

ON FUNCTIONAL EQUATIONS RELATED TO
DERIVATIONS IN SEMIPRIME RINGS AND STANDARD
OPERATOR ALGEBRAS

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ABSTRACT. In this paper functional equations related to derivations on semiprime rings and standard operator algebras are investigated. We prove, for example, the following result, which is related to a classical result of Chernoff. Let X be a real or complex Banach space, let $L(X)$ be the algebra of all bounded linear operators of X into itself and let $A(X) \subset L(X)$ be a standard operator algebra. Suppose there exist linear mappings $D, G : A(X) \rightarrow L(X)$ satisfying the relations $D(A^3) = D(A^2)A + A^2G(A)$, $G(A^3) = G(A^2)A + A^2D(A)$ for all $A \in A(X)$. In this case there exists $B \in L(X)$ such that $D(A) = G(A) = [A, B]$ holds for all $A \in A(X)$.

Throughout, R will represent an associative ring with center $Z(R)$. As usual we write $[x, y]$ for $xy - yx$. Given an integer $n \geq 2$, a ring R is said to be n -torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that a ring R is prime, if for $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_s the symmetric Martindale ring of quotients. For the explanation of Q_s we refer the reader to [2]. Let A be an algebra over the real or complex field and let B be a subalgebra of A . A linear mapping $D : B \rightarrow A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in B$. In case we have a ring R , an additive mapping $D : R \rightarrow R$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [x, a]$ holds

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for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([9]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [5]. Cusack ([8]) generalized Herstein's result to 2-torsion free semiprime rings (see also [3] for an alternative proof). For results in the spirit of Herstein's theorem we refer to [1, 6, 10, 11]. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring, is called a Jordan triple derivation in case $D(xy x) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation D on an arbitrary 2-torsion free ring R is a Jordan triple derivation (see, for example, [5]). Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem.

Brešar ([4]) has proved the following result.

THEOREM 1. *Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ be a Jordan triple derivation. In this case D is a derivation.*

Motivated by Theorem 1, Vukman, Kosi-Ulbl and Eremita ([16]) have proved the following result.

THEOREM 2. *Let R be a 2-torsion free semiprime ring. Suppose there exists an additive mapping $T : R \rightarrow R$ such that*

$$T(xy x) = T(x)yx - xT(y)x + xyT(x)$$

for all $x, y \in R$. Then there exists $q \in Q_s$ such that $2T(x) = qx + xq$ for all $x \in R$.

In the same article, Vukman, Kosi-Ulbl and Eremita proved the result below, which is an immediate consequence of Theorem 1 and Theorem 2.

COROLLARY 3. *Let R be a 2-torsion free semiprime ring. If $D, G : R \rightarrow R$ are additive mappings such that*

$$D(xy x) = D(x)yx - xG(y)x + xyD(x),$$

$$G(xy x) = G(x)yx - xD(y)x + xyG(x)$$

for all $x \in R$, then there exists a derivation $S : R \rightarrow R$ and $q \in Q_s$ such that

$$4D(x) = qx + xq + S(x),$$

$$4G(x) = qx + xq - S(x)$$

for all $x \in R$.

Theorem 1 also motivated Vukman ([20]), who has recently proved the following result.

THEOREM 4. Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ be an additive mapping. Suppose that either

$$D(xy x) = D(xy)x + xyD(x)$$

or

$$D(xy x) = D(x)yx + xD(yx)$$

holds for all pairs $x, y \in R$. In both cases D is a derivation.

It is our aim in this paper to generalize Theorem 4, but first we will prove the following theorem, which is conceptually linked with Theorem 2.

THEOREM 5. Let R be a 2-torsion free semiprime ring and let $T : R \rightarrow R$ be an additive mapping. Suppose that either

$$(1) \quad T(xy x) = T(xy)x - xyT(x)$$

or

$$(2) \quad T(xy x) = T(x)yx - xT(yx)$$

holds for all pairs $x, y \in R$. In both cases $T = 0$.

PROOF. We will restrict our attention on the relation (1), the proof in case we have the relation (2) is similar and will therefore be omitted.

Linearization of the relation (1) gives

$$T(xyz + zyx) = T(xy)z + T(zy)x - xyT(z) - zyT(x), x, y, z \in R$$

and in particular for $z = x^2$

$$(3) \quad T(xy x^2 + x^2 yx) = T(xy)x^2 + T(x^2 y)x - xyT(x^2) - x^2 yT(x), x, y \in R.$$

Putting $xy + yx$ for y in (1) and applying the relation (1) we obtain

$$\begin{aligned} T(xy x^2 + x^2 yx) &= T(x^2 y + xyx)x - x^2 yT(x) - xyxT(x) \\ &= T(x^2 y)x + T(xy)x^2 - xyT(x)x - x^2 yT(x) - xyxT(x), x, y \in R. \end{aligned}$$

We have therefore

$$(4) \quad T(xy x^2 + x^2 yx) = T(x^2 y)x + T(xy)x^2 - xyT(x)x - x^2 yT(x) - xyxT(x), x, y \in R.$$

By comparing (3) and (4) we obtain

$$(5) \quad xyA(x) = 0, x, y \in R,$$

where $A(x) = T(x^2) - T(x)x - xT(x)$. Right multiplication of the above relation by x and left multiplication by $A(x)$ gives $A(x)xyA(x)x = 0$ for all pairs $x, y \in R$, whence it follows

$$(6) \quad A(x)x = 0, x \in R$$

by semiprimeness of R . The substitution $A(x)yx$ for y in the relation (5) gives $xA(x)yxA(x) = 0$ for all pairs $x, y \in R$, which gives

$$(7) \quad xA(x) = 0, x \in R.$$

The linearization of the relation (6) gives

$$B(x, y)x + A(x)y + B(y, x)y + A(y)x = 0$$

for all pairs $x, y \in R$, where $B(x, y)$ denotes $T(xy + yx) - T(x)y - xT(y) - T(y)x - yT(x)$. Putting in the above relation $-x$ for x and comparing the relation so obtained with the above relation, one obtains

$$B(x, y)x + A(x)y = 0.$$

Right multiplication of the above relation by $A(x)$ gives, because of the relation (7), $A(x)yA(x) = 0$ for all pairs $x, y \in R$, whence it follows $A(x) = 0$ for all $x \in R$. In other words, T is a Jordan derivation. By Cusack's generalization of Herstein's theorem T is a derivation. From the relation (1) we obtain $T(x)yx + xT(y)x + xyT(x) = T(x)yx + xT(y)x - xyT(x)$, which reduces to

$$(8) \quad xyT(x) = 0, x, y \in R.$$

From the relation (8) one obtains

$$(9) \quad T(x)x = 0, x \in R$$

and

$$(10) \quad xT(x) = 0, x \in R,$$

(see how the relations (6) and (7) were obtained from the relation (5)). The linearization of the relation (9) gives

$$T(y)x + T(x)y = 0, x, y \in R.$$

Right multiplication of the above relation by $T(x)$ gives because of the relation (10) $T(x)yT(x) = 0$ for all pairs $x, y \in R$, which gives $T = 0$. \square

We proceed with the following corollary, which is a generalization of Theorem 4.

COROLLARY 6. *Let R be a 2-torsion free semiprime ring and let $D, G : R \rightarrow R$ be additive mappings satisfying either the relations*

$$(11) \quad \begin{aligned} D(xyx) &= D(xy)x + xyG(x), \\ G(xyx) &= G(xy)x + xyD(x) \end{aligned}$$

or the relations

$$(12) \quad \begin{aligned} D(xyx) &= D(x)yx + xG(yx), \\ G(xyx) &= G(x)yx + xD(yx) \end{aligned}$$

for all pairs $x, y \in R$. In both cases D and G are derivations and $D = G$.

PROOF. We will restrict our attention on the relations (11), the proof in case we have the relations (12) is similar and will therefore be omitted. We have therefore the relations

$$(13) \quad D(xyx) = D(xy)x + xyG(x), x, y \in R,$$

$$(14) \quad G(xyx) = G(xy)x + xyD(x), x, y \in R.$$

Subtracting the relation (14) from the relation (13) one obtains

$$(15) \quad T(xyx) = T(xy)x - xyT(x), x, y \in R,$$

where $T(x)$ denotes $D(x) - G(x)$. Applying Theorem 5 gives $D = G$, which reduces the relation (13) to

$$D(xyx) = D(xy)x + xyD(x), x, y \in R.$$

Now, it follows from Theorem 4 that D is a derivation, which completes the proof. \square

We proceed with the following conjecture.

CONJECTURE 7. *Let R be a semiprime ring with suitable torsion restrictions and let $D, G : R \rightarrow R$ be additive mappings satisfying either the relations*

$$D(x^3) = D(x^2)x + x^2G(x),$$

$$G(x^3) = G(x^2)x + x^2D(x)$$

or the relations

$$D(x^3) = D(x)x^2 + xG(x^2),$$

$$G(x^3) = G(x)x^2 + xD(x^2)$$

for all $x \in R$. In both cases D and G are derivations and $D = G$.

Our next result is related to the conjecture above.

THEOREM 8. *Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X . Suppose there exist linear mappings $D, G : A(X) \rightarrow L(X)$ satisfying either the relations*

$$(16) \quad D(A^3) = D(A^2)A + A^2G(A),$$

$$G(A^3) = G(A^2)A + A^2D(A)$$

or the relations

$$(17) \quad D(A^3) = D(A)A^2 + AG(A^2),$$

$$G(A^3) = G(A)A^2 + AD(A^2)$$

for all $A \in A(X)$. In both cases there exists some fixed $B \in L(X)$, such that $D(A) = G(A) = [A, B]$ holds for all $A \in A(X)$, which means that D and G are linear derivations.

Theorem 8 is related to the result below first proved by Chernoff ([7]) (see also [12–14, 17–19]).

THEOREM 9. *Let X be a real or complex Banach space, let $A(X)$ be a standard operator algebra on X and let $D : A(X) \rightarrow L(X)$ be a linear derivation. In this case D is of the form $D(A) = [A, B]$ for all $A \in A(X)$ and some fixed $B \in L(X)$.*

In the proof of Theorem 8 we use Herstein's theorem, Theorem 9, Lemma 10 and methods which are similar to those used in [17 - 20].

LEMMA 10. *Let R be a semiprime ring and let $f : R \rightarrow R$ be an additive mapping. If either*

$$f(x)x = 0$$

or

$$xf(x) = 0$$

holds for all $x \in R$, then $f = 0$.

For the proof of Lemma 10 we refer to [15].

PROOF OF THEOREM 8. We will focus our attention to the relations (16). The proof in case we have the relations (17) is similar and will therefore be omitted. We have therefore the relations

$$(18) \quad D(A^3) = D(A^2)A + A^2G(A),$$

$$(19) \quad G(A^3) = G(A^2)A + A^2D(A)$$

for all $A \in A(X)$. Subtracting the relation (19) from the relation (18), we obtain

$$(20) \quad T(A^3) = T(A^2)A - A^2T(A)$$

for all $A \in A(X)$, where $T(A)$ denotes $D(A) - G(A)$. It is our aim to show that $T(A) = 0$ for all $A \in A(X)$. Let us first consider the restriction of T on $F(X)$. Let A be from $F(X)$ and $P \in F(X)$ a projection with $AP = PA = A$. Putting P for A in relation (20), we obtain

$$(21) \quad T(P) = T(P)P - PT(P).$$

Right multiplication of the relation (21) by P gives

$$(22) \quad PT(P)P = 0.$$

Left multiplication of the relation (21) by P gives because of (22)

$$(23) \quad PT(P) = 0.$$

Putting $A + P$ for A in the relation (20), we obtain after some calculations

$$\begin{aligned} 3T(A^2) + 3T(A) &= T(A^2)P + 2T(A)A + 2T(A)P + T(P)A \\ &\quad - A^2T(P) - 2AT(A) - 2AT(P) - PT(A). \end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation, we obtain

$$3T(A) = 2T(A)P + T(P)A - 2AT(P) - PT(A).$$

Left multiplication of the relation (23) by A and considering $A = AP$ gives $AT(P) = APT(P) = 0$, which reduces the relation above to

$$(24) \quad 3T(A) = 2T(A)P + T(P)A - PT(A).$$

Multiplying the relation (24) from both sides by P and considering (23), we obtain

$$(25) \quad PT(A)P = 0.$$

Left multiplication of the relation (24) by P and considering both (23) and (25) leads us to

$$(26) \quad PT(A) = 0.$$

Left multiplication of the relation (26) by A gives

$$(27) \quad AT(A) = 0.$$

Applying (26) in the relation (24) leads us to

$$(28) \quad 3T(A) = 2T(A)P + T(P)A.$$

Right multiplication of the relation (28) by P gives $T(A)P = T(P)A$, which reduces the relation (28) to

$$(29) \quad T(A) = T(A)P.$$

From the relation (29) one can conclude that T maps $F(X)$ into itself. We have therefore a linear mapping T , which maps $F(X)$ into itself, satisfying the relation (27). Applying Lemma 10 we can conclude that $T(A) = 0$ for all $A \in F(X)$. It remains to prove that $T(A) = 0$ holds for all $A \in A(X)$ as well. The mapping T on $A(X)$ is linear, satisfies the relation (20) and, as we already showed, vanishes on $F(X)$. Let $A \in A(X)$, let P be a one-dimensional projection and let us introduce $S \in A(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to see that $T(S) = T(A)$ and $T(S^2) = T(A^2)$. By the equation (20) we now have

$$\begin{aligned} T(S^2)S - S^2T(S) &= T(S^3) = T(S^3 + P) \\ &= T((S + P)^3) = T((S^2)(S + P) - (S + P)^2T(S)) \\ &= T(S^2)S + T(S^2)P - S^2T(S) - PT(S). \end{aligned}$$

We have therefore $T(S^2)P - PT(S) = 0$ and since $T(S) = T(A)$ and $T(S^2) = T(A^2)$, we obtain

$$(30) \quad T(A^2)P - PT(A) = 0.$$

Putting $-A$ for A in the relation (30) and comparing the relation so obtained with (30), we get

$$(31) \quad PT(A) = 0.$$

As P is an arbitrary one-dimensional projection, we have $T(A) = 0$ for all $A \in A(X)$, which was our intention to prove. We have therefore proved that $D = G$. This ascertainment enables us to combine (18) and (19) into one relation

$$(32) \quad D(A^3) = D(A^2)A + A^2D(A).$$

Let us first consider the restriction of D on $F(X)$. Let A be from $F(X)$ and let $P \in F(X)$ be a projection with $AP = PA = A$. Putting P for A in the relation (32), we obtain $D(P) = D(P)P + PD(P)$. Any multiplication of the last relation by P leads us to

$$(33) \quad PD(P)P = 0.$$

The substitution $A + P$ for A in the relation (32) gives

$$(34) \quad \begin{aligned} & 3D(A^2) + 3D(A) + D(P) \\ & = D(A^2)P + 2D(A)P + 2D(A)A + D(P)A + D(P)P \\ & \quad + A^2D(P) + 2AD(A) + 2AD(P) + PD(A) + PD(P). \end{aligned}$$

Previously mentioned relation $D(P) = D(P)P + PD(P)$ reduces the equation (34) to

$$\begin{aligned} 3D(A^2) + 3D(A) & = D(A^2)P + 2D(A)P + 2D(A)A + D(P)A \\ & \quad + A^2D(P) + 2AD(A) + 2AD(P) + PD(A). \end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation, we obtain

$$(35) \quad 3D(A^2) = D(A^2)P + 2D(A)A + A^2D(P) + 2AD(A)$$

and

$$(36) \quad 3D(A) = 2D(A)P + D(P)A + 2AD(P) + PD(A).$$

Right multiplication of the relation (35) by P gives

$$(37) \quad 3D(A^2)P = D(A^2)P + 2D(A)A + A^2D(P)P + 2AD(A)P,$$

which is, after considering (33) and dividing by 2, reduced to

$$(38) \quad D(A^2)P = D(A)A + AD(A)P.$$

The relation (35) gives because of (38)

$$(39) \quad 3D(A^2) = 3D(A)A + AD(A)P + A^2D(P) + 2AD(A).$$

Left multiplication of the relation (36) by A , considering (33) and dividing by 2 leads us to

$$(40) \quad AD(A) = AD(A)P + A^2D(P).$$

The relation (39) gives because of (40)

$$3D(A^2) = 3D(A)A + 3AD(A),$$

which reduces to

$$(41) \quad D(A^2) = D(A)A + AD(A).$$

From the relation (36) one can conclude that D is a linear mapping, which maps $F(X)$ into itself. By the relation (41) D is a Jordan derivation on $F(X)$. Since $F(X)$ is prime, it follows that D is a derivation by Herstein's theorem. Applying Theorem 9 one can conclude that D is of the form

$$(42) \quad D(A) = [A, B]$$

for all $A \in F(X)$ and some fixed $B \in L(X)$. It remains to prove that the relation (42) holds for all $A \in A(X)$ as well. For this purpose we introduce $D_1 : A(X) \rightarrow L(X)$ by $D_1(A) = [A, B]$ and consider $D_0 = D - D_1$. The mapping D_0 is linear, satisfies the relation (32) and it vanishes on $F(X)$. Our aim is to prove that D_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be a one-dimensional projection and let us introduce $S \in A(X)$ by $S = A + PAP - (AP + PA)$. We have $SP = PS = 0$. It is easy to see that $D_0(S) = D_0(A)$ and $D_0(S^2) = D_0(A^2)$. The relation (32) now leads us to

$$\begin{aligned} D_0(S^2)S + S^2D_0(S) &= D_0(S^3) = D_0(S^3 + P) = D_0((S + P)^3) \\ &= D_0(S^2)S + D_0(S^2)P + S^2D_0(S) + PD_0(S). \end{aligned}$$

We have therefore $D_0(S^2)P + PD_0(S) = 0$ and since $D_0(S) = D_0(A)$ and $D_0(S^2) = D_0(A^2)$, we arrive at

$$(43) \quad D_0(A^2)P + PD_0(A) = 0.$$

Putting $-A$ for A in the relation (43) and subtracting the relation so obtained from the relation (43), we get

$$(44) \quad PD_0(A) = 0.$$

Since P is an arbitrary one-dimensional projection, it follows that $D_0(A) = 0$ for all $A \in A(X)$, which concludes the proof of the theorem. \square

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