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ON A FAMILY OF TWO-PARAMETRIC $D(4)$-TRIPLES

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ABSTRACT. Let k be a positive integer. In this paper, we study a parametric family of the sets of integers

$$\{k, A^2k + 4A, (A + 1)^2k + 4(A + 1), d\}.$$

We prove that if d is a positive integer such that the product of any two distinct elements of that set increased by 4 is a perfect square, then

$$\begin{aligned} d = & (A^4 + 2A^3 + A^2)k^3 + (8A^3 + 12A^2 + 4A)k^2 \\ & +(20A^2 + 20A + 4)k + (16A + 8) \end{aligned}$$

for $1 \leq A \leq 22$ and $A \geq 51767$.

1. INTRODUCTION

A set of m distinct positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-m-tuple if $a_ia_j + n$ is a perfect square for $1 \leq i < j \leq m$. Diophantus was the first who has studied such sets and he found the set of four positive rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with the property that the product of any two of them increased by 1 is a square of a rational number. However, the first $D(1)$-quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Later Baker and Davenport ([1]) proved that the set $\{1, 3, 8, 120\}$ cannot be extended to a $D(1)$-quintuple. There are several results of the generalization of that result. In 1997, Dujella ([4]) proved that the $D(1)$-triple of the form $\{k-1, k+1, 4k\}$, for an integer $k \geq 2$, cannot be extended to a quintuple. In 1998, Dujella and Pethő ([6]) proved that the $D(1)$-pair$\{1, 3\}$ cannot be extended to a quintuple. In 2008, Fujita ([13]) obtained more general result by proving that the $D(1)$-pair $\{k - 1, k + 1\}$, for an integer $k \geq 2$ cannot be extended to a quintuple. A folklore conjecture is that there does not exist a $D(1)$-quintuple. Recently,

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Dujella ([5]) proved that there is no $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples and Fujita ([15]) proved that there are at most 10^{276} Diophantine quintuples.

The cases $n = 1$ and $n = 4$ are closely connected. Namely, if we have a $D(4)$-m-tuple with all elements even, dividing those elements by 2 we get a $D(1)$-m-tuple. In 2005, Dujella and Ramasamy ([7]), conjectured that there does not exist a $D(4)$-quintuple. Actually they gave a stronger version of this conjecture.

Conjecture 1.1. *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then*

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2, ac + 4 = s^2, bc + 4 = t^2.$$

Such $D(4)$-quadruples are called regular, and otherwise it is called irregular $D(4)$-quadruples.

The first result of nonextendibility of $D(4)$-m-tuples was proven by Mohanty and Ramasamy (see [21]). They proved that $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Later Kedlaya ([19]) proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$. One generalization of this result was given by Dujella and Ramasamy in [7], where they proved Conjecture for a parametric family of $D(4)$-quadruples. Precisely, they proved that if k and d are positive integers and $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where F_k and L_k are the Fibonacci and Lucas numbers. A second generalization was given by Fujita in [14]. He proved that if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = k^3 - 4k$. All these results support the above Conjecture. The first author studied the size of a $D(4)$-m-tuple. He proved that there does not exist a $D(4)$-sextuple and that there are only finitely $D(4)$-quintuples (see [8–11]).

In [16] and [17], the second and the third author obtained the results on the extension of two-parametric $D(1)$-triple

$$\{k, A^2k + 2A, (A + 1)^2k + 2(A + 1)\}. \tag{1.1}$$

Precisely, they proved that it can be extended to a quadruple on the unique way for $1 \leq A \leq 10$ and $A \geq 52330$. In this paper, for the first time we consider the extension of a two-parametric $D(4)$-triple, i.e.,

$$\{k, A^2k + 4A, (A + 1)^2k + 4(A + 1)\}$$

and prove the following results.

THEOREM 1.2. *Let k be a positive integer. If d is a positive integer such that the product of any two distinct elements of the set*

$$\{k, A^2k + 4A, (A+1)^2k + 4(A+1), d\} \tag{1.2}$$

increased by 4 is a perfect square, then

$$\begin{aligned} d = {} & (A^4 + 2A^3 + A^2)k^3 + (8A^3 + 12A^2 + 4A)k^2 \\ & + (20A^2 + 20A + 4)k + (16A + 8) \end{aligned} \tag{1.3}$$

for $1 \leq A \leq 22$.

THEOREM 1.3. *Let k be a positive integer. If d is a positive integer such that the product of any two distinct elements of the set*

$$\{k, A^2k + 4A, (A+1)^2k + 4(A+1), d\} \tag{1.4}$$

increased by 4 is a perfect square, then d must be as in (1.3) for $A \geq 51767$.

Our family of $D(4)$-triples is closely connected to the $D(1)$-triples the second and the third authors considered. Namely, for k even dividing the elements of our $D(4)$-triple by 2 we get the same $D(1)$-triple as in (1.1). Hence, Theorems 1.2 and 1.3 give the following immediate improvements of the main results of [16] and [17].

COROLLARY 1.4. *Let k be a positive integer and $1 \leq A \leq 22$ or $A \geq 51767$. Then the $D(1)$-triple (1.1) extends on the unique way to a $D(1)$-quadruple.*

The methods we use here are mostly the same as in [16] and [17]. The main difference is in Section 3, where using a modified result of Rickert enables us to prove Theorem 1.2 for more values of A.

The organization of this paper is as follows. In Section 2, we recall some useful results obtained by the first author and we adapt them to our case. Then we use the extension of the result due to Rickert ([22]) on simultaneous approximations of algebraic numbers which are close to 1 to get an upper bound for k. Finally, in Section 4, we use linear forms in logarithms and the Baker-Davenport reduction method to prove Theorem 1.2. Let us mention that the case $A = 1$ was studied by Fujita ([14]). In fact, if we take $A = 1$ and $k = k_1 - 2$, then one obtains the set $\{k_1 - 2, k_1 + 2, 4k_1\}$. So in this paper, we first consider only $2 \leq A \leq 22$ because of the length of the computations. For a larger parameter A it can take a lot of time to verify Theorem 1.2 as it is done in Section 4. After that we consider higher values of the parameter A. In fact, in Section 5, we use another gap principle to get an upper bound for a linear form in logarithms that we reduce into a linear form in two logarithms to prove Theorem 1.3. It is good to specify that one of the most important ingredients of this paper is the use of a linear form in two logarithms. It helps to considerably reduce the size of the bound of A. This method was used for the first time by the second and the third author in [18] and later in [17]. Let us also mention that it would take many years to fill the gap $23 \le A \le 51767$ using the methods we used here.

2. Preliminaries

Let $\{a, b, c\}$ be a $D(4)$-triple and let r, s, t be positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2. \tag{2.1}$$

In order to extend the $D(4)$-triple $\{a, b, c\}$ to a $D(4)$-quadruple $\{a, b, c, d\}$, we have to find integers x, y and z which satisfy

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2. \tag{2.2}$$

Eliminating d, we obtain the following system of Pellian equations.

$$az^2 - cx^2 = 4(a - c), \tag{2.3}$$

$$bz^2 - cy^2 = 4(b - c). \tag{2.4}$$

By [11, Lemma 1], there exists a solution (z_0, x_0) of (2.3) such that $z = v_m$, where

$$v_0 = z_0, \quad v_1 = \frac{1}{2}(sz_0 + cx_0), \quad v_{m+2} = sv_{m+1} - v_m,$$

and $|z_0| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$. Similarly, there exists a solution (z_1, y_1) of (2.4) such that $z = w_n$, where

$$w_0 = z_1, \quad w_1 = \frac{1}{2}(tz_1 + cy_1), \quad w_{n+2} = tw_{n+1} - w_n,$$

and $|z_1| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}}$.

The initial terms z_0 and z_1 are almost completely determined in the following lemma ([9, Lemma 9]).

Lemma 2.1. (i) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Moreover, $|z_0| = 2$, or $|z_0| = \frac{1}{2}(cr - st)$, or $|z_0| < 1.608a^{-\frac{5}{14}}c^{\frac{9}{14}}$.*

(ii) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iii) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.*

(iv) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$, $z_0z_1 > 0$.*

In the present paper, we have

$$a = k, \quad b = A^2k + 4A, \quad c = (A + 1)^2k + 4(A + 1),$$

and using (2.1) we get

$$r = Ak + 2, \quad s = (A + 1)k + 2, \quad t = (A^2 + A)k + (4A + 2).$$

If the second or the fourth item of Lemma 2.1 holds, then

$$bc < bc + 4 = t^2 = |z_0|^2 < \frac{c\sqrt{c}}{\sqrt{a}},$$

and so $b < \sqrt{\frac{c}{a}}$. This implies $b < \sqrt{\frac{c}{a}} < A+3$ and it is impossible. Similarly, if the third item of Lemma 2.1 holds, we obtain

$$ac < ac + 4 = s^2 = |z_1|^2 < \frac{c\sqrt{c}}{\sqrt{b}}.$$

This yields to $a < \frac{\sqrt{c}}{\sqrt{b}} < 2$. So we only have to check what is happening for $k = 1$. But in this case, the first author proved ([9, Lemma 8, 9]) that if we define $d_0 = (z'^2 - 4)/c$, where $|z'| = (cr - st)/2$, then we must have $d_0 > 0$. In this case we would get $d_0 = 0$, a contradiction. Therefore we only consider the first item of Lemma 2.1 with $x_0 = 2$, $y_0 = 2$, and $|z_0| = (cr - st)/2 = 2$, because it is easy to see that we cannot have $|z_0| > 2$. In fact, let us define $d_0 = (z_0^2 - 4)/c$ in the third possibility of the first item. If $|z_0| > 2$, then $d_0 < z_0^2/c < 1.608^2 a^{-5/7} c^{9/7}/c < c$. Thus according to the proof of the above lemma in [9], $\{a, b, c, d_0\}$ is an irregular $D(4)$-quadruple. Also by [8, Proposition 1], if $\{a, b, c, d\}$ is an irregular $D(4)$-quadruple with $a < b < c < d$, then $d > 0.173c^{6.5}a^{5.5}$ or $d > 0.087c^{3.5}a^{2.5}$. Therefore we have $c > 0.173b^{6.5}a^{5.5}$ or $c > 0.087b^{3.5}a^{2.5}$. And when $k \geq 1$, we get a contradiction.

Therefore, we need to solve the system of Pellian equations

$$kz^2 - ((A+1)^2k + 4A + 4)x^2 = -4((A^2 + 2A)k + 4A + 4), \tag{2.5}$$

$$(A^2k + 4A)z^2 - ((A+1)^2k + 4A + 4))y^2 = -4((2A+1)k + 4), \tag{2.6}$$

with $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. This is equivalent to solve the sequence equation

$$z = v_{2m} = w_{2n}. \tag{2.7}$$

Let us specify that the sequences $\{v_m\}_{m\geq 0}$ and $\{w_n\}_{n\geq 0}$ are defined by:

$$\begin{aligned} v_0 = \pm 2, \quad v_1 &= (A+1)^2k + 4(A+1) \pm ((A+1)k + 2), \\ v_{m+2} &= ((A+1)k + 2)v_{m+1} - v_m, \\ w_0 = \pm 2, \quad w_1 &= (A+1)^2k + 4(A+1) \pm ((A^2 + A)k + 4A + 2), \\ w_{n+2} &= ((A^2 + A)k + 4A + 2)w_{n+1} - w_n. \end{aligned}$$

In order to get a gap principle between indices m, n and k, we recall the following lemma, which can be proven by induction.

LEMMA 2.2 ([5, Lemma 9, 1) and 3)]). *We have*

$$\begin{aligned} v_{2m} &\equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2}, \\ w_{2n} &\equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2}. \end{aligned}$$

For the relations between the indices m and n, we have the following.

LEMMA 2.3. *If $v_{2m} = w_{2n}$, then $n \le m \le 2n$.*

PROOF OF LEMMA 2.3. By [9, Lemma 5], if $v_m = w_n$, then $n-1 \le m \le 2n+1$. In our even case, we have $2n-1 \le 2m \le 4n+1$. The result is obtained. ∎

Using Lemma 2.2, we have

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{2.8}$$

In our case, since $c = (A+1)^2k + 4A + 4 = (A+1)((A+1)k+4)$, this congruence implies

$$\begin{aligned} &\pm km^2 + ((A+1)k+2)m \equiv \pm(A^2k+4A)n^2 + \\ &\qquad ((A^2+A)k + 4A + 2)n \pmod{(A+1)k+4}. \end{aligned}$$

As $k+4 = A^2k+4A-(A-1)((A+1)k+4)$, we simplify the above expression to have

$$\pm km^2 - 2m \equiv \pm(k+4)n^2 + 2n \pmod{(A+1)k+4}.$$

Multiplying the above congruence by $\pm(A+1)$ and simplifying it again, we get

$$4m^2 + 4An^2 \pm 2(A+1)m \pm 2(A+1)n \equiv 0 \pmod{(A+1)k+4}. \tag{2.9}$$

By Lemma 2.3, the left side of (2.9) is larger than

$$4n^2 + 4An^2 - 4(A+1)n - 2(A+1)n = (A+1)(4n^2 - 6n).$$

If $n \ge 2$, then $4n^2 - 6n > 0$. Hence from (2.9) we obtain

$$4m^2 + 4An^2 + 2(A+1)m + 2(A+1)n \ge (A+1)k + 4.$$

And using Lemma 2.3 again, we have

$$4(A+1)m^2 + 4(A+1)m \ge (A+1)k + 4,$$

and so $4m^2 + 4m > k$. Thus we have proved the following.

LEMMA 2.4. *If $v_{2m} = w_{2n}$ with $m, n \ge 2$, then*

$$m \ge \frac{\sqrt{k} - 1}{2}.$$

The next result gives a lower bound of z.

Lemma 2.5. *Let x, y, z be positive integer solutions of the system of Pellian equations* (2.5) *and* (2.6) *such that*

$$z \notin \{2, (A^3 + 2A^2 + A)k^2 + (6A^2 + 8A + 2)k + (8A + 6)\}. \tag{2.10}$$

Then

$$\log(z) > \left(\sqrt{k} - 1\right) \log\left((A + 1)k\right).$$

Proof of Lemma 2.5. Using (2.5) with $z_0 = \pm 2, x_0 = 2$ and $z = v_{2m} = w_{2n}$, we have

$$z = \frac{1}{\sqrt{a}} \left((\pm\sqrt{a} + \sqrt{c}) \left(\frac{s + \sqrt{ac}}{2} \right)^{2m} + (\pm\sqrt{a} - \sqrt{c}) \left(\frac{s - \sqrt{ac}}{2} \right)^{2m} \right).$$

If $m = n = 1$ then $z_0 = 2$. Suppose that (2.10) holds, then $m \geq 2$ and $n \geq 2$. We get

$$\begin{aligned} z &> \frac{1}{\sqrt{a}} (\pm\sqrt{a} + \sqrt{c}) \left(\frac{s + \sqrt{ac}}{2} \right)^{2m} - \frac{\sqrt{a} + \sqrt{c}}{\sqrt{a} \left(\frac{s + \sqrt{ac}}{2} \right)^{2m}} \\ &\geq \left(-1 + \sqrt{\frac{c}{a}} \right) (\sqrt{ac})^{2m} - \frac{2\sqrt{c}}{\sqrt{a} (\sqrt{ac})^{2m}} \\ &> \left(-1 + \sqrt{\frac{c}{a}} \right) (\sqrt{ac})^{2m} - 1 > (\sqrt{ac})^{2m}. \end{aligned}$$

Hence, we have

$$\log(z) > 2m \log(\sqrt{ac}) > 2m \log(s - 2) = 2m \log((A + 1)k).$$

Now from Lemma 2.4, we obtain the result. □

3. Application of a modified Rickert's result

In this section, we will use a slightly modified result of Rickert ([22]) (or Bennett ([2])) on simultaneous approximations of algebraic numbers which are close to 1 to get a lower bound for k in order to solve the system of Pellian equations (2.5) and (2.6).

Let $N = \frac{1}{2}(A^2 + A)k + 2A$ and

$$\theta_1 = \sqrt{1 - \frac{2A}{\frac{A^2 + A}{2} k + 2A}} \quad \text{and} \quad \theta_2 = \sqrt{1 + \frac{2}{\frac{A^2 + A}{2} k + 2A}}.$$

Lemma 3.1. *All positive integer solutions (x, y, z) of the simultaneous Pellian equations* (2.5) *and* (2.6) *satisfy*

$$\max \left\{ \theta_1 - \frac{(A^2 + A)x}{Az}, \theta_2 - \frac{(A + 1)y}{Az} \right\} < 2(A^2 + 4A + 3) z^{-2}.$$

Proof of Lemma 3.1. Since $\theta_1 = (A+1)\sqrt{\frac{a}{c}}$ and $\theta_2 = \frac{A+1}{A}\sqrt{\frac{b}{c}}$, one can verify that

$$\begin{aligned}\left|\theta_1 - \frac{(A^2+A)x}{Az}\right| &= (A+1)\left|\sqrt{\frac{a}{c}} - \frac{x}{z}\right| = (A+1)\left|\frac{a}{c} - \frac{x^2}{z^2}\right| \cdot \left|\sqrt{\frac{a}{c}} + \frac{x}{z}\right|^{-1} \\ &< (A+1)\left|\frac{4(a-c)}{cz^2}\right| \cdot \left|2\sqrt{\frac{a}{c}}\right|^{-1} < (A+1)\sqrt{\frac{c}{a}} \cdot \frac{2}{z^2} \\ &< (A+1)(A+3) \cdot \frac{2}{z^2}\end{aligned}$$

and

$$\begin{aligned}\left|\theta_2 - \frac{(A+1)y}{Az}\right| &= \frac{A+1}{A}\left|\sqrt{\frac{b}{c}} - \frac{y}{z}\right| = \frac{A+1}{A}\left|\frac{b}{c} - \frac{y^2}{z^2}\right| \cdot \left|\sqrt{\frac{b}{c}} + \frac{y}{z}\right|^{-1} \\ &< \frac{A+1}{A}\left|\frac{4(b-c)}{cz^2}\right| \cdot \left|2\sqrt{\frac{b}{c}}\right|^{-1} < \frac{A+1}{A}\sqrt{\frac{c}{b}} \cdot \frac{2}{z^2} \\ &< \frac{(A+1)}{A} \cdot \frac{4}{z^2} < (A+1)(A+3)\frac{2}{z^2}.\end{aligned}$$

Therefore, we get the result. □

Now, we recall a result due to Fujita and Bennett. The definitions of $l, p, L, P, p_{ij\kappa}$ are same as those in [12, Lemma 22]. This is slightly different to those stated in Bennett's theorem (see [2, Theorem 3.2]).

Lemma 3.2. *Let $\theta_1, ..., \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer κ, we can find rational numbers $p_{ij\kappa}(0 \le i, j \le m)$ with nonzero determinant such that $f^{-1}D^\kappa p_{ij\kappa}(0 \le i, j \le m)$ are integers and*

$$|p_{ij\kappa}| \le pP^\kappa \ (0 \le i, j \le m), \quad \left|\sum_{j=0}^{m} p_{ij\kappa}\theta_j\right| \le lL^{-\kappa} \ (0 \le i \le m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, ..., \left|\theta_m - \frac{p_m}{q}\right|\right\} > Cq^{-1-\lambda}$$

holds for all integers $p_1, ..., p_m, q$ with $q > 0$, where

$$\lambda = \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad C^{-1} = 2mf^{-1}pDP(\max\{1, 2f^{-1}l\})^\lambda.$$

We will use the above result to prove the following theorem.

THEOREM 3.3. *Let A and N be integers with $A \geq 2$ and $N \geq 0.32A^3(A+1)^4$. Then the numbers*

$$\theta_1 = \sqrt{1 - \frac{2A}{N}} \quad \text{and} \quad \theta_2 = \sqrt{1 + \frac{2}{N}}.$$

satisfy

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > \left(4.13\frac{A^2(A+1)^2}{2A+1}N\right)^{-1} q^{-1-\lambda}, \tag{3.1}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = \frac{\log \frac{2.03A^2(A+1)^2N}{2A+1}}{\log \frac{3.24N^2}{A^2(A+1)^2}} < 1.$$

PROOF OF THEOREM 3.3. All we have to do is to find the real numbers satisfying the assumptions in Lemma 3.2. Using formulas (24) and (25) in [12], we have

$$\left|\sum_{j=0}^{2} p_{ij\kappa}\right| = |I_i(1/N)| < \frac{27}{64}\left(1 - \frac{2A}{N}\right)^{-1}\left(\frac{27}{4}\left(1 - \frac{2A}{N}\right)^2 N^3\right)^{-\kappa} \tag{3.2}$$

and

$$|p_{ij\kappa}|\theta_j \leq \max_{z\in\Gamma_j}\left|\frac{(1+z/N)^{\kappa+1/2}}{|A'(z)|^{\kappa}}\right|, \ (0 \leq j \leq 2), \tag{3.3}$$

where $A'(z) = \Pi_{i=0}^{2}(z - a_i)$ and the contours Γ_j, $(0 \leq j \leq 2)$ are defined by

$$|z - a_j| = \min_{i\neq j}\left\{\frac{|a_j - a_i|}{2}\right\}$$

with $a_0 = 0$, $a_1 = -2A$, $a_2 = 2$. The inequality (3.2) leads to

$$l = \frac{27}{64}\left(1 - \frac{2A}{N}\right)^{-1}, \quad L = \frac{27}{4}\left(1 - \frac{2A}{N}\right)^2 N^3.$$

Comparing the maximum values in (3.3), for $j = 0, 1, 2$ in each contour Γ_j, we have

$$|p_{ij\kappa}| \leq \frac{\max_{z\in\Gamma_j}|1+z/N|^{\kappa+1/2}}{\theta_j \cdot \min_{z\in\Gamma_j}|A'(z)|^{\kappa}} \leq \frac{\left(1+\frac{3}{N}\right)^{1/2}}{\left(1-\frac{2A}{N}\right)^{1/2}}\left(\frac{1}{2A+1}\left(1+\frac{3}{N}\right)\right)^{\kappa},$$

and so we get

$$p = \left(1 + \frac{2A+3}{N-2A}\right)^{1/2}, \quad P = \frac{1}{2A+1}\left(1 + \frac{3}{N}\right).$$

Now let us determine f and D. Note that

$$\prod_{0\le i<j\le 2}(a_i-a_j)=16\frac{A(A+1)}{2}.$$

From the expression (3.7) of $p_{ij}(1/N)$ in [22], we see that

$$2^{l_1}\left(\frac{A(A+1)}{2}\right)^{l_2}N^{\kappa}p_{ij}(1/N)\in\mathbb{Z}$$

for some integers l_1, l_2. We take $p_{ij\kappa}=p_{ij}(1/N)$ for different values of κ. By similar arguments to those in the proof of Lemma 4 in [22], we choose $l_1=3\kappa-1$, $l_2=2\kappa$. Hence we obtain

$$2^{-1}\left(2A^2(A+1)^2N\right)^{\kappa}p_{ij}(1/N)\in\mathbb{Z}.$$

So we take $f=2, D=2A^2(A+1)^2N$. From the assumptions in Theorem 3.3, we have $N\ge 0.32A^3(A+1)^4\ge 207.36$. Therefore we obtain

$$DP<2.03\frac{A^2(A+1)^2N}{2A+1},\quad \frac{L}{D}>\frac{3.24N^2}{A^2(A+1)^2},\quad C^{-1}<4.13\frac{A^2(A+1)^2N}{2A+1}.$$

Theorem 3.3 follows immediately from Lemma 3.2. $\square$

We will use the above result to prove the following proposition that gives us the information on d.

PROPOSITION 3.4. *If $k\ge 8909613$ for $A=2$, $k\ge 7227770$ for $A=3$, $k\ge 6524503$ for $A\ge 4$ and if the set* (1.2) *is a $D(4)$-quadruple, then d is given by* (1.3).

PROOF OF LEMMA 3.4. If d satisfies the condition, then $z^2=cd+4$. Since $d>1$, we have $z\ne 2$. And if d is not as in (1.3), we have

$$z\ne(A^3+2A^2+A)k^2+(6A^2+8A+2)k+(8A+6).$$

Then Lemma 2.5 implies

$$\log(2z)>\left(\sqrt{k}-1\right)\log\left((A+1)k\right). \tag{3.4}$$

We apply Theorem 3.3 with $p_1=A(A+1)x$, $p_2=(A+1)y$, $q=Az$ and $N=\frac{1}{2}A(A+1)k+2A$. If $k>0.64A^2(A+1)^3$, then $N=\frac{1}{2}A(A+1)k+2A>0.32A^3(A+1)^4$. Thus the condition on N stated in Theorem 3.3 is satisfied. Therefore using Theorem 3.3 and Lemma 3.1, we obtain

$$\left(4.13\frac{A^2(A+1)^2}{2A+1}N\right)^{-1}q^{-1-\lambda}<2(A^2+4A+3)z^{-2}. \tag{3.5}$$

Then we have

$$\frac{1}{2}z^2<4.13\frac{A^2(A+1)^2}{2A+1}(A^2+4A+3)(Az)^{1+\lambda}.$$

Since $1 < 1+\lambda < 2$, we get

$$(2z)^2 < 33.04\frac{A^4(A+1)^3}{2A+1}(A+3)z^{1+\lambda} < 16.52A^3(A+1)^3(A+3)z^{1+\lambda}$$
$$< 8.26A^3(A+1)^3(A+3)(2z)^{1+\lambda}.$$

It follows that

$$(1-\lambda)\log(2z) < \log\left(8.26A^3(A+1)^3(A+3)\right).$$

This and (3.4) give

$$(1-\lambda)\left(\sqrt{k}-1\right)\log\left((A+1)k\right) < \log\left(8.26A^3(A+1)^3(A+3)\right).$$

Notice that $k > 0.64A^2(A+1)^3$ and $A \geq 2$. So we have

$$\sqrt{k}-1 < \frac{\log\left(8.26A^3(A+1)^3(A+3)\right)}{\log(0.64A^2(A+1)^4)} \cdot \frac{1}{1-\lambda} =: \frac{\mu(A)}{1-\lambda}, \tag{3.6}$$

where

$$\mu(2) < 1.706, \quad \mu(3) < 1.557, \quad \mu(A) < 1.489 \text{ for } A \geq 4. \tag{3.7}$$

On the other hand, as $N = \frac{1}{2}(A^2+A)k + 2A$, $k > 0.64A^2(A+1)^3$ and $A \geq 2$, we have

$$0.5(A^2+A)k < N < 0.501(A^2+A)k.$$

This leads to

$$\begin{aligned}\lambda &= \frac{\log\frac{2.03A^2(A+1)^2N}{2A+1}}{\log\frac{3.24N^2}{A^2(A+1)^2}} < \frac{\log\frac{1.018A^3(A+1)^3k}{2A+1}}{\log(0.81k^2)} \\ &< \frac{\log(0.509A^2(A+1)^3k)}{\log(0.81k^2)} < \frac{\log(0.7954k^2)}{\log(0.81k^2)}.\end{aligned}$$

Thus we obtain

$$\frac{1}{1-\lambda} < \frac{\log(0.81k^2)}{\log 0.81 - \log 0.7954} < \frac{2\log k - 0.21}{0.01818} < 110.02\log k - 11.55. \tag{3.8}$$

Combining (3.6) and (3.8), we obtain

$$\sqrt{k} < \mu(A)(110.02\log k - 11.55) + 1.$$

When $k \geq 8909613$ for $A = 2$, or $k \geq 7227770$ for $A = 3$, or $k \geq 6524503$ for $A \geq 4$, the above inequality gives a contradiction and completes the proof of Proposition 3.4. $\square$

4. Proof of Theorem 1.2

In this section, we need to consider the remaining cases, i.e., $k \leq 8909612$ for $A = 2$, $k \leq 7227769$ for $A = 3$, $k \leq 6524502$ for $4 \leq A \leq 22$. We will use a theorem of lower bounds to linear forms in logarithms to get an upper bound for m.

Let

$$\alpha_1 = \frac{s + \sqrt{ac}}{2} \quad \text{and} \quad \alpha_2 = \frac{t + \sqrt{bc}}{2}.$$

From equations (2.5) and (2.6), we have

$$v_{2m} = \frac{1}{2\sqrt{a}} \left((z_0\sqrt{a} + x_0\sqrt{c})\alpha_1^{2m} + (z_0\sqrt{a} - x_0\sqrt{c})\alpha_1^{-2m} \right)$$

and

$$w_{2n} = \frac{1}{2\sqrt{b}} \left((z_1\sqrt{b} + y_1\sqrt{c})\alpha_2^{2n} + (z_1\sqrt{b} - y_1\sqrt{c})\alpha_2^{-2n} \right)$$

respectively. Notice $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. Solving equations (2.5) and (2.6) is equivalent to solve $z = v_{2m} = w_{2n}$ with $m, n \neq 0$. So we have (see [9, Lemma 10])

$$0 < \Lambda := 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3 < 2ac\alpha_1^{-4m}, \tag{4.1}$$

where

$$\alpha_3 = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}.$$

It follows that

$$\log |\Lambda| < -4m \log \alpha_1 + \log(2ac) < (2 - 4m) \log \alpha_1. \tag{4.2}$$

In [11], using Baker's method, the first author proved that

$$\frac{2m}{\log(2m+1)} < 6.543 \cdot 10^{15} \log^2 c.$$

By Proposition 3.4, we only need to consider the four cases as above in the range $2 \leq A \leq 22$, and then $c = (A+1)^2 k + 4(A+1) < 3.5 \cdot 10^9$. Then, by

$$\frac{2m}{\log(2m+1)} < 3.16 \cdot 10^{18}$$

we obtain $m < 7.4 \cdot 10^{19}$.

In order to deal with the remaining cases, we will use a Diophantine approximation algorithm so called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of Baker-Davenport reduction method (see [6, Lemma 5a]).

Lemma 4.1. *Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\eta > 0$*, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < EB^{-m}$$

in integers m *and* n *with*

$$\frac{\log(EQ/\eta)}{\log B} \leq m \leq M.$$

We apply Lemma 4.1 with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{2 \log \alpha_2}, \quad E = \frac{ac}{\log \alpha_2}, \quad B = \alpha_1^4$$

and $M = 7.4 \cdot 10^{19}$.

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the condition. We considered the following three cases:

$k \leq 8909612$ for $A = 2$, $k \leq 7227769$ for $A = 3$, $k \leq 6524502$ for $4 \leq A \leq 22$, i.e.,

1) $A = 2, 1 \leq k \leq 8909612$;

2) $A = 3, 1 \leq k \leq 7227769$;

3) $4 \leq A \leq 22$, $1 \leq k \leq 6524502$.

• If $z_0 = z_1 = 2$, then we used the second convergent in 2436033 cases (5.03%), the third convergent in 67217 cases (0.14%), etc., the 10^{th} convergent only in one case (for $(A, k) = (2, 3509101)$). In all cases we obtained $m \leq 7$. From Lemma 2.4, this implies $k \leq 225$. We took $M = 8$ and ran again the program for $1 \leq k \leq 225$ to obtain $m \leq 2$. The third running with $M = 3$ in the range $1 \leq k \leq 25$ gave us $m \leq 1$. The program was run in 20.5 hours.

• With $z_0 = z_1 = -2$, we used the second convergent in 810978 cases (1.67%), the third convergent in 1461562 cases (3.02%), etc., the 24^{th} convergent only in one case (for $(A, k) = (22, 3090024)$). In all cases, we obtained $m \leq 8$. Lemma 2.4 implies $1 \leq k \leq 256$. We ran again the program with $M = 9$ and we got $m \leq 2$. The third running with $M'' = 3$ for $1 \leq k \leq 25$ gave us $m \leq 1$. The computations were done in 19 hours.

Combining this and Proposition 3.4, we have $m = n = 1$ in equation (2.7) ($m = n = 0$ gives the trivial extension with $d = 0$). When $v_0 = w_0 = 2$, we have $v_2 < w_2$. When $v_0 = w_0 = -2$, $z = v_2 = w_2$ implies $d = (A^4 + 2A^3 + A^2)k^3 + (8A^3 + 12A^2 + 4A)k^2 + (20A^2 + 20A + 4)k + (16A + 8)$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

In Section 2, we showed that solving our problem consists in taking $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. This is equivalent to solve the sequence equation $z = v_{2m} = w_{2n}$. On the other hand, all the solutions of the equation (2.4), which make the system (2.3) and (2.4) solvable are also given by $y = u'_{2n}$, where the sequence (u'_n) is given by

$$u'_0 = 2, u'_1 = t \pm b, u'_{n+2} = tu'_{n+1} - u'_n.$$

Moreover, by eliminating z in the system (2.3) and (2.4), we obtain the equation

$$ay^2 - bx^2 = 4(a-b). \tag{5.1}$$

Now, solving this Pellian equation we get that $y = u''_l$ where the sequence (u''_l) is given by

$$u''_0 = y_2, u''_1 = \frac{1}{2}(ry_2 + bx_2), u''_{l+2} = ru''_{l+1} - u''_l,$$

where (y_2, x_2) is the solution of (5.1) which satisfies $|y_2| < \sqrt{\frac{b\sqrt{b}}{\sqrt{a}}}$ and $x_2 \geq 1$. Furthermore, considering congruences modulo b we get

$$u''_{2l} \equiv y_2 \pmod{b},\ u''_{2l+1} \equiv \frac{1}{2}(ry_2 + bx_2) \pmod{b},\ u'_{2n} \equiv y_1 = 2 \pmod{b}.$$

So, if we have the solution $y = u'_{2n} = u''_l$ which would give us the extension of our $D(4)$-triple, we have to consider two cases depending on parity of l. If we have $u'_{2n} = u''_{2l}$, then we get $y_2 \equiv 2 \pmod{b}$. And using the estimate for $|y_2|$ we see that $y_2 = 2$, which implies $x_2 = 2$. On the other hand if we have $u'_{2n} = u''_{2l+1}$, we get

$$\frac{1}{2}(ry_2 \pm bx_2) \equiv 2 \pmod{b}.$$

Furthermore we have

$$|(ry_2 + bx_2)(ry_2 - bx_2)| = 4b(b-a) - 4y_2^2 < 4b^2,$$

which implies (using the estimate for $|y_2|$ again)

$$\frac{1}{2}(bx_2 - r|y_2|) = 2.$$

But using that we now have to consider $A \geq 23$, and that (y_2, x_2) is the solution of (5.1), we conclude

$$|r|y_2| + bx_2| < 2bx_2 < 2b\sqrt{b}$$

and

$$|(ry_2 + bx_2)(ry_2 - bx_2)| = 4b(b-a) - 4y_2^2 > 3b^2,$$

which gives us

$$\frac{1}{2}(bx_2 - r|y_2|) > \frac{3b^2}{4b\sqrt{b}} = 0.75\sqrt{b} > 2,$$

a contradiction.

So, we have just proved that it is enough to consider $y = u'_{2n} = u''_{2l}$, when $y_2 = 2$, because it is the only possibility that can give us the extension of our triple. From now on, we will consider the system of Pellian equations (2.3) and (5.1), i.e.,

$$\begin{aligned} az^2 - cx^2 &= 4(a-c), \\ ay^2 - bx^2 &= 4(a-b). \end{aligned}$$

According to the above analysis, if $\{a,b,c,d\}$ is a $D(4)$-quadruple, then equations (2.3) and (5.1) have common solution x and all solutions of (2.3) are given by $x = W_m$, where

$$W_0 = x_0 = 2, \quad W_1 = \frac{1}{2}(sx_0 + az_0) = s \pm a, \quad W_{m+2} = sW_{m+1} - W_m.$$

In the same way, all solutions of (5.1) are given by $x = V_l$, where

$$V_0 = x_2 = 2, \quad V_1 = \frac{1}{2}(rx_2 + ay_2) = r + a, \quad V_{l+2} = rV_{l+1} - V_l.$$

Also one can notice that $2|m$ and $2|l$ and we have to solve the equation

$$x = W_{2m} = V_{2l}. \tag{5.2}$$

Lemma 5.1. *If $W_{2m} = V_{2l}$, then $m \le l$. Furthermore, $m \ne l$ for $l > 1$.*

Proof of Lemma 5.1. When $z_0 = 2$, it is easy to conclude that $V_2 < W_2$ from

$$V_0 = W_0 = 2,\ r < s, \text{ so } V_1 < W_1.$$

When $z_0 = -2$, then we conclude

$$V_2 = r(r+a) - 2 = A(A+1)k^2 + (4A+2)k + 2 = s(s-a) - 2 = W_2.$$

So we have $V_n \le W_n$, for $n = 0,1,2$, except when $z_0 = -2$ and $n = 1$. Now by induction for $n \ge 3$, we conclude

$$\begin{aligned} V_n &= rV_{n-1} - V_{n-2} < rV_{n-1} \le rW_{n-1} \\ &\le sW_{n-1} - W_{n-1} < sW_{n-1} - W_{n-2} = W_n. \end{aligned}$$

Therefore, if $V_{2l} = W_{2m}$, we have $m < l$ for $l \ge 2$, and $m = l$ for $m = l = 0$, or $m = l = 1$, $z_0 = -2$. $\square$

We also have

$$\begin{aligned} W_{2m} &= \frac{1}{\sqrt{c}}((\pm\sqrt{a} + \sqrt{c})\alpha^{2m} - (\pm\sqrt{a} - \sqrt{c})\alpha^{-2m}), \\ V_{2l} &= \frac{1}{\sqrt{b}}((\sqrt{a} + \sqrt{b})\beta^{2l} - (\sqrt{a} - \sqrt{b})\beta^{-2l}), \end{aligned}$$

where $\alpha = \frac{s+\sqrt{ac}}{2}$ and $\beta = \frac{r+\sqrt{ab}}{2}$. Let

$$\mu = \frac{\sqrt{c}(\sqrt{b} + \sqrt{a})}{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}$$

and let us define

$$\Lambda = 2l\log\beta - 2m\log\alpha + \log\mu.$$

Then we have the following result.

LEMMA 5.2. *If $ml \neq 0$, then $0 < \Lambda < \alpha^{1-4m}$.*

PROOF OF LEMMA 5.2. Let us define

$$P = \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}}\beta^{2l} \quad \text{and} \quad Q = \frac{\sqrt{c}\pm\sqrt{a}}{\sqrt{c}}\alpha^{2m}.$$

Then equation $W_{2m} = V_{2l}$ implies

$$P + \left(1 - \frac{a}{b}\right) P^{-1} = Q + \left(1 - \frac{a}{c}\right) Q^{-1}$$

and

$$P - Q = \left(1 - \frac{a}{c}\right) Q^{-1} - \left(1 - \frac{a}{b}\right) P^{-1}. \tag{5.3}$$

Thus we obtain

$$(P-Q)PQ = \left(1 - \frac{a}{c}\right) P - \left(1 - \frac{a}{b}\right) Q = P - Q + \frac{a}{b}Q - \frac{a}{c}P,$$

and

$$(PQ-1)(P-Q) = \frac{a}{b}Q - \frac{a}{c}P. \tag{5.4}$$

Since $ml \neq 0$, we have $PQ > 1$.

Suppose $P < Q$, then from (5.4) we get $\frac{a}{b}Q < \frac{a}{c}P < \frac{a}{c}Q$. It yields $c < b$, which is impossible. Since $P \neq Q$, we have $P > Q$. Moreover,

$$0 < P - Q < \left(1 - \frac{a}{c}\right) Q^{-1}.$$

Therefore, we have $\Lambda > 0$ and

$$\Lambda = \log\frac{P}{Q} < \frac{P}{Q} - 1 < \left(1 - \frac{a}{c}\right) Q^{-2} \leq \frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} \cdot \alpha^{-4m} < \alpha^{1-4m}.$$

□

The next lemma follows immediately from the proof of Lemma 5.2.

LEMMA 5.3. *If $W_{2m} = V_{2l}$ with $ml \neq 0$, then $\Lambda < \log\beta + \log\mu$.*

From Lemma 5.1, if $l > 1$, then there exists a positive integer ν such that

$$l = m + \nu. \tag{5.5}$$

By Lemma 5.3, we have

$$(2l-1)\log\beta - 2m\log\alpha = \Lambda - \log\beta - \log\mu < 0.$$

It follows that

$$\frac{2l-1}{2m} < \frac{\log\alpha}{\log\beta}.$$

In Section 4, we considered $A \leq 22$, so we assume that $A \geq 23$ now. Hence $r = Ak+2 \geq 25$ and $s = (A+1)k+2 \geq 26$. We get $\sqrt{ac}-\sqrt{ab} < 1.004(s-r)$. From above inequality we obtain

$$\begin{aligned} \frac{2\nu-1}{2m} &< \frac{\log\alpha}{\log\beta} - 1 = \frac{\log(\alpha/\beta)}{\log\beta} < \frac{\alpha-\beta}{\beta\log\beta} \\ &= \frac{s+\sqrt{ac}-r-\sqrt{ab}}{2\beta\log\beta} < \frac{2.004(s-r)}{2\beta\log\beta} < \frac{2.004k}{2\sqrt{ab}\log\beta} \\ &< \frac{2.004k}{2Ak\log\beta} < \frac{1.002}{A\log\beta} \end{aligned} \tag{5.6}$$

(an analogous estimate appeared for the first time in [3, Lemma 3]). Therefore, we proved the following lemma.

Lemma 5.4. *If $ml \neq 0$ and $V_{2l} = W_{2m}$ has a solution, then*

$$m > 0.99(\nu - 0.5)A\log\beta,$$

where $\nu = l - m$ is a positive integer.

Now we will use the following result due to Mignotte (see [20], Corollary of Theorem 2, page 110) on linear forms in two logarithms. For any non-zero algebraic number γ of degree d' over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a'\prod_{j=1}^{d'}\left(X-\gamma^{(j)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d'}\left(\log|a'| + \sum_{j=1}^{d'}\log\max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

its absolute logarithmic height.

Theorem 5.5 (Mignotte). *Consider the linear form*

$$\Lambda = b_1\log\gamma_1 - b_2\log\gamma_2,$$

where b_1 and b_2 are positive integers. Suppose that γ_1, γ_2 are multiplicatively independent. Put

$$D = [\mathbb{Q}(\gamma_1,\gamma_2):\mathbb{Q}]/[\mathbb{R}(\gamma_1,\gamma_2):\mathbb{R}]$$

and let ρ, τ and a_2 be positive real numbers with $\rho \geq 4$, $\tau = \log\rho$,

$$a_i \geq \max\{1, (\rho-1)\log|\gamma_i| + 2Dh(\gamma_i)\} \ \ (i = 1,2)$$

and

$$a_1a_2 \geq \max\{20, 4\tau^2\}.$$

Furthermore suppose h is a real number with

$$h \geq \max\left\{3.5, 1.5\tau, D\left(\log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\tau+1.377\right)+0.023\right\},$$

$\chi = h/\tau$, $\upsilon = 4\chi + 4 + 1/\chi$. *Then we have the lower bound*

$$\log|\Lambda| \geq -(C_0 + 0.06)(\tau + h)^2 a_1 a_2, \tag{5.7}$$

where

$$C_0 = \frac{1}{\tau^3}\left\{\left(2 + \frac{1}{2\chi(\chi+1)}\right)\left(\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{4\tau}{3\upsilon}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{32\sqrt{2}(1+\chi)^{3/2}}{3\upsilon^2\sqrt{a_1a_2}}}\right)\right\}^2.$$

In order to satisfy the conditions of the theorem, we rewrite Λ to

$$\Lambda = \log\left(\beta^{2\nu}\mu\right) - 2m\log(\alpha/\beta). \tag{5.8}$$

Here we take

$$D = 4, \quad b_1 = 1, \quad b_2 = 2m, \quad \gamma_1 = \beta^{2\nu}\mu, \quad \gamma_2 = \alpha/\beta.$$

We know that α and β are multiplicatively independent algebraic units and that μ is not an algebraic unit (see below its characteristic polynomial) so we conclude that γ_1 and γ_2 are multiplicatively independent.

We have $h(\gamma_1) = h(\beta^{2\nu}\mu) \leq h(\beta^{2\nu}) + h(\mu)$. It is easy to see that $h(\beta^{2\nu}) = \nu\log\beta$. Also notice that μ is a root of

$$b^2(c-a)^2x^4 - 4b^2c(c-a)x^3 + 2bc(3bc-ab-ac-a^2)x^2 - 4bc^2(b-a)x + c^2(b-a)^2,$$

and the absolute values of its conjugates greater than 1 are

$$\frac{\sqrt{c}(\sqrt{b}+\sqrt{a})}{\sqrt{b}(\sqrt{c}+\sqrt{a})} \quad \text{and} \quad \frac{\sqrt{c}(\sqrt{b}+\sqrt{a})}{\sqrt{b}(\sqrt{c}-\sqrt{a})}.$$

Then we have

$$h(\mu) \leq \frac{1}{4}\log\left(b^2(c-a)^2 \cdot \frac{c}{b} \cdot \frac{(\sqrt{b}+\sqrt{a})^2}{c-a}\right) < \frac{1}{2}\log(2bc).$$

Therefore, $h(\gamma_1) \leq \nu\log\beta + \log\sqrt{2bc}$. Since $\alpha = \frac{s+\sqrt{ac}}{2}$ and $\beta = \frac{r+\sqrt{ab}}{2}$, then $\gamma_2 = \frac{s+\sqrt{ac}}{r+\sqrt{ab}}$ is a root of

$$X^4 - rsX^3 + (r^2 + s^2 - 2)X^2 - rsX + 1.$$

The absolute values of its conjugates greater than 1 are α/β and $\alpha/\overline{\beta} = \alpha\beta$. Hence $h(\gamma_2) = \frac{1}{4}\left(\log(\alpha/\beta) + \log(\alpha\beta)\right) = \frac{1}{2}\log\alpha$.

We can choose $\rho = 5.0$. Then, all the above computations allow us to take

$$a_1 = 16(\nu + 1.06)\log\beta, \quad a_2 = 4.16\log\alpha.$$

As $A \geq 23$, we get $r = Ak + 2 \geq 25$ and $s = (A+1)k + 2 \geq 26$. Thus we have $a_1 > 106.04$ and $a_2 > 13.54$. Therefore, the condition $a_1a_2 > \max\{20, 4\tau^2\}$

holds. This and Lemma 5.4 implies

$$\begin{aligned}\frac{1}{4.16\log\alpha} &< \frac{1}{4.16\log\beta} < \frac{\nu+1.06}{4.16\cdot 0.99(\nu-0.5)A\log\beta}\cdot\frac{16m}{16(\nu+1.06)\log\beta}\\ &< 0.078\cdot\frac{2m}{16(\nu+1.06)\log\beta}.\end{aligned}$$

It implies $b_1/a_2 < 0.078 b_2/a_1$.

Let us consider

$$h = 4\log\left(\frac{b_2}{a_1}\right) + 7.735 = 4\log\left(\frac{m}{8(\nu+1.06)\log\beta}\right) + 7.735.$$

First, assume that $h \geq 36.59$. Then we find $C_0 < 0.438$, thus we have

$$\log|\Lambda| > -33.147\left(4\log\left(\frac{m}{8(\nu+1.06)\log\beta}\right) + 9.345\right)^2(\nu+1.06)\log\alpha\log\beta.$$

From Lemma 5.2, we get $\log|\Lambda| < (1-4m)\log\alpha$. Combining these bounds of $\log|\Lambda|$, we have

$$m - 0.25 < 8.287\left(4\log\left(\frac{m}{8(\nu+1.06)\log\beta}\right) + 9.345\right)^2(\nu+1.06)\log\beta.$$

Since $(\nu+1.06)\log\beta > 6.627$, then the above inequality implies

$$\frac{m}{(\nu+1.06)\log\beta} < 0.038 + 8.287\left(4\log\left(\frac{m}{(\nu+1.06)\log\beta}\right) + 1.028\right)^2.$$

It follows that

$$\frac{m}{(\nu+1.06)\log\beta} \leq 12439. \tag{5.9}$$

Otherwise, if $h < 36.59$, then we get

$$\frac{m}{(\nu+1.06)\log\beta} < e^{9.294} < 10873.$$

Now by Lemma 5.4, we have $m > 0.99(\nu-0.5)A\log\beta$. This and (5.9) imply

$$\frac{0.99(\nu-0.5)A}{\nu+1.06} \leq 12439.$$

Now from $\nu \geq 1$ and the above inequality, we obtain $A < 51767$. This completes the proof of Theorem 1.3.

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