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ON THE FAMILY OF ELLIPTIC CURVES $Y^2 = X^3 - T^2X + 1$

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Abstract. Let E be the elliptic curve over $\mathbb{Q}(T)$ given by the equation

$$E : Y^2 = X^3 - T^2X + 1.$$

We prove that the torsion subgroup of the group $E(\mathbb{C}(T))$ is trivial, $\mathrm{rank}_{\mathbb{Q}(T)}(E) = 3$ and $\mathrm{rank}_{\mathbb{C}(T)}(E) = 4$. We find a parametrization of E of rank at least four over the function field $\mathbb{Q}(a, i, s, n, k)$ where $s^2 = i^3 - a^2i$. From this we get a family of rank ≥ 5 over the field of rational functions in two variables and a family of rank ≥ 6 over an elliptic curve of positive rank. We also found particular elliptic curves with rank ≥ 11.

1. Introduction

Let E be the elliptic curve over $\mathbb{Q}(T)$ given by the equation

$$Y^2 = X^3 - T^2X + 1.$$

In [1, Theorem 3.11] it is proven that if $t \geq 2$ is an integer, the elliptic curve $E_t : Y^2 = X^3 - t^2X + 1$ has rank at least 2 over $\mathbb{Q}$, with independent points $(0, 1)$ and $(-1, t)$. It is proven that the rank of E_t is at least 3, for integers $t \equiv 0 \pmod 4$ and $t = 7$. Here the third independent point is $(-t, 1)$. Additionally, the torsion subgroup of $E_t(\mathbb{Q})$ is trivial, for all integer values $t \geq 1$.

In this paper we prove that the rank of the elliptic curve E over $\mathbb{Q}(T)$ is equal 3. We find the generators $(0, 1)$, $(-1, T)$, $(-T, 1)$ of the finitely generated Abelian group $E(\mathbb{Q}(T))$ and prove that its torsion subgroup is trivial. Since the rank of $E(\mathbb{Q}(T))$ is equal three, by the Silverman's specialization theorem [12, p. 271, Theorem 11.4] we obtain that rank $E_t(\mathbb{Q}) \geq 3$, for all but finitely many rational values t. We also compute the rank of E over $\mathbb{C}(T)$ and find the

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generators. In addition we find parametrisations of E for which the generic rank is ≥ 4, ≥ 5 and ≥ 6. We also search for particular high rank curves in the family E_t. We find several curves with rank ≥ 9 for integer values of the parameter t, and several curves with rank ≥ 11 for rational values of the parameter t.

2. The rational elliptic surface $Y^2 = X^3 - T^2X + 1$

In this section we give results regarding the elliptic curve $Y^2 = X^3 - T^2X + 1$ over $\mathbb{Q}(T)$. We will find the torsion subgroup, calculate the rank over $\mathbb{Q}(T)$ and $\mathbb{C}(T)$, find the generators and find parametrizations of generic rank ≥ 4 and ≥ 5.

Proposition 2.1. *Let E be the elliptic curve over $\mathbb{Q}(T)$ given by the equation*

$$E : Y^2 = X^3 - T^2X + 1.$$

(i) *The associated elliptic surface (denoted $\mathcal{E}$) is rational.*

(ii) *$rank_{\mathbb{C}(T)}E = 4$,*

(iii) *The generators of the group $E(\mathbb{C}(T))$ are the points*

$$(0,1),\ (-1,T),\ (-T,1),\left(\frac{1+\sqrt{-3}}{2},\frac{1-\sqrt{-3}}{2}T\right),$$

and the torsion subgroup of $E(\mathbb{C}(T))$ is trivial.

(iv) *$rank_{\mathbb{Q}(T)}E = 3$, the generators over $\mathbb{Q}(T)$ are*

$$(0,1),\ (-1,T),\ (-T,1),$$

and the torsion subgroup of $E(\mathbb{Q}(T))$ is trivial.

(v) *For*

$$T(a,i,s,n,k) = an^2 + \left(2ak + \frac{s}{i}\right)n + ak^2 + \frac{s}{i}\cdot k + \frac{a^3 - 2ai^2}{i^3 - a^2i}$$

the elliptic curve $Y^2 = X^3 - T(a,i,s,n,k)^2X + 1$ over the function field $\mathbb{Q}(a,i,s,n,k)$ where $s^2 = i^3 - a^2i$, has rank ≥ 4, with an extra independent point with the first coordinate

$$X_C(a,i,s,n,k) = i(n+k)^2 + \frac{i^2}{a^2 - i^2}.$$

Proof of Proposition 2.1. The elliptic curve E over $\mathbb{Q}(T)$ is written in short Weierstrass form

$$E : Y^2 = X^3 + A(T)X + B(T),$$

where we have

$$A(T) = -T^2,$$
$$B(T) = 1,$$
$$\Delta(T) = 16(4T^6 - 27),$$

here Δ is the discriminant.

(i) Since $\deg(A) = 2$ and $\deg(B) = 0$, from [11, Equation 10.14] we find that the associated elliptic surface $\mathcal{E}$ is rational.

(ii) For the proof we will use Shioda's formula. Since from (i) we know that the associated elliptic surface $\mathcal{E}$ is rational, by [11, Lemma 10.1] we find that the rank of the Néron-Severi group (denoted $NS(\mathcal{E}, \mathbb{C})$) of $\mathcal{E}$ over $\mathbb{C}$ is equal 10. From the discriminant $\Delta(T)$, we see that the singular fibres are at $\varphi_1, \ldots, \varphi_6$ and ∞, where the φ_i are the roots of the equation $4T^6 - 27 = 0$. We determine the numbers m_s (of irreducible components of the fibre over s) from Kodaira types of singular fibres [7, Section 4]:

s	coefficients				
	$ord_{T=s}(A)$	$ord_{T=s}(B)$	$ord_{T=s}(\Delta)$	Kodaira type	$m_s - 1$
φ_i	0	0	1	I_1	0
∞	2	6	6	I_0^*	4

Now we compute $\text{rank}_{\mathbb{C}(T)}(E)$ using Shioda's formula [11, Corollary 5.3]

$$\text{rank}_{\mathbb{C}(T)}\mathcal{E} = \text{rank}NS(\mathcal{E}, \mathbb{C}) - 2 - \sum_s (m_s - 1) = 10 - 2 - 6 \cdot 0 - 4 = 4.$$

(iii) The group $E(\mathbb{C}(T))$ is generated, by [11, Theorem 10.10], with the points of the form $(a_2T^2 + a_1T + a_0, b_3T^3 + b_2T^2 + b_1T + b_0)$, $a_i,\ b_i \in \mathbb{C}$. We list all such points

$$(0, \pm 1) =: \pm P,$$
$$(-1, \pm T) =: \pm Q,$$
$$(-T, \pm 1) =: \pm R,$$
$$(T, \pm 1) = \mp P \mp R,$$
$$(T + 2, \pm 2T \pm 3) = \mp Q \mp R,$$
$$(-T + 2, \pm 2T \mp 3) = \mp P \mp Q \mp R,$$
$$(T^2 - 1, \pm T^3 \mp 2T) = \pm P \pm Q \pm 2R,$$
$$(T^2 + 2T + 2, \pm T^3 \pm 3T^2 \pm 4T \pm 3) = \mp P \pm Q,$$
$$(T^2 - 2T + 2, \pm T^3 \mp 3T^2 \pm 4T \mp 3) = \pm P \pm Q,$$
$$\left(\frac{1 + \sqrt{-3}}{2}, \pm \frac{1 - \sqrt{-3}}{2} T\right) =: \pm S,$$

$$\left(\frac{1-\sqrt{-3}}{2}, \pm\frac{1+\sqrt{-3}}{2}T\right) = \pm P \pm Q \pm 2R \mp S,$$
$$\big(T-1-\sqrt{-3}, \pm(1-\sqrt{-3})T \mp 3\big) = \pm R \mp S,$$
$$\big(T-1+\sqrt{-3}, \pm(1+\sqrt{-3})T \mp 3\big) = \mp P \mp Q \mp R \pm S,$$
$$\big(-T-1-\sqrt{-3}, \pm(1-\sqrt{-3})T \pm 3\big) = \pm P \pm R \mp S,$$
$$\big(-T-1+\sqrt{-3}, \pm(1+\sqrt{-3})T \pm 3\big) = \mp Q \mp R \pm S,$$
$$\left(-\frac{1}{3}T^2-1, \pm\frac{\sqrt{-3}}{9}T^3\right) = \pm P \pm Q \pm 2R \mp 2S,$$
$$\left(\frac{1}{6}(1+\sqrt{-3})T^2+\frac{1}{2}(1-\sqrt{-3}), \pm\frac{\sqrt{-3}}{9}T^3\right) = \pm Q \pm S,$$
$$\left(\frac{1}{6}(1-\sqrt{-3})T^2+\frac{1}{2}(1+\sqrt{-3}), \pm\frac{\sqrt{-3}}{9}T^3\right) = \pm P \pm 2Q \pm 2R \mp S,$$
$$\left(-\frac{1}{2}(1-\sqrt{-3})T^2+\frac{1}{2}(1+\sqrt{-3}), \pm T^3 \pm (1-\sqrt{-3})T\right) = \pm P \pm 2R \mp S,$$
$$\left(-\frac{1}{2}(1+\sqrt{-3})T^2+\frac{1}{2}(1-\sqrt{-3}), \pm T^3 \pm (1+\sqrt{-3})T\right) = \mp Q \pm S,$$
$$\left(-\frac{1}{2}(1-\sqrt{-3})T^2+2T-(1+\sqrt{-3}), \pm T^3 \mp \frac{3}{2}(1+\sqrt{-3})T^2 \mp 2(1-\sqrt{-3})T \pm 3\right)$$
$$= \mp P \mp S,$$
$$\left(-\frac{1}{2}(1-\sqrt{-3})T^2-2T-(1+\sqrt{-3}), \pm T^3 \pm \frac{3}{2}(1+\sqrt{-3})T^2 \mp 2(1-\sqrt{-3})T \mp 3\right)$$
$$= \pm P \mp S,$$
$$\left(-\frac{1}{2}(1+\sqrt{-3})T^2+2T-(1-\sqrt{-3}), \pm(T^3-\frac{3}{2}(1-\sqrt{-3})T^2-2(1+\sqrt{-3})T+3)\right)$$
$$= \mp 2P \mp Q \mp 2R \pm S,$$
$$\left(-\frac{1}{2}(1+\sqrt{-3})T^2-2T-(1-\sqrt{-3}), \pm T^3 \pm \frac{3}{2}(1-\sqrt{-3})T^2 \mp 2(1+\sqrt{-3})T \mp 3\right)$$
$$= \mp Q \mp 2R \pm S.$$

We see that all of the listed points which generate $E(\mathbb{C}(T))$ can be written as a combination of the four points P, Q, R, S, so the points P, Q, R, S generate the group $E(\mathbb{C}(T))$. From (ii) we conclude that P, Q, R, S are independent points of infinite order that generate the group $E(\mathbb{C}(T))$, and the torsion subgroup $E(\mathbb{C}(T))_{\mathrm{Tors}}$ is trivial.

(iv) Since the torsion subgroup of $E(\mathbb{C}(T))$ is trivial, we conclude that the torsion subgroup of $E(\mathbb{Q}(T))$ is trivial.

We have to prove that the generators of the group $E(\mathbb{Q}(T))$ are the three independent points $P, Q, R \in E(\mathbb{Q}(T))$ in (iii). So we have to prove that these three points generate the subgroup $E(\mathbb{Q}(T))$ of the group $E(\mathbb{C}(T))$, where the later is generated by the four points P, Q, R, S in (iii). It is obvious that the point $S \in E(\mathbb{C}(T))\backslash E(\mathbb{Q}(T))$ will not give a point of the group $E(\mathbb{Q}(T))$, by the action of the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$ and using that the torsion subgroup of $E(\mathbb{C}(T))$ is trivial. Precisely, since $aP+bQ+cR+dS \in E(\mathbb{Q}(T))$ $(a,b,c,d \in \mathbb{Z})$, would mean $0 = (aP+bQ+cR+dS)^\sigma - (aP+bQ+cR+dS) = dS^\sigma - dS = d(S^\sigma - S)$ for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{-3},T)/\mathbb{Q}(T))$. Since the torsion subgroup is trivial and $S = \left(\frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}T\right)$, this would give $d = 0$. Thus, it would mean that if $d \in \mathbb{Z}\backslash\{0\}$, then $aP+bQ+cR+dS \notin E(\mathbb{Q}(T))$. So we conclude that the points P, Q, R generate the subgroup $E(\mathbb{Q}(T))$ of the group $E(\mathbb{C}(T))$.

(v) We look for parameters T of the form an^2+bn+c $(a,b,c \in \mathbb{Q})$ for which the curve

$$Y^2 = X^3 - T^2X + 1$$

has a point of the form $(i_2n^2+i_1n+i_0, j_3n^3+j_2n^2+j_1n+j_0)$, for i_0, i_1, i_2, j_0, j_1, j_2, $j_3 \in \mathbb{Q}$. In other words we search for a, b, c, i_0, i_1, i_2 for which the polynomial

$$(i_2n^2+i_1n+i_0)^3 - (an^2+bn+c)^2(i_2n^2+i_1n+i_0) + 1$$

is a square in the field $\mathbb{Q}(n)$. Such an observation leads to the listed subfamily.

We have

$$\begin{aligned} Y_C(a,i,s,n,k) = {} & sn^3 + (3sk-a)n^2 \\ & + \frac{3ik^2s^2 - a^2i - 2aisk - 2s^2}{is}n \\ & + \frac{ai^2 + a^2sk - 2i^2sk - as^2k^2 + s^3k^3}{i(i^2-a^2)}. \end{aligned}$$

And we have

$$\begin{aligned} & X_C(a,i,s,n,k)^3 - T(a,i,s,n,k)^2X_C(a,i,s,n,k) + 1 - Y_C(a,i,s,n,k)^2 \\ & \quad = (-s^2 - a^2i + i^3)q(a,i,s,n,k) = 0, \end{aligned}$$

where $q \in \mathbb{Q}(a,i,s,n,k)$. This proves that the listed point is on the elliptic curve $Y^2 = X^3 - T(a,i,s,n,k)^2 + 1$ over $\mathbb{Q}(a,i,s,n,k)$ where $s^2 = i^3 - a^2i$.

For the specialization $(a,i,s,n,k) \mapsto (6,-3,9,1,1)$ we have that on the curve $E_{T(6,-3,9,1,1)} : Y^2 = X^3 - (\frac{58}{3})^2X + 1$ over $\mathbb{Q}$ the four corresponding points $(0,1)$, $(-1, \frac{58}{3})$, $(-\frac{58}{3}, 1)$, $(-\frac{35}{3}, \frac{158}{3})$ are independent, this shows that the points from the claim of the proposition are independent since the specialization map is a homomorphism. ∎

Remark 2.2. • Most of the technical claims in (i)-(iv) from the above proposition can be extracted from [5], but for the sake of completeness we have given proofs here.

- From Proposition 2.1 (iv) and the Silverman's specialization theorem [12, p. 271,Theorem 11.4] the rank of $E_t(\mathbb{Q}) \geq 3$, for all but finitely many rational values t.

From the subfamily in Proposition 2.1 (v) we get subfamilies of rank ≥ 5. We write the family in Proposition 2.1 (v) as

$$T(a,i,s,n,k) = a\left(n+k+\frac{s}{2ai}\right)^2 + \frac{8a^2i^3 - 4a^4i - a^2s^2 + i^2s^2}{4ai^2(a^2-i^2)}.$$

Now we look at the solution of

$$T(a,i,s,n,k) = T(a,i_2,s_2,n_2,k_2),$$

where

$$i_2 = \frac{a(i+a)}{i-a},\ \ s_2 = 2\frac{a^2s}{(i-a)^2}.$$

Actually, $(i_2,s_2) = (i,-s)+(a,0)$ on the curve $Y^2 = X^3 - a^2X$ over $\mathbb{Q}(a)$. So we look for solutions of

$$a\left(n+k+\frac{s}{2ai}\right)^2 + \frac{8a^2i^3 - 4a^4i - a^2s^2 + i^2s^2}{4ai^2(a^2-i^2)}$$
$$= a\left(n_2+k_2+\frac{s_2}{2ai_2}\right)^2 + \frac{8a^2i_2^3 - 4a^4i_2 - a^2s_2^2 + i_2^2s_2^2}{4ai_2^2(a^2-i_2^2)},$$

more precisely

$$\left(n+k+\frac{s}{2ai}\right)^2 - \left(n_2+k_2+\frac{s_2}{2ai_2}\right)^2$$
$$= \frac{1}{a}\left(\frac{8a^2i_2^3 - 4a^4i_2 - a^2s_2^2 + i_2^2s_2^2}{4ai_2^2(a^2-i_2^2)} - \frac{8a^2i^3 - 4a^4i - a^2s^2 + i^2s^2}{4ai^2(a^2-i^2)}\right).$$

This leads to the solution of the equation $A^2 - B^2 = p$, which is

$$(A,B) = \left(\frac{1}{2}\frac{u^2p - 2up + 2u + u^2 + p + 1}{u^2-1}, -\frac{1}{2}\frac{u^2p - u^2 - 2up - 2u + p - 1}{(u^2-1)}\right),$$

where on the other hand $A = n + k + \frac{s}{2ai}$ and $B = n_2 + k_2 + \frac{s_2}{2ai_2}$ (coming from the squares in $T(a,i,s,n,k)$ and $T(a,i_2,s_2,n_2,k_2)$ respectively), and p is as below.

Proposition 2.3. *Let*

$$T(a,i,s,u) = a\left(\frac{1}{2}\frac{u^2p - 2up + 2u + u^2 + p + 1}{u^2-1}\right)^2$$
$$+ \frac{8a^2i^3 - 4a^4i - a^2s^2 + i^2s^2}{4ai^2(a^2-i^2)},$$

where

$$i_2(a,i) = \frac{a(i+a)}{i-a}, \qquad s_2(a,i,s) = 2\frac{a^2s}{(i-a)^2},$$

$$p(a,i,s,u) = \frac{1}{4a}\left(\frac{8a^2i_2^3 - 4a^4i_2 - a^2s_2^2 + i_2^2s_2^2}{ai_2^2(a^2-i_2^2)} - \frac{8a^2i^3 - 4a^4i - a^2s^2 + i^2s^2}{ai^2(a^2-i^2)}\right)$$
$$= \frac{1}{4}\frac{(a^2+2ai-i^2)(i^5+6ai^4-6a^2i^3-i^2s^2-6i^2a^3+5a^4i-2ias^2+a^2s^2)}{a^2i^2(a^2-i^2)^2}.$$

The elliptic curve $Y^2 = X^3 - T(a,i,s,u)^2X + 1$ over the function field $\mathbb{Q}(a,i,s,u)$ where $s^2 = i^3 - a^2i$ has rank ≥ 5 with five independent points: the three generators $(0,1)$, $(-1,T)$, $(-T,1)$ provided in Proposition 2.1 (iv) and two additional points $C(a,i,s,u)$ and $D(a,i,s,u)$ (notion from Proposition 2.1 (v)) where the first coordinates are

$$X_C(a,i,s,u) = \frac{i}{4}\left(\frac{u^2p - 2up + 2u + u^2 + p + 1}{u^2-1} - \frac{s}{ia}\right)^2 + \frac{i^2}{a^2-i^2},$$
$$X_D(a,i,s,u) = \frac{a(i+a)}{i-a}\left(\frac{1}{2}\frac{u^2p - u^2 - 2up - 2u + p - 1}{u^2-1} + \frac{s}{i^2-a^2}\right)^2 - \frac{(i+a)^2}{4ai}.$$

Proof of Proposition 2.3. For the second coordinate of C and D we can take the second coordinate from the proof of Proposition 2.1 (v), specifically $Y(a,i,s,0,A-\frac{s}{2ai})$ and $Y(a,i_2,s_2,0,B-\frac{s}{i^2-a^2})$, respectively. With the specialisation $(a,i,s,u) \mapsto (6,-2,8,3)$ we prove that the above listed five points on the elliptic curve (over $\mathbb{Q}(a,i,s,u)$ where $s^2 = i^3 - a^2i$) are independent, since the specialization gives the elliptic curve

$$E_{T(6,-2,8,3)} : Y^2 = X^3 - \left(\frac{239}{32}\right)^2 X + 1$$

with the corresponding five independent points $(0,1)$, $(-1,\frac{239}{32})$, $(-\frac{239}{32},1)$, $(-\frac{925}{288},\frac{20953}{1728})$, $(-\frac{299}{64},\frac{6469}{512})$. ∎

Remark 2.4. The variety

$$s^2 = i^3 - a^2i$$

from Proposition 2.1 (v) can be considered as an elliptic curve over the field $\mathbb{Q}(a)$. In fact, it is the well-known "congruent number" elliptic curve. The torsion subgroup of this elliptic curve is equal $\{O,\ (0,0),\ (a,0),\ (-a,0)\}$. Nontrivial points on this variety $s^2 = i^3 - a^2i$ can easily be obtained, for example $(a,i,s) = (6,-2,8)$ is a point on the variety. We also have parametrisations of this variety (see e.g. [3]), for example:

$$\begin{cases} a(t) = t(t^2-1), \\ i(t) = -t^2+1, \\ s(t) = (t^2-1)^2, \end{cases} .$$

For this parametrization we get that Proposition 2.1 (v) and Proposition 2.3 transform into:

COROLLARY 2.5. (i) *Let*

$$T(t,n,k) = (t^3-t)n^2 + (t^2-1)(2tk-1)n + (t^3-t)k^2 - (t^2-1)k + \frac{(t^3-2t)}{(t^2-1)},$$

and

$$X_C(t,n,k) = -(t^2-1)n^2 - 2(t^2-1)kn - (t^2-1)k^2 + \frac{1}{t^2-1}.$$

The elliptic curve

$$Y^2 = X^3 - T(t,n,k)^2X + 1$$

over $\mathbb{Q}(t,n,k)$ *has rank* ≥ 4 *with four independent points with first coordinates* $0, -1, -T(t,n,k), X_C(t,n,k)$.

(ii) *Let*
$T(t,u) = ((t^8 - 2t^7 + 4t^5 - 3t^4 - 2t^3 - 2t^2 + 8t)u^4 + (4t^8 - 8t^7 - 4t^6 + 16t^5 - 8t^4 + 8t^3 - 4t^2 - 16t - 4)u^3 + (6t^8 - 12t^7 - 8t^6 + 24t^5 + 6t^4 - 44t^3 + 28t^2 + 16t + 8)u^2 + (4t^8 - 8t^7 - 4t^6 + 16t^5 - 8t^4 + 8t^3 - 4t^2 - 16t - 4)u + t^8 - 2t^7 + 4t^5 - 3t^4 - 2t^3 - 2t^2 + 8t)/(4t(u^2-1)^2(t+1)(t-1)^3)$,
and
$X_C(t,u) = -((t^4 - 5t^2 + 4t + 2)u^2 + (2t^4 - 2t^3 - 2t - 2)u + t^4 - 2t^3 + t^2 + 2t)((t^4 - t^2 + 2)u^2 + (2t^4 - 2t^3 - 2t - 2)u + t^4 - 2t^3 - 3t^2 + 6t)/(4t^2(u^2 - 1)^2(t+1)(t-1)^3)$,
$X_D(t,u) = -((t^4 - 2t^3 + 3t^2 - 2t - 2)u^2 + (2t^4 - 2t^3 - 4t^2 + 6t + 2)u + t^4 - 3t^2)((t^4 + t^2 - 4t)u^2 + (2t^4 - 2t^3 - 4t^2 + 6t + 2)u + t^4 - 2t^3 - t^2 + 2t - 2)/(4t(u^2-1)^2(t^2-1)^2)$.
The elliptic curve

$$Y^2 = X^3 - T(t,u)^2X + 1$$

over $\mathbb{Q}(t,u)$ *has rank* ≥ 5 *with five independent points with first coordinates* 0, -1, $-T(t,u)$, $X_C(t,u)$, $X_D(t,u)$.

PROOF OF COROLLARY 2.5. (i) For the specialization $(t,n,k) \mapsto (2,1,1)$ the elliptic curve $E_{T(2,1,1)} : Y^2 = X^3 - \left(\frac{58}{3}\right)^2 X + 1$ is a curve over $\mathbb{Q}$ for which the four listed points are $(0,1)$, $(-1, \frac{58}{3})$, $(-\frac{58}{3}, 1)$, $(-\frac{35}{3}, \frac{158}{3})$ and are independent. This proves that for the elliptic curve $Y^2 = X^3 - T(t,n,k)^2X + 1$ over $\mathbb{Q}(t,n,k)$ the four points from the claim of the corollary are independent.

(ii) The specialization $(t, u) \mapsto (2, 2)$ gives the elliptic curve $E_{T(2,2)} : Y^2 = X^3 - \left(\frac{404}{27}\right)^2 X + 1$ over $\mathbb{Q}$ for which the five listed points $(0, 1)$, $(-1, \frac{404}{27})$, $(-\frac{404}{27}, 1)$, $(-\frac{247}{27}, \frac{2902}{81})$, $(-\frac{518}{81}, \frac{24949}{729})$ are independent. This proves that for the elliptic curve $Y^2 = X^3 - T(t, u)^2X + 1$ over $\mathbb{Q}(t, u)$ the five points from the claim of the corollary are independent. □

3. A subfamily of higher rank

- In [5], rational functions

$$M_1(m) = \frac{2}{75}m^2 + m + 8,$$

$$M_2(m) = \frac{1830m^4 - 64641m^3 + 907768m^2 - 5882331m + 15154230}{30(m^2 - 91)^2},$$

  are given such that the rank of $E_{M(m)}$ over $\mathbb{Q}(m)$ is ≥ 4 and ≥ 5, respectively.
- The first rational function $M_1(m)$ is equal $T\left(6, 18, 72, m, -\frac{16}{15}m - \frac{19}{12}\right)$ from Proposition 2.1 (v) and the fourth listed point $(\frac{4}{75}m^2 + \frac{8}{5}m + 12, \frac{4}{375}m^3 + \frac{34}{75}m^2 + \frac{32}{5}m + 31)$ from [5, Propositon 5.2.1.] is equal $-(0, 1) - (-T, 1) - (\frac{2}{25}m^2 + \frac{19}{5}m + 44, \frac{1}{375}(8m^3 + 580m^2 + 13800m + 107625))$, where the last point is the fourth independent point from Proposition 2.1 (v).
- The second rational function $M_2(m)$ is equal

$$T\left(120, 180, 1800, \frac{1}{60}\frac{16m^2 - 743m + 6461}{m^2 - 91}, \frac{5}{12}\right)$$

  and it has two extra points R_4 with first coordinate

$$\frac{1}{2}\frac{130m^4 - 4785m^3 + 70188m^2 - 469539m + 1222158}{(m^2 - 91)^2}$$

  and R_5 with first coordinate

$$-\frac{1}{150}\frac{(57m^2 - 743m + 2730)(42m^2 - 743m + 4095)}{(m^2 - 91)^2}.$$

  The fifth point R_5 in [5] is equal $(0, 1) + (-1, T) + (-T, 1) - C$, where C is the fourth independent point from Proposition 2.1 (v).
- In [5] an elliptic surface over a curve is found for which the Mordell-Weil group has rank ≥ 6. Here we give a new example of an infinite family of curves with rank ≥ 6.

Corollary 3.1. *The elliptic curve given by the equation*

$$Y^2 = X^3 - \left(210n^2 + 187n + \frac{275}{7}\right)^2 X + 1$$

over the function field $\mathbb{Q}(m,n)$ *where*

$$\big((m^2-91)(420n+187)\big)^2 = 53209m^4 - 1809948m^3 + 25059146m^2 - 164705268m + 440623729,$$

has rank at least six with six independent points with first coordinates $0, -1,$

$$-\left(210n^2+187n+\frac{275}{7}\right),$$

$$\frac{1}{2}\frac{130m^4-4785m^3+70188m^2-469539m+1222158}{(m^2-91)^2},$$

$$-\frac{1}{150}\frac{(57m^2-743m+2730)(42m^2-743m+4095)}{(m^2-91)^2}, 294n^2+245n+49.$$

Proof of Corollary 3.1. Here we will intersect $M_2(m)$ with $T(210, 294, 3528, n, \frac{5}{12})$ from Proposition 2.1 (v), to get a subfamily of higher rank:

$$M_2(m) = T\left(210, 294, 3528, n, \frac{5}{12}\right) = 210n^2+187n+\frac{275}{7}$$

gives

$$\big((m^2-91)(420n+187)\big)^2 = 53209m^4 - 1809948m^3 + 25059146m^2 - 164705268m + 440623729, \tag{3.1}$$

so (m,n) on (3.1) give six points listed in the claim of the corollary (where the fourth and fifth come from [5] and the last is from Proposition 2.1 (v)). The curve

$$V^2 = 53209U^4 - 1809948U^3 + 25059146U^2 - 164705268U + 440623729$$

has a rational point $(\frac{71}{9}, \frac{346490}{81})$, so it transforms into the elliptic curve

$$Y^2 = X^3 - X^2 - 312055478905X - 66993477540839303.$$

of rank 1 generated by the point $\left(\frac{21246300582064}{12649337}, \frac{257606668421579637}{1264933}\right)$, which corresponds to $(m,n) = (\frac{819}{71}, -\frac{3911}{4893})$.

The specialization $(m,n) \mapsto \left(\frac{819}{71}, -\frac{3911}{4893}\right)$ gives the elliptic curve

$$E_{T(210,294,3528,-\frac{3911}{4893},\frac{5}{12})} = E_{M_2(\frac{819}{71})} : Y^2 = X^3 - \left(\frac{1301974}{54289}\right)^2 X + 1$$

with corresponding six independent points with first coordinates 0, -1, $-\frac{1301974}{54289}$, $\frac{265691040}{9174841}$, $-\frac{397026}{54289}$, $\frac{2226040}{54289}$. It proves that the six points from the claim of the corollary are independent. $\square$

REMARK 3.2. Points (m, n) in the above corollary can be obtained by the transformation

$$m = \frac{1730382402X + 2460079Y + 533615416078542}{50252248X + 311841Y + 13066029905888},$$

where (X, Y) is a point on the curve

$$Y^2 = X^3 - X^2 - 312055478905X - 66993477540839303.$$

This elliptic curve has rank 1 with generator $\left(\frac{21246300582064}{12649337}, \frac{25760668421579637}{1264933}\right)$ and torsion subgroup of order four generated by $(644929, 0)$ and $(-312181, 0)$. The value n can be obtained from (3.1).

4. Specializations with high rank

The highest rank found of the elliptic curve $E_t : Y^2 = X^3 - t^2X + 1$ over $\mathbb{Q}$ is ≥ 11 and is obtained for $t = \frac{23687}{3465}, \frac{86444}{833}, \frac{72269}{123}$. For example, for $t = \frac{72269}{123}$ we get the elliptic curve

$$Y^2 = X^3 - \left(\frac{72269}{123}\right)^2 X + 1$$

and eleven independent points

$$\left(-1, \frac{72269}{123}\right), \left(-\frac{601}{123}, \frac{159743}{123}\right), \left(-\frac{793}{123}, \frac{61163}{41}\right), \left(-\frac{7025}{123}, \frac{543577}{123}\right),$$
$$\left(\frac{72269}{123}, 1\right), \left(\frac{72515}{123}, \frac{144907}{123}\right), \left(\frac{1889}{3}, \frac{698807}{123}\right), \left(\frac{24568}{41}, \frac{354287}{123}\right),$$
$$\left(\frac{226895}{369}, \frac{4977013}{1107}\right), \left(-\frac{1133}{5043}, \frac{57582829}{206763}\right), \left(-\frac{328949}{5043}, \frac{975094283}{206763}\right).$$

The rank is actually equal 11 with the assumption of the Birch-Swinnerton-Dyer conjecture and the Generalized Riemann hypothesis. This was proved using the commands `Roha` which gives that the rank is odd and `Mest` which gives rank < 12.85 (both in Apecs [2]), where the later uses the Birch-Swinnerton-Dyer conjecture and the Generalized Riemann hypothesis. The same holds for $t = \frac{86444}{833}, \frac{23687}{3465}$.

This curve is found using the sieve method ([4, 6, 8]) which states that one may expect that high rank curves have large Mestre-Nagao sum, which is given by the formula

$$S(N, E) = \sum_{p \leq n, p \text{ prime}} \frac{2 - a_p}{p + 1 - a_p} \log(p),$$

where $a_p = a_p(E) = p + 1 - \sharp E(\mathbb{F}_p)$. This expectation has been experimentally verified, and it is related to the Birch and Swinnerton-Dyer conjecture. Provided N is not too large, $S(N, E)$ can be calculated using Pari ([9]). Here we observed $t = \frac{t_1}{t_2}$ $(1 \leq t_2 \leq 10000,\ 1 \leq t_1 \leq 90000)$, and elliptic curves E_t

with $S(523, E_t) > 23$ for which $S(1979, E_t) \geq 47$. The lower bound for the rank was calculated using the command `Seek1` in Apecs [2]. We also observed integers $1 \leq t \leq 21819$, and elliptic curves E_t with $S(523, E_t) > 23$ for which $S(1979, E_t) \geq 34$. Here is a list of values t we obtained for which the rank is ≥ 8:

rank	t
≥ 8	$\frac{3665}{4374}, \frac{6355}{2809}, \frac{21507}{8125}, \frac{833}{10}$, 3778, 4972, 5476, 5846, 5901, 6569, 7324, 7609, 8255, 8617, 8627, 8951, 9598, 10804, 12755, 13143, 14137, 14358, 14401, 15052, 17671, 19406, 19489, 19744, 21168
≥ 9	$\frac{1663}{5547}, \frac{1187}{1800}, \frac{1609}{1330}, \frac{3317}{2523}, \frac{2647}{1920}, \frac{4639}{3362}, \frac{3104}{1445}, \frac{9127}{2625}, \frac{28793}{7920}, \frac{12589}{2873}, \frac{33233}{7098}, \frac{34859}{2738}, \frac{59973}{3325}, \frac{39725}{1083}, \frac{29049}{224}, \frac{18907}{104}, \frac{48808}{75}$, 11416, 16228, 20529
≥ 10	$\frac{317}{7000}, \frac{5443}{2662}, \frac{24733}{7680}, \frac{4951}{966}, \frac{3581}{416}, \frac{49049}{1632}, \frac{85717}{2625}, \frac{15121}{352}, \frac{56263}{950}, \frac{14179}{138}, \frac{10343}{65}, \frac{20798}{105}$
≥ 11	$\frac{23687}{3465}, \frac{86444}{833}, \frac{72269}{123}$

For the integer values t in the above table the exact value of the rank was calculated using *mwrank* and it is equal to the lower bound in the table.

In [5] the highest rank is obtained for $t = 347, 443$ and is equal 7.

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