$ ([8]). It is easy to construct a computable sequence of rational numbers β such that $\beta(\mathbb{N}) = \mathbb{Q} \cap [0, b]$. Then the tuple $([0, b], d', \beta)$ is a computable metric space, where d' is the Euclidean metric on $[0, b]$. On the other hand, let $\alpha : \mathbb{N} \to \mathbb{Q}$ be any computable surjection and let d be the Euclidean metric on $\mathbb{R}$. Then β is a computable sequence in $(\mathbb{R}, d, \alpha)$. Therefore $([0, b], d', \beta)$ is a subspace of $(\mathbb{R}, d, \alpha)$. The set $\{b\}$ is co-c.e. in $([0, b], d', \beta)$ since

$$[0, b] \setminus \{b\} = \bigcup_{i \in \mathbb{N}} [0, f(i+1)\rangle.$$

On the other hand, $\{b\}$ is not co-c.e. in $(\mathbb{R}, d, \alpha)$; if it were, it would be computable, namely $(\mathbb{R}, d, \alpha)$ has the effective covering property and compact closed balls and in such computable metric space each co-c.e. singleton set is computable ([5]), however $\{b\}$ is not computable, the fact that $\{b\}$ is c.e. would imply that b is a computable number.

Let us note that $x \in [0, b]$ is a computable point in $([0, b], d', \beta)$ if and only if x is a computable number. Therefore b is not a computable point in $([0, b], d', \beta)$ which implies that $\{b\}$ is not c.e. (and consequently not computable) in this computable metric space. Using the previous argument we conclude that $([0, b], d', \beta)$ does not have the effective covering property.

PROPOSITION 3.3. *Let (X, d, α) be a computable metric space and let K be a nonempty compact set in (X, d). Then K is computable compact in (X, d, α) if and only if there exist a metric d' on K and a sequence β in K such that (K, d', β) is a subspace of (X, d, α) and (K, d', β) has the effective covering property.*

PROOF. Suppose K is computable compact. First, we claim that there exists a computable sequence in (X, d, α) dense in K. If (X, d) is complete, this follows immediately from the fact that K is c.e.; otherwise we can take its completion (X', d'), note that (X', d', α) is a computable metric space in which K is also c.e. (the fact that it is closed follows from its compactness) and apply the fact that a computable sequence (x_i) in (X', d', α) such that $x_i \in X$ for each $i \in \mathbb{N}$ is computable in (X, d, α).

So, let β be a computable sequence dense in K. With the restriction $d' : K \times K \to \mathbb{R}$ of the metric $d : X \times X \to \mathbb{R}$, the triple (K, d', β) is a computable metric space. It remains to prove that this space has the effective covering property. For $i \in \mathbb{N}$ let $I_i^\beta = B(\beta_{\langle i \rangle_1}, q_{\langle i \rangle_2})$ and for $j \in \mathbb{N}$ let $J_j^\beta = \bigcup_{i \in [j]} I_i^\beta$. Let A be the set associated to the sequences $(\beta_{\langle i \rangle_1})$, $(\alpha_{\langle j \rangle_1})$, $(q_{\langle i \rangle_2})$, $(q_{\langle j \rangle_2})$ as in Lemma 2.4. Hence $(j, i) \in A$ implies $I_j \subseteq I_i^\beta$. Let

$$\mathcal{A} = \{(m, n) \in \mathbb{N}^2 \mid \forall j \in [m]\ \exists i \in [n] \text{ such that } (j, i) \in A\}.$$

Then $(m,n) \in \mathcal{A}$ implies $J_m \subseteq J_n^\beta$. It follows readily from Proposition 2.6(iii) that $\mathcal{A}$ is c.e.

Suppose $n \in \mathbb{N}$ is such that $K \subseteq J_n^\beta$. It is easy to conclude from Lemma 2.4(iii) and the fact that K is compact that then there exists $m \in \mathbb{N}$ such that $K \subseteq J_m$ and $(m,n) \in \mathcal{A}$. Therefore, for $n \in \mathbb{N}$ we have the following equivalence:

$$K \subseteq J_n^\beta \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } K \subseteq J_m \text{ and } (m,n) \in \mathcal{A}.$$

Therefore $\{n \in \mathbb{N} \mid K \subseteq J_n^\beta\}$ is a c.e. set. Proposition 2.9 now implies that (K, d', β) has the effective covering property.

Conversely, suppose that (K, d', β) is a subspace of (X, d, α) which has the effective covering property. Using the same notation J_n^β we can find in the same way a c.e. subset $\mathcal{B}$ of $\mathbb{N}^2$ such that for $m \in \mathbb{N}$ the equivalence

$$K \subseteq J_m \Leftrightarrow \exists n \in \mathbb{N} \text{ such that } K \subseteq J_n^\beta \text{ and } (n,m) \in \mathcal{B}$$

holds, which implies computable enumerability of the set $\{m \in \mathbb{N} \mid K \subseteq J_m\}$. On the other hand, K is c.e. in (X, d, α) since

$$K \cap I_i \neq \emptyset \Leftrightarrow \exists j \in \mathbb{N} \text{ such that } d(\alpha_{\langle i \rangle_1}, \beta_j) < q_{\langle i \rangle_2}.$$

Therefore K is computable compact. ∎

As a consequence of Proposition 3.1 and Proposition 3.3 we have the following theorem.

THEOREM 3.4. *Let (X, d, α) be a computable metric space which is locally computable. Then the statements (a), (b) and (c) of Theorem 1.1 hold.*

PROOF. Let S be a co-c.e. set in (X, d, α) and suppose S is compact. Then $S \subseteq K$, where K is computable compact in (X, d, α). By Proposition 3.3 there exist d' and β such that (K, d', β) is a subspace of (X, d, α) and such that (K, d', β) has the effective covering property. Note that the metric on S induced by d is same as the metric on S induced by d'.

Now, if S is a circularly chainable, but not a chainable continuum, then S is computable in (K, d', β) by Theorem 1.1. Therefore S is c.e. in (K, d', β) which implies that S is c.e. in (X, d, α) (Proposition 3.1). This and the fact that S is co-c.e. in (X, d, α) give that S is computable in (X, d, α).

In the same way we prove the statements (b) and (c). ∎

4. LOCAL COMPACTNESS AND THE EFFECTIVE COVERING PROPERTY

In this section we prove that each computable metric space which has the effective covering property and which is locally compact is locally computable. This would mean that Theorem 3.4 is a generalization of Theorem 1.1.

Let (X, d, α) be computable metric space which has the effective covering property and which is locally compact. In order to prove that (X, d, α) is locally computable, it would be enough to prove that for each $x \in X$ there exists a computable compact set K whose interior contain x. Namely, in that case for each compact set $A \subseteq X$ we can find finitely many computable compact sets $K_1, \dots, K_n$ such that $A \subseteq K_1 \cup \dots K_n$. It is easy to prove that the union of finitely many computable compact sets is computable. Hence A is contained in a computable compact set.

Since (X, d) is locally compact, each $x \in X$ is contained in some compact ball $\widehat{B}(\alpha_i, r)$, where $i \in \mathbb{N}$, $r \in \mathbb{Q}$, $r > 0$. So it would be enough to prove that each compact closed rational ball is computable compact. However, this does not have to be true.

Example 4.1. Let (λ_i) be a computable sequence of real numbers such that $0 \leq \lambda_i \leq 2^{-i}$ for each $i \in \mathbb{N}$ and such that the set $\{i \in \mathbb{N} \mid \lambda_i = 0\}$ is not computable ([8]). Let $A = (0, 0)$ and for $i \in \mathbb{N}$ let $B_i = (1 + \lambda_i, 2^{-i})$. Let

$$X = \{A\} \cup \{B_i \mid i \in \mathbb{N}\} \cup \{(1, 0)\}$$

and let d be the metric on X defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}. \tag{4.1}$$

Let $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ be the sequence defined by $\alpha_0 = A$, $\alpha_i = B_{i-1}$, $i \geq 1$. Then (X, d, α) is a computable metric space. This space is clearly compact and it follows easily from Proposition 2.9 that this space has the effective covering property.

Suppose that the set $\widehat{B}(A, 1)$ is c.e. in (X, d, α). Then there exists a computable sequence (z_i) in (X, d, α) dense in $\widehat{B}(A, 1)$. Let (x_i) and (y_i) be sequences of real numbers such that $z_i = (x_i, y_i)$ for each $i \in \mathbb{N}$. Then (x_i) and (y_i) are computable sequences in $\mathbb{R}$. For $i \in \mathbb{N}$ we have

$$\lambda_i = 0 \Leftrightarrow B_i \in \widehat{B}(A, 1) \Leftrightarrow \exists j \in \mathbb{N} \text{ such that } z_j = B_i \Leftrightarrow$$
$$\Leftrightarrow \exists j \in \mathbb{N} \text{ such that } y_j = 2^{-i}.$$

Since $y_j \in \{2^{-k} \mid k \in \mathbb{N}\} \cup \{0\}$ for each $j \in \mathbb{N}$, equality $y_j = 2^{-i}$ is equivalent to $|y_j - 2^{-i}| < 2^{-(i+1)}$. Therefore

$$\lambda_i = 0 \Leftrightarrow \exists j \in \mathbb{N} \text{ such that } |y_j - 2^{-i}| < 2^{-(i+1)}.$$

It follows (Proposition 2.2) that the set $\{i \in \mathbb{N} \mid \lambda_i = 0\}$ is c.e. and since the complement of this set is also c.e. by Proposition 2.2, we have that $\{i \in \mathbb{N} \mid \lambda_i = 0\}$ is a computable set, which is impossible.

So, $\widehat{B}(A, 1)$ is not c.e. and therefore not computable compact.

On the other hand, in each computable metric space (X, d, α) the closure of the open ball $B(\alpha_i, r)$ is c.e. set for all $i \in \mathbb{N}$, $r \in \mathbb{Q}$, $r > 0$, namely for $k \in \mathbb{N}$ we have

$$\overline{B(\alpha_i, r)} \cap I_k \neq \emptyset \Leftrightarrow \exists j \in \mathbb{N} \text{ such that } d(\alpha_j, \alpha_i) < r \text{ and } d(\alpha_j, \alpha_{\langle k \rangle_1}) < q_{\langle k \rangle_2}.$$

Nevertheless, $\overline{B(\alpha_i, r)}$ need not be computable compact.

Example 4.2. Let (X, d, α) be a computable metric space defined as in Example 4.1 with the exception that we now take $B_i = (1 - \lambda_i, 2^{-i})$ for each $i \in \mathbb{N}$. Let us suppose that $\overline{B(A,1)}$ is computable compact. Note that $\overline{B(A,1)} = B(A,1) \cup \{(1,0)\}$. By Proposition 3.3 and Proposition 2.9 there exists a computable sequence (z_i) in (X, d, α) and a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $z_i \in \overline{B(A,1)}$ for each $i \in \mathbb{N}$ and

$$\overline{B(A,1)} \subseteq B(z_0, 2^{-k}) \cup \cdots \cup B(z_{f(k)}, 2^{-k})$$

for each $k \in \mathbb{N}$. We have $z_i = (x_i, y_i)$, where (x_i) and (y_i) are computable sequences in $\mathbb{R}$. Since $y_j \in \{2^{-i} \mid i \in \mathbb{N}\} \cup \{0\}$ for each $j \in \mathbb{N}$, for $j, i \in \mathbb{N}$ we have

$$\Big(y_j = 2^{-i} \Leftrightarrow |y_j - 2^{-i}| < 2^{-(i+1)}\Big) \text{ and } \Big(y_j \neq 2^{-i} \Leftrightarrow |y_j - 2^{-i}| > 2^{-(i+1)}\Big).$$

This implies that the set $S = \{(j,i) \in \mathbb{N}^2 \mid y_j = 2^{-i}\}$ is c.e. and that its complement is also c.e. Therefore S is computable.

Let $i \in \mathbb{N}$, $i \geq 1$. If $\lambda_i > 0$, then $B_i \in B(A,1)$ and therefore $B_i \in B(z_0, 2^{-(i+1)}) \cup \cdots \cup B(z_{f(i+1)}, 2^{-(i+1)})$, which implies $2^{-i} = y_j$ for some $j \in \{0, \ldots, f(i+1)\}$. On the other hand, if $2^{-i} = y_j$ for some $j \in \mathbb{N}$, then $z_j = (1 - \lambda_i, 2^{-i})$ and the fact that $z_j \neq (1,0)$ gives, together with $z_j \in \overline{B(A,1)}$, that $z_j \in B(A,1)$ which implies $\lambda_i > 0$. So we have the following conclusion:

$$\lambda_i > 0 \;\Leftrightarrow\; \exists j \in \{0, \ldots, f(i+1)\} \text{ such that } (j,i) \in S.$$

It follows that $\{i \in \mathbb{N} \mid \lambda_i > 0\}$ is a computable set which is impossible.

Theorem 4.3. *Let (X, d, α) be a computable metric space which has the effective covering property and such that (X, d) is locally compact. Then (X, d, α) is locally computable.*

Proof. Suppose $a \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$, are such that the closed ball $\widehat{B}(\alpha_a, 2r)$ is compact. Let S be the set of all $(j, n) \in \mathbb{N}^2$ such that

(1) $\widehat{B}(\alpha_a, r + r \cdot 2^{-n}) \subseteq J_j$;
(2) $q_{\langle (j)_k \rangle_1} < 2^{-n}$ for each $k \in \{0, \ldots, \overline{j}\}$;
(3) $d(\alpha_a, \alpha_{\langle (j)_k \rangle_0}) < r + 2r \cdot 2^{-n}$.

As the intersection of c.e. sets, S is c.e. (the set determined by (1) is c.e. since (X, d, α) has the effective covering property and that the sets determined by (2) and (3) are c.e. we conclude similarly as in Example 2.8). For each $n \in \mathbb{N}$ the set $\widehat{B}(\alpha_a, r + r \cdot 2^{-n})$ is compact and therefore for each $n \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $(j, n) \in S$. Proposition 2.2(iv) now implies that there exists a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $(\varphi(n), n) \in S$ for each $n \in \mathbb{N}$. We define the functions $G : \mathbb{N}^2 \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ by

$$G(n,k) = \langle (\varphi(n))_k \rangle_0, \;\; g(n) = \overline{\varphi(n)},$$

$n, k \in \mathbb{N}$. It follows that for all $n \in \mathbb{N}$ and $k \in \{0, \dots, g(n)\}$ we have

$$\alpha_{G(n,k)} \in B(\alpha_a, r + 2r \cdot 2^{-n}). \tag{4.2}$$

If $n \in \mathbb{N}$, then, for $j = \varphi(n)$, we have

$$\widehat{B}(\alpha_a, r + r \cdot 2^{-n}) \subseteq I_{(j)_0} \cup \dots \cup I_{(j)_{\overline{j}}} = B(\alpha_{\langle (j)_0 \rangle_0}, q_{\langle (j)_0 \rangle_1}) \cup \dots$$
$$\dots \cup B(\alpha_{\langle (j)_{\overline{j}} \rangle_0}, q_{\langle (j)_{\overline{j}} \rangle_1}) \subseteq B(\alpha_{G(n,0)}, 2^{-n}) \cup \dots \cup B(\alpha_{G(n,g(n))}, 2^{-n}),$$

hence

$$\widehat{B}(\alpha_a, r + r \cdot 2^{-n}) \subseteq B(\alpha_{G(n,0)}, 2^{-n}) \cup \dots \cup B(\alpha_{G(n,g(n))}, 2^{-n}). \tag{4.3}$$

For $n \in \mathbb{N}$ let

$$A_n = \{\alpha_{G(n,k)} \mid k \in \{0, \dots, g(n)\}\}.$$

Let $\mathcal{A} = \bigcup_{n \in \mathbb{N}} A_n$ and for $n \in \mathbb{N}$ let $A_{\leq n} = \bigcup_{n' \leq n} A_{n'}$, $A_{>n} = \bigcup_{n' > n} A_{n'}$.

Computable functions $h : \mathbb{N}^2 \to \mathbb{N}$ and $\psi : \mathbb{N} \to \mathbb{N}$ such that $h(\mathbb{N}) = \{(n, k) \in \mathbb{N}^2 \mid k \leq g(n)\}$ and

$$\{(n', k) \in \mathbb{N}^2 \mid n' \leq n, k \leq g(n')\} \subseteq h(\{0, \dots, \psi(n)\})$$

are easily seen to exist. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(m) = G(h(m))$, $m \in \mathbb{N}$. We have

$$A_{\leq n} \subseteq \{\alpha_{f(m)} \mid m \leq \psi(n)\}, \tag{4.4}$$

for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Now (4.3) implies that $\widehat{B}(\alpha_a, r + r \cdot 2^{-n}) \subseteq \bigcup_{0 \leq m \leq \psi(n)} B(\alpha_{f(m)}, 2^{-n})$. On the other hand, by (4.2), we have $A_{>n} \subseteq \widehat{B}(\alpha_a, r + r \cdot 2^{-n})$. Therefore,

$$A_{>n} \subseteq \bigcup_{0 \leq m \leq \psi(n)} B(\alpha_{f(m)}, 2^{-n}).$$

This, together with (4.4), implies

$$\mathcal{A} \subseteq \bigcup_{0 \leq m \leq \psi(n)} B(\alpha_{f(m)}, 2^{-n}).$$

If we take $n + 1$ instead of n, we get

$$\mathcal{A} \subseteq \bigcup_{0 \leq m \leq \psi(n+1)} B(\alpha_{f(m)}, 2^{-(n+1)}) \text{ and } \overline{\mathcal{A}} \subseteq \bigcup_{0 \leq m \leq \psi(n+1)} \overline{B(\alpha_{f(m)}, 2^{-(n+1)})},$$

which gives

$$\overline{\mathcal{A}} \subseteq \bigcup_{0 \leq m \leq \psi(n+1)} B(\alpha_{f(m)}, 2^{-n}). \tag{4.5}$$

The sequence $(\alpha_{f(m)})_{m \in \mathbb{N}}$ is clearly computable in (X, d, α) and it is dense in $\overline{\mathcal{A}}$ since $\mathcal{A} = \{\alpha_{f(m)} \mid m \in \mathbb{N}\}$. So, if $d' : \overline{\mathcal{A}} \times \overline{\mathcal{A}} \to \mathbb{R}$ is the restriction of the metric d, then $(\overline{\mathcal{A}}, d', (\alpha_{f(m)})_{m \in \mathbb{N}})$ is a computable metric space which, by (4.5) and Proposition 2.9, has the effective covering property. Therefore,

$\overline{\mathcal{A}}$ is computable compact by Proposition 3.3 (note that compactness of $\overline{\mathcal{A}}$ follows from $\overline{\mathcal{A}} \subseteq \widehat{B}(\alpha_a, 2r)$). Finally, it follows from (4.3) that for each $\varepsilon > 0$ and each $x \in \widehat{B}(\alpha_a, r)$ there exists $y \in \mathcal{A}$ such that $d(x, y) < \varepsilon$. Therefore $\widehat{B}(\alpha_a, r) \subseteq \overline{\mathcal{A}}$.

We have proved the following: if $a \in \mathbb{N}$ and $r \in \mathbb{Q}$, $r > 0$, are such that $\widehat{B}(\alpha_a, 2r)$ is compact, then $\widehat{B}(\alpha_a, r)$ is contained in some computable compact set. From this and the fact that (X, d) is locally compact we easily conclude that each $x \in X$ is contained in the interior of some computable compact set. As we have already noticed, this implies that (X, d, α) is locally computable.

∎

The following example shows that the converse of the previous theorem does not hold in general.

Example 4.4. Let $([0, b], d', \beta)$ be the computable metric space constructed in Example 3.2. With the Euclidean metric d on $[0, b\rangle$, the triple $([0, b\rangle, d, \beta)$ is also a computable metric space. This space is locally computable since each compact subset A of $[0, b\rangle$ is contained in some $[0, b']$, where $b' \in \mathbb{Q}$, $b' < b$, and this segment is computable compact in $[0, b\rangle$; namely, it is easy to construct a computable sequence (x_i) of rational numbers in $[0, b']$ and a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $[0, b'] \subseteq B(x_0, 2^{-k}) \cup \cdots \cup B(x_{g(k)}, 2^{-k})$ for each $k \in \mathbb{N}$ (here $B(x, r)$ denotes the open ball in $\mathbb{R}$), which, by Proposition 2.9 and Proposition 3.3, implies that $[0, b']$ is computable compact.

Suppose that $([0, b\rangle, d, \beta)$ has the effective covering property. Then, in the same way as in Example 2.8, we get a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $[0, b\rangle \subseteq B(\beta_0, 2^{-k}) \cup \cdots \cup B(\beta_{f(k)}, 2^{-k})$ for each $k \in \mathbb{N}$. It follows $[0, b] \subseteq B(\beta_0, 2 \cdot 2^{-k}) \cup \cdots \cup B(\beta_{f(k)}, 2 \cdot 2^{-k})$ for each $k \in \mathbb{N}$ and therefore $[0, b] \subseteq B(\beta_0, 2^{-k}) \cup \cdots \cup B(\beta_{f(k+1)}, 2^{-k})$ for each $k \in \mathbb{N}$. Now Proposition 2.9 implies that $([0, b], d', \beta)$ has the effective covering property, which is impossible (Example 3.2). Hence $([0, b\rangle, d, \beta)$ does not have the effective covering property.

5. Local Non-computability and Computability of Co-c.e. Sets

We have seen in Section 3 that the statements of Theorem 1.1 hold under the assumption that (X, d, α) is locally computable. By Theorem 4.3 this result is indeed a generalization of Theorem 1.1. Now we will show that the statements of Theorem 1.1 does not have to be true if (X, d, α) is not locally computable. In fact, we will show that neither of the two assumptions of Theorem 1.1 can be omitted.

Example 5.1. Let f, b and β be as in Example 3.2. We may assume $\frac{1}{2} < b$. Let $X = [-b, b] \times [-b, b]$, let d be the metric on X defined by (4.1) and let $\alpha = (\alpha_i)$ be the sequence in X defined by $\alpha_i = ((-1)^{(i)_2}\beta_{(i)_0}, (-1)^{(i)_3}\beta_{(i)_1})$, $i \in \mathbb{N}$. Then $\alpha(\mathbb{N}) = \mathbb{Q}^2 \cap X$ and (X, d, α) is a computable metric space in which computable points are exactly those computable points in $\mathbb{R}^2$ which belong to X.

We have

$$X \setminus (\{b\} \times [-b, b]) = \bigcup_{i \in \mathbb{N}} B\left((-\frac{1}{2}, 0), f(i) + \frac{1}{2}\right)$$

and it is easy to conclude from this that $\{b\} \times [-b, b]$ is co-c.e. in (X, d, α). So we have a co-c.e. arc which does not contain any computable point.

We get similarly that the sets $\{-b\} \times [-b, b]$, $[-b, b] \times \{-b\}$, $[-b, b] \times \{b\}$ are co-c.e. and so

$$S = ([-b, b] \times \{-b, b\}) \cup (\{-b, b\} \times [-b, b])$$

is a co-c.e. topological circle which does not contain any computable point and which, in particular, is not computable.

Furthermore, it is not hard to check that

$$T = (\{0\} \times [\frac{1}{2}, b]) \cup (\{b\} \times [-b, b]) \cup ([0, b] \times \{-b, b\}) \cup (\{0\} \times [-b, -\frac{1}{2}])$$

is a co-c.e. arc (as the union of co-c.e. arcs) which serves as an example that the statement c) of Theorem 1.1 does not hold for (X, d, α).

Note that the computable metric space constructed in the previous example has compact closed balls, but it does not have the effective covering property (which follows from Theorem 1.1). The rest of this section is devoted to showing that the statements (a), (b) and (c) of Theorem 1.1 do not have to be true if (X, d, α) is a computable metric space which has the effective covering property, but which is not locally compact.

A function $f : [0, 1] \to \mathbb{R}$ is said to be *sequentially computable* if $(f(x_j))$ is a computable sequence for each computable sequence (x_j) in $[0, 1]$. We say that f is *effectively uniformly continuous* if there exists a computable function $\delta : \mathbb{N} \to \mathbb{N}$ such that for each $k \in \mathbb{N}$ and $x, y \in [0, 1]$ the inequality $|x - y| < 2^{-\delta(k)}$ implies $|f(x) - f(y)| < 2^{-k}$. We say that f is *computable* if f is sequentially computable and effectively uniformly continuous ([8]).

A sequence of functions (f_i), $f_i : [0, 1] \to \mathbb{R}$, is said to be *computable* ([8]) if the following two conditions are satisfied:

(1) the function $\mathbb{N}^2 \to \mathbb{R}$, $(i, j) \mapsto f_i(x_j)$, is computable for each computable sequence (x_j) in $[0, 1]$;

(2) there exists a computable function $\delta : \mathbb{N}^2 \to \mathbb{N}$ such that for each $i, k \in \mathbb{N}$ and $x, y \in [0, 1]$ the inequality $|x - y| < 2^{-\delta(i,k)}$ implies $|f_i(x) - f_i(y)| < 2^{-k}$.

Let $C[0, 1]$ denote the set of all continuous functions $[0, 1] \to \mathbb{R}$ and let d_∞ be the metric of uniform convergence on $C[0, 1]$, defined by $d_\infty(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}$.

The proof of the following proposition is straightforward (see [8]).

PROPOSITION 5.2. *(i) If (f_i) is a sequence of functions $[0,1] \to \mathbb{R}$ which satisfies condition (2) in the definition of computable sequence of functions and such that the function $\mathbb{N}^2 \to \mathbb{N}$, $(i,j) \mapsto f_i(x_j)$, is computable for some computable sequence (x_j) in $[0,1]$ which is dense in $[0,1]$, then (f_i) is a computable sequence.*

(ii) If (f_i) and (g_j) are computable sequences of functions $[0,1] \to \mathbb{R}$, then the function $\mathbb{N}^2 \to \mathbb{R}$, $(i,j) \mapsto d_\infty(f_i, g_j)$, is computable.

Recall that each finite sequence $(a_0, \dots, a_n)$ in $\mathbb{N}$ is of the form $((i)_0, \dots, (i)_{\overline{i}})$ for some $i \in \mathbb{N}$. For $i \in \mathbb{N}$ let $\widehat{i} = \max\{1, \overline{i}\}$ and let $f_i : [0,1] \to \mathbb{R}$ be the function defined by

1) $f_i\left(\frac{k}{\widehat{i}}\right) = q_{(i)_k}$, $k \in \{0, \dots \widehat{i}\}$;

2) f_i is linear on $\left[\frac{k}{\widehat{i}}, \frac{k+1}{\widehat{i}}\right]$, $k \in \{0, \dots \widehat{i} - 1\}$.

It is straightforward to check that the sequence (f_i) satisfies condition (2) from the definition of a computable sequence and that the function $\mathbb{N}^2 \to \mathbb{Q}$, $(i,j) \mapsto f_i(r_j)$ is computable for each computable sequence (r_j) of rational numbers in $[0,1]$. Therefore, by Proposition 5.2(i), the sequence (f_i) is computable.

Clearly, the set $\{f_i \mid i \in \mathbb{N}\}$ is dense in $(C[0,1], d_\infty)$ and therefore $(C[0,1], d_\infty, (f_i))$ is a computable metric space (Proposition 5.2(ii)). We have that g is a computable point in this computable metric space if and only if g is a computable function $[0,1] \to \mathbb{R}$.

PROPOSITION 5.3. *Suppose $g, h \in C[0,1]$ are sequentially computable functions such that $h(x) \neq 0$ for each $x \in [0,1]$. Let*

$$S = \{g + th \mid t \in [0,1]\}.$$

Then S is c.e. in $(C[0,1], d_\infty, (f_i))$.

PROOF. Let (r_j) be some fixed computable sequence of rational numbers such that $\{r_j \mid j \in \mathbb{N}\} = \mathbb{Q} \cap [0,1]$. Let $T = \{g + th \mid t \in \mathbb{R}\}$. Then $f \in T$ if and only if

$$\frac{f(r_j) - g(r_j)}{h(r_j)} = \frac{f(0) - g(0)}{h(0)}$$

for each $j \in \mathbb{N}$ and this is equivalent to

$$f(r_j)h(0) - f(0)h(r_j) = g(r_j)h(0) - g(0)h(r_j)$$

for each $j \in \mathbb{N}$. Let $M \in \mathbb{N}$ be such that $|h(x)| < M$ for each $x \in [0,1]$. Let A be the set of all $(i,j) \in \mathbb{N}^2$ such that

$$|(f_{\langle i \rangle_0}(r_j)h(0) - f_{\langle i \rangle_0}(0)h(r_j)) - (g(r_j)h(0) - g(0)h(r_j))| > 2Mq_{\langle i \rangle_1}.$$

Let $(i,j) \in A$ and suppose that there exists $f' \in I_i \cap T$. Then

$$f'(r_j)h(0) - f'(0)h(r_j) = g(r_j)h(0) - g(0)h(r_j)$$

and

$$|(f_{\langle i\rangle_0}(r_j)h(0) - f_{\langle i\rangle_0}(0)h(r_j)) - (f'(r_j)h(0) - f'(0)h(r_j))| =$$

$$= |(f_{\langle i\rangle_0}(r_j) - f'(r_j))h(0) - (f_{\langle i\rangle_0}(0) - f'(0))h(r_j)| \leq 2d_\infty(f_{\langle i\rangle_0}, f')M.$$

Hence

$$|(f_{\langle i\rangle_0}(r_j)h(0) - f_{\langle i\rangle_0}(0)h(r_j)) - (g(r_j)h(0) - g(0)h(r_j))| < 2q_{\langle i\rangle_1}M,$$

which is impossible. Therefore, $I_i \cap T = \emptyset$ whenever $i \in \mathbb{N}$ is such that $(i,j) \in A$ for some $j \in \mathbb{N}$. Let

$$B = \{i \in \mathbb{N} \mid \exists j \in \mathbb{N} \text{ such that } (i,j) \in A\}.$$

Since A is c.e. (Proposition 2.2(iv)), B is c.e. We have $\bigcup_{i\in B} I_i \subseteq C[0,1] \setminus T$. On the other hand, if $f \in C[0,1] \setminus T$, then there exists $j \in \mathbb{N}$ and $\lambda > 0$ such that

$$|(f(r_j)h(0) - f(0)h(r_j)) - (g(r_j)h(0) - g(0)h(r_j))| > 4\lambda M.$$

If we now take $i \in \mathbb{N}$ such that $d_\infty(f, f_{\langle i\rangle_0}) < q_{\langle i\rangle_1} < \lambda$, then we easily get $(i,j) \in A$. Hence $i \in B$, $f \in I_i$ and we conclude that

$$\bigcup_{i\in B} I_i = C[0,1] \setminus T.$$

So T is c.e.

Let a and b, $a < b$, be computable numbers such that $[a,b] = \{g(0) + th(0) \mid t \in [0,1]\}$. Similarly as above, we easily get that the set $\{f \in C[0,1] \mid f(0) \in [a,b]\}$ is co-c.e. Now the fact that S is co-c.e. follows from

$$S = T \cap \{f \in C[0,1] \mid f(0) \in [a,b]\}.$$

□

To see that the computable metric space $(C[0,1], d_\infty, (f_i))$ can serve as a counter-example to Theorem 1.1, consider a sequentially computable continuous function $g : [0,1] \to \mathbb{R}$ which is not computable; see [8].

Let $h : [0,1] \to \mathbb{R}$, $h(x) = 3$, $x \in [0,1]$. The set $S = \{g + th \mid t \in [0,1]\}$ is c.e. by Proposition 5.3. It is clear that S is an arc. Suppose that $f \in S$ is a computable function. Then $g = f - th$, where $t \in [0,1]$. Using the fact that f and h are effectively uniformly continuous functions, we conclude easily that g also must be effectively uniformly continuous (see [8]), which is impossible since g is not computable. Hence S is a co-c.e. arc in $(C[0,1], d_\infty, (f_i))$ which does not contain any computable point and which, in particular, does not contain any computable nonempty set.

Let $h_1, h_2 : [0,1] \to \mathbb{R}$, $h_1(x) = 1 + x$, $h_2(x) = 2 - x$, $x \in [0,1]$. We conclude in the same way that $S_1 = \{g + th_1 \mid t \in [0,1]\}$, $S_2 = \{(g + h_1) + th_2 \mid t \in [0,1]\}$ are co-c.e. arcs which do not contain any computable point. Note that $S \cap S_1 = \{g\}$, $S \cap S_2 = \{g + h\}$, $S_1 \cap S_2 = \{g + h_1\}$. Therefore $S \cup S_1 \cup S_2$ is a co-c.e. topological circle and it does not contain any computable point.

Finally, let $T_1 = \{0 + tg \mid t \in [0,1]\}$, $T_2 = \{1 + t(g-1) \mid t \in [0,1]\}$ (here 0 denotes the constant function $x \mapsto 0$ and 1 the constant function $x \mapsto 1$). Then T_1 and T_2 are co-c.e. (we may assume here that $g(x) > 1$ for each $x \in [0,1]$, namely continuity of g implies that there exists $M \in \mathbb{N}$ such that $g(x) + M > 1$ for each $x \in [0,1]$ and the function $g + M$ is also sequentially computable, but not computable); the only computable point in T_1 is the constant function 0 and the only computable point in T_2 is the constant function 1. The union $T_1 \cup T_2$ is a co-c.e. arc whose endpoints are only computable points in it. Consequently, $T_1 \cup T_2$ is not computable.

We conclude the paper with a somewhat technical verification that $(C[0,1], d_\infty, (f_i))$ has the effective covering property.

LEMMA 5.4. *Let $f \in C[0,1]$, let $r > 0$, $n \in \mathbb{N}$, $n \geq 1$, and let $x_1, \dots, x_n$, $y_1, \dots, y_n$ be such that $0 \leq x_1 < \cdots < x_n \leq 1$ and $y_i \in [f(x_i) - r, f(x_i) + r]$ for each $i \in \{1, \dots, n\}$. Then there exists $h \in \widehat{B}(f, r)$ such that $h(x_i) = y_i$ for each $i \in \{1, \dots, n\}$.*

PROOF. Certainly there exists $g \in C[0,1]$ such that $g(x_i) = y_i$ for each $i \in \{1, \dots, n\}$. Now the function $h : [0,1] \to \mathbb{R}$ defined by $h(x) = \max\{f(x) - r, \min\{f(x) + r, g(x)\}\}$ has the desired property. ∎

If $f \in C[0,1]$ and $m \in \mathbb{N}$, $m \geq 1$, are such that f is linear on $\left[\frac{i}{m}, \frac{i+1}{m}\right]$ for each $i \in \{0, \dots, m-1\}$, then we will say that f is $m-$linear. A map $f \in C[0,1]$ is said to be piece-wise linear if f is $m-$linear for some $m \in \mathbb{N}$.

Suppose that $f, g_1, \dots, g_n \in C[0,1]$ are piece-wise linear and let $r, s_1, \dots, s_k \in \langle 0, \infty \rangle$. Then $f, g_1, \dots, g_n$ are $m-$linear for some $m \in \mathbb{N}$. For $i \in \{0, \dots, m\}$ let $x_i = \frac{i}{m}$.

In order to determine whether

$$\widehat{B}(f, r) \subseteq \bigcup_{k=1}^{n} B(g_k, s_k) \tag{5.1}$$

holds, it is enough to observe only those $k \in \{1, \dots, n\}$ for which (5.2), (5.3) and (5.4) hold:

$$g_k(x_i) - s \leq f(x_i) - r \text{ and } f(x_i) + r \leq g_k(x_i) + s \tag{5.2}$$

for each $i \in \{0, \dots, m\}$,

$$g_k(x_i) - s < f(x_i) - r \text{ or } g_k(x_{i+1}) - s < f(x_{i+1}) - r \tag{5.3}$$

and

$$f(x_i) + r < g_k(x_i) + s \text{ or } f(x_{i+1}) + r < g_k(x_{i+1}) + s \tag{5.4}$$

for each $i \in \{0, \dots, m-1\}$. Namely, if $k \in \{1, \dots, n\}$ is such that (5.2), (5.3) or (5.4) does not hold, then there exist infinitely many $x \in [0,1]$ such that

$$f(x) - r \leq g_k(x) - s \text{ or } f(x) + r \geq g_k(x) + s.$$

This implies $\widehat{B}(f,r) \subseteq \bigcup_{i \neq k} B(g_i, s_i)$, otherwise it would be possible (using Lemma 5.4) to construct $h \in \widehat{B}(f,r)$ such that $h \notin \bigcup_{i \neq k} B(g_i, s_i)$ and $h \notin B(g_k, s_k)$.

So let us suppose that for each $k \in \{1, \ldots, n\}$ (5.2), (5.3) and (5.4) hold. We claim that then (5.1) is equivalent to the following: for all $a_0, \ldots, a_m \in \{0,1\}$, there exists $k \in \{1, \ldots, n\}$ such that

$$f(x_i) + (-1)^{a_i} \cdot r \in \langle g_k(x_i) - s_k, g_k(x_i) + s_k \rangle \tag{5.5}$$

for each $i \in \{0, \ldots, m\}$. Indeed, if (5.1) holds and if $a_0, \ldots, a_m \in \{0,1\}$, then by Lemma 5.4 there exists $h \in \widehat{B}(f,r)$ such that $h(x_i) = f(x_i) + (-1)^{a_i} \cdot r$ for each $i \in \{0, \ldots, m\}$. We have $h \in B(g_k, s_k)$ for some $k \in \{1, \ldots, n\}$ and (5.5) follows.

Conversely, suppose that for all $a_0, \ldots, a_m \in \{0,1\}$ there exists k such that (5.5) holds. Let $h \in \widehat{B}(f,r)$. Let $a_0, \ldots, a_m \in \{0,1\}$ be such that $h(x_i) = f(x_i) + (-1)^{a_i} \cdot r$ for those $i \in \{0, \ldots, m\}$ for which $h(x_i) \in \{f(x_i) - r, f(x_i) + r\}$. Let k be such that (5.5) holds. It follows from (5.2) that $h \in B(g_k, s_k)$.

For functions $f, g_1, \ldots, g_n$ of the form f_i, $i \in \mathbb{N}$, and for rational numbers $r, s_1, \ldots, s_n$ we can effectively check whether (5.2), (5.3) , (5.4) and (5.5) hold. We conclude that the set $\{(i,j) \in \mathbb{N}^2 \mid \widehat{I}_i \subseteq J_j\}$ is computable. Hence the computable metric space $(C[0,1], d_\infty, (f_i))$ has the effective covering property.

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