GLASNIK MATEMATIČKI
Vol. 47(67)(2012), 53 – 59

A NOTE ON THE SIMULTANEOUS PELL EQUATIONS $x^2 - ay^2 = 1$ AND $z^2 - by^2 = 1$

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Abstract. Let m, n be positive integers with $1 < m < n$. Let δ be a positive number with $\frac{1}{2} < \delta < 1$. In this paper we prove that if $\gcd(m, n) > n^{\delta}$ and $n > (8 \times 10^{16}(\log(10^{16}/\theta^3))^3/\theta^3)^{1/\theta}$, where $\theta = \min(1-\delta,\ 2\delta-1)$, then the simultaneous Pell equations $x^2-(m^2-1)y^2 = 1$ and $z^2 - (n^2 - 1)y^2 = 1$ have only one positive integer solution $(x, y, z) = (m, 1, n)$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a, b be distinct positive integers. The simultaneous Pell equations

$$x^2 - ay^2 = 1,\ z^2 - by^2 = 1,\ x, y, z \in \mathbb{N} \tag{1.1}$$

arise in connection with a variety of classical problems on number theory and arithmetic algebraic geometry (see [7]). Let $N(a, b)$ denote the number of solutions (x, y, z) of (1.1). As early as the 1920s, using the diophantine approximation method of A. Thue ([12]), C. L. Siegel ([11]) proved that $N(a, b)$ is always finite. However, his result is ineffective. An effective upper bound for $N(a, b)$ was given by H. P. Schlickewei ([9]). Using the Subspaces Theorem of W. M. Schmidt ([10]), he proved that $N(a, b) < 4 \times 8^{2^{78}}$. In 1996, using the Padé approximation method (see [8]), D.W. Masser and J.H. Rickert ([6]) improved considerably the above mentioned upper bound; they proved that $N(a, b) \leq 16$. One year later, M.A. Bennett ([2]) further proved that $N(a, b) \leq 3$. Simultaneously, since there is no known pair (a, b) which makes $N(a, b) = 3$, he proposed the following conjecture:

2010 *Mathematics Subject Classification.* 11D09.
Key words and phrases. Simultaneous Pell equations; number of solutions.
Supported by the National Natural Science Foundation of China (No.10971184).

Conjecture A. $N(a,b) \le 2$.

In 2001, P.-Z. Yuan ([13]) and the author ([4]) independently proved that if $\max(a,b) > C$, where C is an effectively computable constant, then $N(a,b) \le 2$. Recently, M. A. Bennett, M. Cipu, M. Mignotte and R. Okazaki ([3]) completely verified Conjecture A, namely, they unconditionally proved that $N(a,b) \le 2$.

By [2], if (1.1) has solutions and (x_1, y_1, z_1) is the solution of (1.1) with $y_1 \le y$, where y through over all solutions (x,y,z) of (1.1), then $y_1 \mid y$. Therefore, if (1.1) has solutions, then it is equivalent to the equations

$$X^2 - (m^2-1)Y^2 = 1,\ Z^2 - (n^2-1)Y^2 = 1,\ X, Y, Z \in \mathbb{N}, \tag{1.2}$$

where m and n are distinct positive integers with $\min(m,n) > 1$. Obviously, (1.2) has a solution $(X,Y,Z) = (m,1,n)$. In this respect, M. A. Bennett ([1]) showed that if

$$n = \frac{\alpha^{2l} - \bar{\alpha}^{2l}}{4\sqrt{m^2-1}},\ l \in \mathbb{N}, \tag{1.3}$$

where

$$\alpha = m + \sqrt{m^2-1}\ ,\ \bar{\alpha} = m - \sqrt{m^2-1}\ , \tag{1.4}$$

then (1.2) has an other solution $(X,Y,Z) = ((\alpha^{2l} + \bar{\alpha}^{2l})/2, 2n, 2n^2-1)$. Thus, P.-Z. Yuan ([14]) proposed a stronger conjecture as follows:

Conjecture B. *If $N(m^2-1, n^2-1) \ge 2$, then n must satisfy (1.3).*

The above mentioned conjecture has not been solved yet. In this paper, we verify Conjecture B for m and n are sufficiently large and they have sufficiently large common divisor, namely, we prove the following result:

Theorem 1.1. *Let δ be a positive number with $\frac{1}{2} < \delta < 1$. If $\gcd(m,n) > \max(m^\delta, n^\delta)$ and*

$$\max(m,n) > \left(\frac{8 \times 10^{16}}{\theta^3}\left(\log \frac{10^{16}}{\theta^3}\right)^3\right)^{1/\theta},\ \theta = \min(1-\delta,\ 2\delta - 1), \tag{1.5}$$

then (1.2) has only one solution $(X,Y,Z) = (m,1,n)$.

2. Preliminaries

Lemma 2.1 ([5, Formula 1.76]). *For any positive integer k and any complex numbers α and $\bar{\alpha}$, we have*

$$\alpha^k + \bar{\alpha}^k = \sum_{i=0}^{[k/2]} (-1)^i \begin{bmatrix} k \\ i \end{bmatrix} (\alpha + \bar{\alpha})^{k-2i} (\alpha\bar{\alpha})^i\ ,$$

where $[k/2]$ is the integral part of $k/2$,

$$\begin{bmatrix} k \\ i \end{bmatrix} = \frac{(k-i-1)\,!\,k}{(k-2i)\,!\,i\,!} \in \mathbb{N},\ i = 0, 1, \ldots, \begin{bmatrix} \frac{k}{2} \end{bmatrix}\ .$$

Let m, n be positive integers with $1 < m < n$. Let $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ be defined as in (1.4) and

$$\beta = n + \sqrt{n^2 - 1}\ ,\ \bar{\beta} = n - \sqrt{n^2 - 1}\ , \tag{2.1}$$

respectively. For any positive integer k, let

$$u_k + v_k\sqrt{m^2 - 1} = \alpha^k,\ u'_k + v'_k\sqrt{n^2 - 1} = \beta^k. \tag{2.2}$$

It is a well known fact that $(u, v) = (u_k, v_k)$ $(k = 1, 2, \ldots)$ and $(u', v') = (u'_k, v'_k)$ $(k = 1, 2, \ldots)$ are all solutions of Pell equations

$$u^2 - (m^2 - 1)v^2 = 1,\ u, v \in \mathbb{N} \tag{2.3}$$

and

$$u'^2 - (n^2 - 1)v'^2 = 1,\ u', v' \in \mathbb{N}, \tag{2.4}$$

respectively.

LEMMA 2.2. *For any positive integer k with $k > 1$, we have $v_k < v'_k$.*

PROOF OF LEMMA 2.2. By (1.4), (2.1) and (2.2), $\{v_k\}_{k=1}^{\infty}$ and $\{v'_k\}_{k=1}^{\infty}$ are increasing sequences satisfying $v_1 = v'_1 = 1$ and

$$v_{k+1} = 2mv_k - v_{k-1},\ v'_{k+1} = 2nv'_k - v'_{k-1},\ k \in \mathbb{N}, \tag{2.5}$$

where $v_0 = v'_0 = 0$. We now assume that l is the least positive integer such that $v_l \geq v'_l$. Since $1 < m < n$, we get from (2.5) that $l > 1$, $v_{l-1} < v'_{l-1}$ and $(2n-2)v'_{l-1} \geq 2mv'_{l-1} > 2mv_{l-1} \geq 2mv_{l-1} - v_{l-2} = v_l \geq v'_l = 2nv'_{l-1} - v'_{l-2} > (2n-1)v'_{l-1}$, a contradiction. Thus, the lemma is proved. □

LEMMA 2.3. *Let r and s be positive integers with* $\min(r, s) > 1$. *If*

$$v_r = v'_s, \tag{2.6}$$

then we have:

(i) $r > s$.
(ii) $r \equiv s \pmod 2$.
(iii) *If $2 \nmid r$, then $r \equiv s \pmod 4$.*

PROOF OF LEMMA 2.3. By Lemma 2.2, we have $v_s < v'_s$. Therefore, if (2.6) holds, then $r > s$. We see from (1.4), (2.1) and (2.2) that $v_k \equiv k \pmod 2$ and $v'_k \equiv k \pmod 2$. It implies that $r \equiv s \pmod 2$ by (2.6).

Since $\alpha - \bar{\alpha} = 2\sqrt{m^2 - 1}$ and $\alpha\bar{\alpha} = 1$, by Lemma 2.1, if $2 \nmid r$, then

$$v_r = \frac{\alpha^r - \bar{\alpha}^r}{2\sqrt{m^2 - 1}} = \frac{\alpha^r - \bar{\alpha}^r}{\alpha - \bar{\alpha}} = \sum_{i=0}^{(r-1)/2} \begin{bmatrix} r \\ i \end{bmatrix} (\alpha - \bar{\alpha})^{r-2i-1}(\alpha\bar{\alpha})^i$$

$$= \sum_{i=0}^{(r-1)/2} \begin{bmatrix} r \\ i \end{bmatrix} (4(m^2 - 1))^{(r-1)/2-i}, \tag{2.7}$$

whence we get

$$v_r \equiv r \pmod 4. \tag{2.8}$$

Similarly, since $r \equiv s \pmod 2$, we have

$$v'_s \equiv s \pmod 4. \tag{2.9}$$

Therefore, if (2.6) holds, then from (2.8) and (2.9) we get $r \equiv s \pmod 4$. Thus, the lemma is proved. □

Let $d = \gcd(m, n)$. Then we have

$$m = dm_1\ ,\ n = dn_1,\ m_1, n_1 \in \mathbb{N},\ \gcd(m_1, n_1) = 1. \tag{2.10}$$

LEMMA 2.4. *If $d > n^\delta$ and (2.6) holds, where δ is a positive number with $\frac{1}{2} < \delta < 1$, then $r > n^\theta$, where*

$$\theta = \min(1 - \delta,\ 2\delta - 1). \tag{2.11}$$

PROOF OF LEMMA 2.4. For $2 \mid r$, we have

$$v_r = \frac{\alpha^r - \bar{\alpha}^r}{2\sqrt{m^2 - 1}} = m \sum_{i=0}^{r/2-1} \binom{r}{2i+1} m^{r-2i-1}(m^2 - 1)^i, \tag{2.12}$$

whence we get

$$v_r \equiv rm(m^2 - 1)^{r/2-1} \equiv (-1)^{r/2-1}\ rm \pmod{m^3}. \tag{2.13}$$

Similarly, since $2 \mid s$, we have

$$v'_s \equiv (-1)^{s/2-1}\ sn \pmod{n^3}. \tag{2.14}$$

Therefore, by (2.6), (2.13) and (2.14), we obtain

$$rm_1 \equiv \lambda sn_1 \pmod{d^2}, \lambda \in \{\pm 1\}. \tag{2.15}$$

We find from (2.15) that either

$$rm_1 = sn_1 \tag{2.16}$$

or

$$rm_1 + sn_1 \geq d^2. \tag{2.17}$$

When (2.16) holds, since $\gcd(m_1, n_1) = 1$, we get

$$r = n_1 t,\ s = m_1 t,\ t \in \mathbb{N}. \tag{2.18}$$

It implies that $r \geq n_1 = n/d > n^{1-\delta} \geq n^\theta$ by (2.11). When (2.17) holds, since $n_1 > m_1$, we have $r > (rm_1 + sn_1)/2n_1 \geq d^2/2n_1 > n^{3\delta-1}/2 = n^{2\delta-1} \cdot n^\delta/2 \geq n^\theta$. Thus, the lemma holds for $2 \mid r$.

For $2 \nmid r$, we have

$$v_r = \sum_{i=0}^{(r-1)/2} \binom{r}{2i+1} m^{r-2i-1}(m^2 - 1)^i, \tag{2.19}$$

whence we get

(2.20) $$v_r \equiv (-1)^{(r-3)/2}(-1+((r^2-1)/2)m^2) \pmod{m^4}.$$

Further, by Lemma 2.3, we have $2 \nmid s$ and $r \equiv s \pmod{n^4}$. Hence, we get

(2.21) $$v'_s \equiv (-1)^{(r-3)/2}(-1+((s^2-1)/2)n^2) \pmod{n^4}.$$

Furthermore, by (2.6), (2.20) , and (2.21), we obtain

(2.22) $$(r^2-1)m_1^2 \equiv (s^2-1)n_1^2 \pmod{2d^2}.$$

We find from (2.22) that either

(2.23) $$(r^2-1)m_1^2 = (s^2-1)n_1^2$$

or

(2.24) $$\max((r^2-1)m_1^2,\ (s^2-1)n_1^2) > 2d^2.$$

When (2.23) holds, we have

(2.25) $$(r^2-1) = n_1^2 t,\ (s^2-1) = m_1^2 t,\ t \in \mathbb{N},$$

whence we get $r > \sqrt{r^2-1} \geq n_1 > n^{1-\delta} \geq n^\theta$ by (2.11). When (2.24) holds, since $r > s$ and $n_1 > m_1$, we get $r > \max(m_1\sqrt{r^2-1},\ n_1\sqrt{s^2-1})/n_1 > 2d^2/n_1 > 2n^{3\delta-1} > 2n^\theta$. To sum up, the lemma is proved. □

Lemma 2.5. *Let c, c_1, c_2, c_3 be positive numbers.*

(i) *If $c_2 > 2e^{c_1/c_2} \log c_2$, then $c > c_1 + c_2 \log c$ for $c \geq 2c_2 \log c_2$.*
(ii) *If $c_3 > 8(\log c_3)^3$, then $c > c_3(\log c)^3$ for $c > 8c_3(\log c_3)^3$.*

Proof of Lemma 2.5. Let

(2.26) $$f(c) = c - (c_1 + c_2 \log c).$$

Since $f'(c) = 1 - c_2/c$, we have $f'(c) > 0$ for $c > c_2$. It implies that $f(c)$ is an increasing function for $c > c_2$. On the other hand, if $f(2c_2 \log c_2) \leq 0$, then from (2.26) we get

(2.27) $$2c_2 \log c_2 \leq c_1 + c_2(\log 2 + \log c_2 + \log\log c_2),$$

whence we obtain $c_2 \leq 2e^{c_1/c_2} \log c_2$, which contradicts the assumption. Therefore, we have $f(2c_2 \log c_2) > 0$. Thus, by (2.26), the result (i) is proved. Using the same method, we can deduce the result (ii). The lemma is proved. □

Lemma 2.6 ([3, Formula (11)]). *If (2.6) holds, then*

$$r < 4.26 \times 10^{13}(\log\beta)^2\ (\log(er)).$$

3. Proof of Theorem 1.1

We may assume that $1 < m < n$. If (1.2) has two solutions,then it has a solution (X, Y, Z) with $Y > 1$. By (1.3), (2.2), (2.3) and (2.4), we have

$$Y = v_r = v'_s,\ r, s \in \mathbb{N},\ \min(r, s) > 1. \tag{3.1}$$

By Lemma 2.3, we have $r > s$. Since $\beta = n + \sqrt{n^2 - 1} < 2n$, by Lemma 2.6, we get

$$r < 4.26 \times 10^{13} (\log n)^2 (1 + \log r). \tag{3.2}$$

Put $c_1 = c_2 = 4.26 \times 10^{13} (\log n)^2$. Since $c_1/c_2 = 1$ and $c_2 > 2\mathrm{e} \log c_2$, by (i) of Lemma 2.5, we see from (3.2) that

$$r < 2c_2 \log c_2 < 8.52{\times}10^{13} (\log 2n)^2 (31.39{+}2 \log\log 2n) < 10^{16} (\log n)^3. \tag{3.3}$$

On the other hand, by Lemma 2.4, we have $r > n^\theta$. Substitute it into (3.3), we get

$$n^\theta < 10^{16} (\log n)^3 = \frac{10^{16}}{\theta^3} (\log n^\theta)^3. \tag{3.4}$$

Put $c_3 = 10^{16}/\theta^3$. Since $c_3 > 10^{16}$, we have $c_3 > 8(\log c_3)^3$. Therefore, by (ii) of Lemma 5, we see from (3.4) that

$$n^\theta < 8c_3 (\log c_3)^3 < \frac{8 \times 10^{16}}{\theta^3}\ (\log \frac{10^{16}}{\theta^3})^3. \tag{3.5}$$

It implies that if $\gcd(m, n) > n^\delta$ and (1.5) holds, then (1.2) has only one solution $(X, Y, Z) = (m, 1, n)$. Thus, the theorem is proved.

Acknowledgements.

The author thanks the referee for his valuable suggestions.

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Received: 27.4.2010.
Revised: 31.10.2010.