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ON (ANTI-)MULTIPLICATIVE GENERALIZED DERIVATIONS

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Abstract. Let R be a semiprime ring and let $F, f : R \to R$ be (not necessarily additive) maps satisfying $F(xy) = F(x)y + xf(y)$ for all $x, y \in R$. Suppose that there are integers m and n such that $F(uv) = mF(u)F(v) + nF(v)F(u)$ for all u, v in some nonzero ideal I of R. Under some mild assumptions on R, we prove that there exists $c \in C(I^{\perp\perp})$ such that $c = (m+n)c^2$, $nc[I^{\perp\perp}, I^{\perp\perp}] = 0$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. The main result is then applied to the case when F is multiplicative or anti-multiplicative on I.

1. Introduction

Let R be an associative ring not necessarily with an identity element. Recall that a ring R is a *prime ring* if $aRb = 0$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and R is a *semiprime ring* if $aRa = 0$ (where $a \in R$) implies $a = 0$. A ring R is said to be *n-torsion free* (n is an integer) if $na = 0$ (where $a \in R$) implies $a = 0$. For $a, b \in R$ we shall write $[a, b] = ab - ba$.

Let M be an R-bimodule. Recall that an additive map $d : R \to M$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive map $D : R \to M$ is a *generalized derivation* if there exists a derivation $d : R \to M$ such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in R$ (this notion was introduced by Brešar in [4]). Obviously, each derivation is also a generalized derivation.

A map φ from R to a ring R' is called *multiplicative* (resp. *anti-multiplicative*) if $\varphi(xy) = \varphi(x)\varphi(y)$ (resp. $\varphi(xy) = \varphi(y)\varphi(x)$) for all $x, y \in$

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R. Thus, $\varphi : R \to R'$ is a *homomorphism* (resp. an *anti-homomorphism*) of rings if it is both additive and multiplicative (resp. anti-multiplicative). An additive map $\varphi : R \to R'$ is called a *Jordan homomorphism* if $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for all $x, y \in R$. If R' is 2-torsion free then φ is a Jordan homomorphism if and only if $\varphi(x^2) = \varphi(x)^2$ for all $x \in R$.

In 1989 Bell and Kappe ([3]) obtained the following result: if d is a derivation and d is also a homomorphism or an anti-homomorphism of a semiprime ring R, then $d = 0$. In case R is prime they have proved that a derivation $d : R \to R$, which is a homomorphism or an anti-homomorphism on some nonzero right ideal I of R, must be the zero map. This result was later generalized and extended by many authors ([1,6–8,11,12], etc.) In 2004 Rehman ([11]) treated the problem of describing a generalized derivation D of a prime ring R which is also a homomorphism or an anti-homomorphism on a nonzero ideal I of R. Later, Gusić in [8] considered a slightly more general problem and obtained the following result.

THEOREM 1.1 (I. Gusić). *Let F and f be arbitrary maps of a prime ring R such that*

$$F(xy) = F(x)\, y + x f(y) \qquad \text{for all } x, y \in R. \tag{1.1}$$

Suppose that I is a nonzero ideal of R. Then the following holds.

(a) *If F is multiplicative on I then $f = 0$, and $F = 0$ or $F = id$.*
(b) *If F is anti-multiplicative on I then $f = 0$, and $F = 0$ or $F = id$. Moreover, in the latter case R is commutative.*

Note that additivity of the maps F and f is not assumed in Theorem 1.1. However, assuming that F and f satisfy (1.1) and F is multiplicative or anti-multiplicative on a nonzero ideal I of R implies in particular that both F and f are automatically additive.

The aim of this paper is to generalize the result of Gusić ([8]) to semiprime rings. Moreover, instead of assuming that F is either multiplicative or anti-multiplicative on a nonzero ideal I of a semiprime ring R we consider the following more general condition:

$$F(uv) = mF(u)\, F(v) + nF(v)\, F(u) \quad \text{for all } u, v \in I, \tag{1.2}$$

where m and n are fixed integers (see Theorem 3.3). In particular, we shall see that both F and f are automatically additive on $I^{\perp\perp}$. Typical maps satisfying (1.1) and (1.2) are those of the form $x \mapsto cx$ when restricted to I, with c satisfying $c = (m+n)c^2$, $[c, I] = 0$ and $nc[I, I] = 0$. We shall prove that, under certain mild conditions, these maps are basically the only examples of maps satisfying (1.1) and (1.2) and that (1.2) holds for all $u, v \in I^{\perp\perp}$ as well (see Theorem 3.3).

2. Preliminaries

From now on R denotes an arbitrary semiprime ring. Our main result relies on the following characterization of generalized derivations of a semiprime ring which was obtained by Lee ([10, Theorem 3]).

Theorem 2.1 (T.-K. Lee). *Let I be a dense right ideal of a semiprime ring R. Suppose that $D : I \to Q_{mr}(R)$ is a generalized derivation with its associated derivation d. Then both D and d can be uniquely extended to a generalized derivation and a derivation of $Q_{mr}(R)$, respectively, and there exists $q \in Q_{mr}(R)$ such that*

$$D(x) = qx + d(x)$$

for all $x \in Q_{mr}(R)$.

Recall that a right ideal I of R is said to be *dense* if given any $0 \neq r_1 \in R$, $r_2 \in R$ there exists $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in I$. One defines a dense left ideal in an analogous fashion. Let us also mention that an ideal I of R is called *essential* if for every nonzero ideal J of R we have $I \cap J \neq 0$. Let I be any ideal of a semiprime ring R. Then I is dense as a right ideal if and only if I is dense as a left ideal if and only if I is essential ideal. Moreover, the left, the right and the two–sided annihilator of I in R coincide. We denote this annihilator by $I^{\perp}$. We remark that $I \cap I^{\perp} = 0$ and also that $I \oplus I^{\perp}$ is always an essential ideal of R. Thus, I is essential if and only if $I^{\perp} = 0$. We write $I^{\perp\perp}$ for $(I^{\perp})^{\perp}$. Note that each nonzero ideal I of a semiprime ring R is an essential ideal of $I^{\perp\perp}$.

By $Q_{mr}(R)$ we denote the *maximal right ring of quotients* (or *Utumi right ring of quotients*) of R. For an account on the theory of maximal rings of quotients of semiprime rings the reader is referred to [2]. Let us just recall here that any semiprime ring R can be considered as a subring of its maximal right ring of quotients $Q_{mr}(R)$. It turns out that $Q_{mr}(R)$ is a semiprime ring (or a prime ring if R is prime) with the identity element. By $C(R)$ we denote the center of $Q_{mr}(R)$, which is called the *extended centroid* of R. It turns out that $C(R)$ is a field if and only if R is prime. Furthermore, for any essential ideal I of R and any $q \in Q_{mr}(R)$, $qIq = 0$ implies $q = 0$. Namely, assume that $qIq = 0$ for some $q \neq 0$. Then there would exist $x \in I$ such that $0 \neq qx \in R$ (see [2, Proposition 2.1.7]). Therefore, $0 \neq (qx) R (qx) \subseteq qIqx$ and this would yield $qIq \neq 0$, a contradiction.

3. The results

Lemma 3.1. *Let R be a semiprime ring and suppose that $F : R \to R$ and $f : R \to R$ are maps satisfying*

$$F(xy) = F(x)y + xf(y) \qquad \text{for all } x, y \in R.$$

Then $f(xy) = f(x)y + xf(y)$ *for all* $x, y \in R$.

PROOF. For all $x, y, z \in R$ we have

$$x\big(f(yz) - f(y)z - yf(z)\big) = F(xyz) - F(x)yz - F(xy)z + F(x)yz - F(xyz) + F(xy)z = 0.$$

Since R is semiprime it follows that $f(xy) = f(x)y + xf(y)$ for all $x, y \in R$. ∎

LEMMA 3.2. *Let* R *be a semiprime ring and suppose that* $F : R \to R$ *and* $f : R \to R$ *are maps satisfying*

$$F(xy) = F(x)y + xf(y) \qquad \text{for all } x, y \in R.$$

Then for each ideal I *of* R *the following holds:*

(i) $F\left(I^{\perp}\right) \subseteq I^{\perp}$ *and* $f\left(I^{\perp}\right) \subseteq I^{\perp}$,
(ii) $F\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$ *and* $f\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$,
(iii) *if* F *is additive on* I *then* F *and* f *are additive on* $I^{\perp\perp}$.

PROOF. Let $u \in I^{\perp}$ and $v \in I$. Then $F(u)v \in I$. Since $F(0) = 0$, we have $F(u)v + uf(v) = 0$. This implies $F(u)v = -uf(v) \in I \cap I^{\perp} = 0$. Since $u \in I^{\perp}$ and $v \in I$ are arbitrary, it follows $F(I^{\perp}) \subseteq I^{\perp}$. Similarly, $vf(u) = -F(v)u \in I \cap I^{\perp} = 0$. Hence, $If(I^{\perp}) = 0$ and so $f\left(I^{\perp}\right) \subseteq I^{\perp}$. Thus, (i) holds true.

Replacing I by $I^{\perp}$ in (i) we obtain (ii).

Next, suppose that F is additive on I. Consequently, for all $x, y \in I^{\perp\perp}$ and $u \in I$ we have

$$\begin{aligned}\big(F(x+y) - F(x) - F(y)\big)u &= F((x+y)u) - (x+y)f(u) - F(xu) + xf(u) \\ &\quad - F(yu) + yf(u) \\ &= F(xu+yu) - F(xu) - F(yu) = 0.\end{aligned}$$

Since $F\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$ and since I is an essential ideal of a semiprime ring $I^{\perp\perp}$ it follows that $F(x+y) = F(x) + F(y)$ for all $x, y \in I^{\perp\perp}$. Therefore, F is additive on $I^{\perp\perp}$. Consequently,

$$\begin{aligned}x\left(f(y+z) - f(y) - f(z)\right) &= F(x(y+z)) - F(x)(y+z) - F(xy) + F(x)y \\ &\quad - F(xz) + F(x)z \\ &= 0\end{aligned}$$

for all $x \in R$, $y, z \in I^{\perp\perp}$. Thus, f is additive on $I^{\perp\perp}$. ∎

We are now ready to prove our main result.

THEOREM 3.3. *Let* I *be a nonzero ideal of a semiprime ring* R. *Let* $F : R \to R$ *and* $f : R \to R$ *be maps satisfying*

$$F(xy) = F(x)y + xf(y) \qquad \text{for all } x, y \in R.$$

Suppose that there are integers m and n such that R is $(m+n)$-torsion free and

$$F(uv) = mF(u)F(v) + nF(v)F(u) \qquad \text{for all } u, v \in I. \tag{3.3}$$

Then F and f are additive on $I^{\perp\perp}$.

Furthermore, suppose that at least one of the following holds:

(i) $m = 0$,
(ii) $n = 0$,
(iii) *R is 2-torsion free and m-torsion free,*
(iv) *R is 2-torsion free and n-torsion free.*

Then $f(I^{\perp\perp}) = 0$ and there exists $c \in C(I^{\perp\perp})$ such that $c = (m+n)c^2$, $nc[I^{\perp\perp}, I^{\perp\perp}] = 0$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. In particular, identity (3.3) holds for all $u, v \in I^{\perp\perp}$.

The proof of Theorem 3.3 consists of several steps.

Step 1. *Maps F and f are additive on $I^{\perp\perp}$.*

Proof. For $x, y \in R$, by $G(x,y)$ we denote $F(x+y) - F(x) - F(y)$. First notice that $G(xz, yz) = G(x,y)z$ for all $x, y, z \in R$. For all $x, y, z \in I$ we have

$$\begin{aligned} mG(x,y)F(z) + nF(z)G(x,y) &= F(xz+yz) - F(xz) - F(yz) \\ &= G(xz, yz) = G(x,y)z. \end{aligned}$$

This further implies

$$\begin{aligned} &mG(x,y)G(z,u) + nG(z,u)G(x,y) \\ &\quad = G(x,y)(z+u) - G(x,y)z - G(x,y)u = 0 \end{aligned} \tag{3.4}$$

for all $x, y, z, u \in I$. In particular, $(m+n)G(x,y)^2 = 0$. Since R is $(m+n)$-torsion free we get $G(x,y)^2 = 0$. Obviously, (3.4) implies

$$(m+n)(G(x,y)G(z,u) + G(z,u)G(x,y)) = 0$$

and hence

$$G(x,y)G(z,u) + G(z,u)G(x,y) = 0 \tag{3.5}$$

for all $x, y, z, u \in I$. Let $w \in R$. Setting $x = zw$, $y = uw$ in (3.5) we get

$$G(z,u)wG(z,u) = 0.$$

Consequently, $G(z,u)RG(z,u) = 0$ for all $z, u \in I$ and hence $G(I, I) = 0$. Thus, F is additive on I. According to Lemma 3.2 we may now conclude that both F and f are additive on $I^{\perp\perp}$. □

Step 2. *There exists $q \in Q_{mr}(I^{\perp\perp})$ such that $F(x) = qx + f(x)$ for all $x \in I^{\perp\perp}$.*

Proof. Since F and f are additive on $I^{\perp\perp}$ and since $F\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$, and $f\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$ (see Lemma 3.2), we may consider $F\mid_{I^{\perp\perp}}$ as a generalized derivation and $f\mid_{I^{\perp\perp}}$ as its corresponding derivation, both mapping from $I^{\perp\perp}$ to $I^{\perp\perp}$. Hence we may apply T.-K. Lee's result [10, Theorem 3] (see also Theorem 2.1) to conclude that there exists $q \in Q_{mr}\left(I^{\perp\perp}\right)$ such that $F(x) = qx + f(x)$ for all $x \in I^{\perp\perp}$. ∎

Step 3. *If $f(I^{\perp\perp}) = 0$, then there exists $q \in C(I^{\perp\perp})$ such that $q = (m+n)q^2$ and $nq^2[I^{\perp\perp}, I^{\perp\perp}] = 0$.*

Proof. We have $F(x) = qx$ for all $x \in I^{\perp\perp}$. Thus,

$$quv = mquqv + nqvqu \tag{3.6}$$

for all $u, v \in I$. Interchanging the roles of u and v, we get

$$qvu = mqvqu + nquqv, \tag{3.7}$$

and adding up (3.6) and (3.7) we obtain

$$quv + qvu = (m+n)quqv + (m+n)qvqu. \tag{3.8}$$

Multiplying (3.8) with $w \in R$ from the right, we get

$$quvw + qvuw = (m+n)quqvw + (m+n)qvquw. \tag{3.9}$$

Replacing v by vw in (3.8), we get

$$quvw + qvwu = (m+n)quqvw + (m+n)qvwqu. \tag{3.10}$$

Subtracting (3.10) from (3.9) yields

$$qv[u,w] = (m+n)qv[qu,w],$$

that is

$$qv([u,w] - (m+n)[qu,w]) = 0.$$

Replacing u by ur, $r \in R$, we get

$$qv([u,w]r + u[r,w] - (m+n)[qu,w]r - (m+n)qu[r,w]) = 0$$

which implies

$$(qv - (m+n)qvq)u[r,w] = 0. \tag{3.11}$$

Replacing v by vx, $x \in I^{\perp\perp}$, in (3.11), we get

$$(qvx - (m+n)qvxq)u[r,w] = 0. \tag{3.12}$$

Replacing u by xu in (3.11), we get

$$(qvx - (m+n)qvqx)u[r,w] = 0. \tag{3.13}$$

Subtracting (3.13) from (3.12) yields, since R is $(m+n)$-torsion free,

$$qv[q,x]u[r,w] = 0 \tag{3.14}$$

for all $u, v \in I$, $w, r \in R$ and $x \in I^{\perp\perp}$. Setting $r = qx$ and $w = x$ in (3.14) we obtain

$$qv\,[q,x]\,u\,[q,x]\,x = 0$$

for all $u, v \in I$ and $x \in I^{\perp\perp}$. In particular,

$$qv\,[q,x]\,xuqv\,[q,x]\,x = 0$$

for all $u, v \in I$ and $x \in I^{\perp\perp}$. This implies $qI\,[q,x]\,x = 0$ for each $x \in I^{\perp\perp}$. Consequently,

$$[q,x]\,xI\,[q,x]\,x = 0$$

and hence $[q,x]\,x = 0$ for each $x \in I^{\perp\perp}$. Linearizing this identity we obtain

$$[q,x]\,y + [q,y]\,x = 0 \tag{3.15}$$

for all $x, y \in I^{\perp\perp}$. Setting $y = qx$ in (3.15) we obtain $[q,x]\,qx = 0$ and hence $[q,x]^2 = 0$ for all $x \in I^{\perp\perp}$. Now, (3.15) implies $[q,x]\,[q,y]\,x = 0$ which in turn gives

$$[q,x]\,[q,y]\,z + [q,z]\,[q,y]\,x = 0 \tag{3.16}$$

for all $x, y, z \in I^{\perp\perp}$. Setting $y = zx$ in (3.16) we get

$$\begin{aligned} 0 &= [q,x]\,[q,zx]\,z + [q,z]\,[q,zx]\,x \\ &= [q,x]\,z\,[q,x]\,z + [q,x]\,[q,z]\,xz + [q,z]\,z\,[q,x]\,x + [q,z]\,[q,z]\,x^2. \end{aligned}$$

Since $[q,x]\,x = 0$, $[q,x]^2 = 0$, and $[q,x]\,[q,z]\,x = 0$ it follows that

$$[q,x]\,z\,[q,x]\,z = 0 \tag{3.17}$$

for all $x, z \in I^{\perp\perp}$. Let $t \in R$. Replacing z by $z + zt$ in (3.17) we see that

$$[q,x]\,zt\,[q,x]\,z = 0$$

for all $x, z \in I^{\perp\perp}$, $t \in R$. Since R is semiprime and $[q, I^{\perp\perp}]I^{\perp\perp} \subseteq R$ it follows that $\left[q, I^{\perp\perp}\right] I^{\perp\perp} = 0$. Since $[q, I^{\perp\perp}] \in Q_{mr}(I^{\perp\perp})$, this yields $[q, I^{\perp\perp}] = 0$. Thus, $q \in C\left(I^{\perp\perp}\right)$.

Now, (3.3) implies

$$\left(q - (m+n)\,q^2\right) u^2 = 0 \tag{3.18}$$

and hence

$$\left(q - (m+n)\,q^2\right)(uv + vu) = 0 \tag{3.19}$$

for all $u, v \in I$. Let $\alpha = q - (m+n)\,q^2 \in C(I^{\perp\perp})$. Setting $v = ur$ in (3.19), where $r \in R$, we get $\alpha uru = 0$. Consequently, $\alpha uR\alpha u = 0$ for all $u \in I$. This implies $\alpha I = 0$ and hence $\alpha = 0$. Thus, $q = (m+n)\,q^2$.

Since $q \in C(I^{\perp\perp})$ and $q = (m+n)q^2$, (3.6) implies

$$mq^2uv + nq^2vu = quv = mq^2uv + nq^2uv \tag{3.20}$$

for all $u, v \in I$. Hence, $nq^2[I, I] = 0$. Now, for all $u, v \in I$ and $w, w' \in I^{\perp\perp}$ we have

$$0 = nq^2[u, wv] = nq^2[u, w]v + nq^2 w[u, v] = nq^2[u, w]v.$$

This implies $nq^2[u, w] = 0$. Thus, $nq^2 \left[I, I^{\perp\perp}\right] = 0$ and so

$$0 = nq^2[uw', w] = nq^2[u, w]w' + nq^2 u[w', w] = nq^2 u[w', w]$$

for all $u \in I$ and $w, w' \in I^{\perp\perp}$. Hence, $nq^2 I[I^{\perp\perp}, I^{\perp\perp}] = 0$, which further implies $Inq^2[I^{\perp\perp}, I^{\perp\perp}] = 0$ and finally $nq^2[I^{\perp\perp}, I^{\perp\perp}] = 0$. ∎

STEP 4. *If $n = 0$, then $f\left(I^{\perp\perp}\right) = 0$ and there exists $c \in C(I^{\perp\perp})$ such that $c = mc^2$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$.*

PROOF. By Step 2, there exists $c \in Q_{mr}\left(I^{\perp\perp}\right)$ such that $F(x) = cx + f(x)$ for all $x \in I^{\perp\perp}$. Calculating $F(xyz)$ in two different ways:

$$\begin{aligned} F(xyz) &= mF(xy)F(z) = m\big((cxy + f(xy))(cz + f(z))\big) \\ &= m\big((cxy + f(xy))cz + cxyf(z) + f(xy)f(z)\big), \end{aligned}$$

and

$$\begin{aligned} F(xyz) &= mF(x)F(yz) = m\big((cx + f(x))(cyz + f(yz))\big) \\ &= m\big((cx + f(x))cyz + cxf(yz) + f(x)f(yz)\big), \end{aligned}$$

we get, since R is m-torsion free,

$$\begin{aligned} 0 = &\big((cxy + f(xy))c - (cx + f(x))cy\big)z + cxyf(z) - cxf(yz) + f(xy)f(z) \\ &- f(x)f(yz) \end{aligned}$$

for all $x, z \in I$, $y \in I^{\perp\perp}$. Recall that $f|_{I^{\perp\perp}}$ is a derivation (see Lemma 3.1, Lemma 3.2 and Step 1). Consequently,

$$\begin{aligned} 0 &= \big((cxy + f(xy))c - (cx + f(x))cy\big)z - cxf(y)z + f(x)yf(z) \\ &\quad + xf(y)f(z) - f(x)yf(z) - f(x)f(y)z \\ &= \big((cxy + f(xy))c - (cx + f(x))cy - cxf(y) - f(x)f(y)\big)z + xf(y)f(z) \end{aligned}$$

for all $x, z \in I$, $y \in I^{\perp\perp}$. Let

$$\begin{aligned} G(x, y, z) = &\big((cxy + f(xy))c - (cx + f(x))cy - cxf(y) - f(x)f(y)\big)z \\ &+ xf(y)f(z). \end{aligned}$$

Since $G\left(I, I^{\perp\perp}, I\right) = 0$, we have

$$\begin{aligned} 0 &= G(x, y, zy) - G(x, y, z)y \\ &= xf(y)f(zy) - xf(y)f(z)y \\ &= xf(y)zf(y) \end{aligned}$$

for all $x, z \in I$, $y \in I^{\perp\perp}$. Thus, $If(y)If(y) = 0$ for each $y \in I^{\perp\perp}$. Since $f\left(I^{\perp\perp}\right) \subseteq I^{\perp\perp}$ it follows that $f\left(I^{\perp\perp}\right) = 0$. It remains to apply Step 3. ∎

STEP 5. *If* $m = 0$, *then* $f\left(I^{\perp\perp}\right) = 0$ *and there exists* $c \in C(I^{\perp\perp})$ *such that* $c = nc^2$, $c[I^{\perp\perp}, I^{\perp\perp}] = 0$ *and* $F(x) = cx$ *for all* $x \in I^{\perp\perp}$.

PROOF. By Step 2, there exists $c \in Q_{mr}\left(I^{\perp\perp}\right)$ such that $F(x) = cx + f(x)$ for all $x \in I^{\perp\perp}$. We can express $F(uxv)$, with $u, v, x \in I$, in the following two ways:

$$\begin{aligned} F(uxv) &= nF(v)\,F(ux) = nF(v)\,F(u)\,x + nF(v)\,uf(x) \\ &= F(uv)\,x + nF(v)\,uf(x) \end{aligned}$$

and

$$F(uxv) = F(u)\,xv + uf(xv) = F(u)\,xv + uf(x)\,v + uxf(v)\,.$$

Consequently,

$$F(uv)\,x = F(u)\,xv + uf(x)\,v + uxf(v) - nF(v)\,uf(x)\,.$$

On the other hand

$$F(uv)\,x = F(u)\,vx + uf(v)\,x.$$

Comparing the last two identities we obtain

$$F(u)\,[v, x] = u\,(f(x)\,v + xf(v) - f(v)\,x) - nF(v)\,uf(x) \tag{3.21}$$

for all $u, v, x \in I$. Let $w, z \in I$. Since $F(I^2) \subseteq I$, inserting $x = v$ and $u = F(wz)$ in (3.21) we get

$$\begin{aligned} 0 &= nF(v)\,F(wz)\,f(v) - F(wz)\,vf(v) \\ &= (F(wzv) - F(wz)\,v)\,f(v) \\ &= wzf(v)\,f(v)\,. \end{aligned}$$

Thus, $I^2 f(v)^2 = 0$ and hence $f(v)^2 = 0$ for each $v \in I$. Since f is additive on I it follows that $f(u)\,f(v) + f(v)\,f(u) = 0$ for all $u, v \in I$. Consequently,

$$\begin{aligned} & f(u)\,f(w)\,wf(v)\,f(u) \\ &\quad = f(u)\,f(w)\,f(wv)\,f(u) - f(u)\,f(w)\,f(w)\,vf(u) = 0 \end{aligned} \tag{3.22}$$

and similarly $f(u)\,f(v)\,wf(w)\,f(u) = 0$ for all $u, v, w \in I$. Replacing w by $w + v$ in (3.22) we obtain

$$f(u)\,f(w)\,vf(v)\,f(u) + f(u)\,f(v)\,wf(v)\,f(u) = 0$$

which yields

$$f(u)\,f(v)\,wf(u)\,f(v) = 0$$

for all $u, v, w \in I$. Consequently, $f(I)\,f(I) = 0$. Thus, for all $u, v \in I$ we have

$$f(u)\,vf(u) = f(u)\,f(vu) - f(u)\,f(v)\,u = 0$$

which in turn implies $f(I) = 0$. Hence, $f(x)\,u = f(xu) - xf(u) = 0$ for all $x \in I^{\perp\perp}$ and so $f\left(I^{\perp\perp}\right) = 0$. Step 3 finishes the proof. ∎

Step 6. *The map $(m+n)F : I \to R$ is a Jordan homomorphism. Furthermore, if R is 2-torsion free, then $F(xyx) = (m+n)^2 F(x)F(y)F(x)$ for all $x, y \in I$.*

Proof. By (3.3), for all $u \in I$,

$$F(u^2) = (m+n)F(u)^2.$$

According to Step 1, the map F is additive on I. Hence, the map $(m+n)F : I \to R$ is a Jordan homomorphism. The second statement follows from e.g. [9, Lemma 3.1]. □

Step 7. *If R is 2-torsion free and $w \in I$ is such that $mnF(wxw) = 0$ for all $x \in I$, then $mnF(w) = 0$.*

Proof. By (3.3), for all $x, z \in I$, $mnF(wxwz) = 0$. Thus,

$$0 = mnF(wxwz) = mnF(wxw)z + mnwxwf(z) = mnwxwf(z),$$

which implies

$$(mnwf(z))I(mnwf(z)) = 0.$$

Since $mnwf(z) \in I$, this yields $mnwf(z) = 0$. Therefore, $mnF(wz) = mnF(w)z$ for all $z \in I$. Then, by Step 6, for all $z \in I$,

$$\begin{aligned} 0 &= mnF(w(wz)w) = (m+n)^2 mnF(w)F(wz)F(w) \\ &= (m+n)^2 mnF(w)^2 zF(w). \end{aligned}$$

Since $F(I) \subseteq I^{\perp\perp}$, and R is $(m+n)$-torsion free, we conclude $mnF(w)^2 = 0$. Now we have

$$\begin{aligned} 0 &= mnF(wzw) = m^2nF(wz)F(w) + mn^2F(w)F(wz) \\ &= m^2nF(w)zF(w) + mn^2F(w)^2z = m^2nF(w)zF(w). \end{aligned}$$

Finally, $(mnF(w))I(mnF(w)) = 0$ which implies $mnF(w) = 0$. □

Step 8. *If $mnF([u,v]) = 0$ for all $u, v \in I$, then $mn[F(u), F(v)] = 0$ for all $u, v \in I$.*

Proof. By (3.3), $mnF([u,v]w) = 0$ for all $u, v, w \in I$. Consequently, for all $u, v, x \in I$ and $w \in I^{\perp\perp}$ we have

$$mn[u,x]vf(w) = mn(F([u,x]vw) - F([u,x]v)w) = 0.$$

Hence, $(mn[u,x]f(w))I(mn[u,x]f(w)) = 0$ and so $mn[u,x]f(w) = 0$ for all $u, x \in I$, $w \in I^{\perp\perp}$. According to Lemma 3.2, $f(I) \subseteq I^{\perp\perp}$. Consequently, for all $u, v, x, z \in I$ we have

$$\begin{aligned} 0 &= mnf([u,x]vf(z)) \\ &= mnf([u,x])vf(z) + mn[u,x]f(v)f(z) + mn[u,x]vf(f(z)) \\ &= mnf([u,x])vf(z) \end{aligned}$$

which implies $mnf([u,x]) = 0$ for all $u, x \in I$. Since, by (3.3), $mnF(w[u,v]) = 0$ for all $u, v, w \in I$, we have $mnF(w)[u,v] = 0$ for all $u, v, w \in I$. This implies $mnF(w)u[x,v] = 0$ for all $x \in I^{\perp\perp}$, and $u, v, w \in I$. In particular, we have $mnF(w)u[mnF(w), v] = 0$ for all $u, v, w \in I$, since $F(I) \subseteq I^{\perp\perp}$. Therefore, $[mnF(w), I]I[mnF(w), I] = 0$ and so $[mnF(w), I] = 0$. Hence, $[mnF(w), II^{\perp\perp}] = 0$, which in turn implies $I[mnF(w), I^{\perp\perp}] = 0$. This yields $[mnF(w), I^{\perp\perp}] = 0$. In particular, $[mnF(u), F(v)] = 0$ for all $u, v \in I$. ∎

STEP 9. *If R is 2-torsion free, then $mn[F(u), F(v)] = 0$ for all $u, v \in I$.*

PROOF. According to Step 6, the map $(m+n)F : I \to R$ is a Jordan homomorphism. By e.g. [9, Lemma 3.4], for all $u, v, x \in I$,

$$\begin{aligned}&\big((m+n)F(uv) - (m+n)^2F(u)F(v)\big)(m+n)F(x)\\ &\times \big((m+n)F(uv) - (m+n)^2F(v)F(u)\big)\\ &+ \big((m+n)F(uv) - (m+n)^2F(v)F(u)\big)(m+n)F(x)\\ &\times \big((m+n)F(uv) - (m+n)^2F(u)F(v)\big) = 0.\end{aligned}$$

Since R is $(m+n)$-torsion free, this implies

$$\begin{aligned}&\big(F(uv) - (m+n)F(u)F(v)\big)F(x)\big(F(uv) - (m+n)F(v)F(u)\big)\\ &+ \big(F(uv) - (m+n)F(v)F(u)\big)F(x)\big(F(uv) - (m+n)F(u)F(v)\big) = 0.\end{aligned}$$

By (3.3), this yields

$$mn\big([F(v), F(u)]F(x)[F(u), F(v)] + [F(u), F(v)]F(x)[F(v), F(u)]\big) = 0,$$

that is, since R is 2-torsion free,

$$mn\big([F(u), F(v)]F(x)[F(u), F(v)]\big) = 0 \tag{3.23}$$

for all $u, v, x \in I$. By (3.3), for all $u, v \in I$,

$$(m-n)[F(u), F(v)] = F([u,v])$$

and so (3.23) yields

$$mnF([u,v])F(x)F([u,v]) = 0$$

for all $u, v, x \in I$. By Step 6,

$$mnF([u,v]x[u,v]) = 0$$

for all $u, v, x \in I$. Step 7 implies $mnF([u,v]) = 0$ for all $u, v \in I$. Finally Step 8 yields $mn[F(u), F(v)] = 0$ for all $u, v \in I$. ∎

STEP 10. *If R is 2-torsion free and m-torsion free, then $f\left(I^{\perp\perp}\right) = 0$ and there exists $c \in C\left(I^{\perp\perp}\right)$ such that $c = (m+n)c^2$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. Furthermore, $nc[I^{\perp\perp}, I^{\perp\perp}] = 0$ and (3.3) holds for all $u, v \in I^{\perp\perp}$.*

Proof. By Step 9, $nF(v)F(u) = nF(u)F(v)$ for all $u, v \in I$. Then (3.3) implies

$$F(uv) = (m+n)F(u)F(v)$$

for all $u, v \in I$. Using Step 4 with the integers $(m+n)$ and 0, we see that $f\left(I^{\perp\perp}\right) = 0$ and there exists $c \in C(I^{\perp\perp})$ such that $c = (m+n)c^2$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. Then

$$nc^2vu = n(cv)(cu) = nF(v)F(u) = nF(u)F(v) = n(cu)(cv) = nc^2uv$$

for all $u, v \in I$. Thus $nc^2[I, I] = 0$. In particular, $nc^2[I, II^{\perp\perp}] = 0$, which further implies $Inc^2[I, I^{\perp\perp}] = 0$. Hence, $nc^2[I, I^{\perp\perp}] = 0$. This yields, using a similar argument as before, that $nc^2[I^{\perp\perp}, I^{\perp\perp}] = 0$. Obviously, this implies that the ideal generated by $nc[I^{\perp\perp}, I^{\perp\perp}]$ is a nilpotent ideal of the semiprime ring $I^{\perp\perp}$ and so it is the zero ideal. Thus, $nc[I^{\perp\perp}, I^{\perp\perp}] = 0$. □

Step 11. *If R is 2-torsion free and n-torsion free, then $f\left(I^{\perp\perp}\right) = 0$ and there exists $c \in C\left(I^{\perp\perp}\right)$ such that $c = (m+n)c^2$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. Furthermore, $c[I^{\perp\perp}, I^{\perp\perp}] = 0$ and (3.3) holds for all $u, v \in I^{\perp\perp}$.*

Proof. By Step 9, $mF(u)F(v) = mF(v)F(u)$ for all $u, v \in I$. Then (3.3) implies

$$F(uv) = (m+n)F(v)F(u)$$

for all $u, v \in I$. Using Step 5 with the integers 0 and $(m+n)$, we see that $f\left(I^{\perp\perp}\right) = 0$ and there exists $c \in C(I^{\perp\perp})$ such that $c = (m+n)c^2$, $c[I^{\perp\perp}, I^{\perp\perp}] = 0$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. □

Hence, we have proved Theorem 3.3.

We now consider two special cases, when F is multiplicative or anti-multiplicative on I. Using Theorem 3.3 we obtain the following.

Corollary 3.4. *Let I be a nonzero ideal of a semiprime ring R. Let $F: R \to R$ and $f: R \to R$ be maps satisfying*

$$F(xy) = F(x)y + xf(y) \qquad \text{for all } x, y \in R.$$

Then the following holds.

(i) *If F is multiplicative on I then $f\left(I^{\perp\perp}\right) = 0$ and there exists an idempotent $c \in C\left(I^{\perp\perp}\right)$ such that $F(x) = cx$ for all $x \in I^{\perp\perp}$. Furthermore, F is multiplicative on $I^{\perp\perp}$.*

(ii) *If F is anti-multiplicative on I, then $f\left(I^{\perp\perp}\right) = 0$ and there exists an idempotent $c \in C\left(I^{\perp\perp}\right)$ such that $c\left[I^{\perp\perp}, I^{\perp\perp}\right] = 0$ and $F(x) = cx$ for all $x \in I^{\perp\perp}$. Furthermore, F is multiplicative and anti-multiplicative on $I^{\perp\perp}$.*

Suppose that I is an essential ideal of a semiprime ring R. Then $I^{\perp} = 0$ and $I^{\perp\perp} = R$. Thus, Corollary 3.4 yields the following result.

Corollary 3.5. *Let I be an essential ideal of a semiprime ring R. Let $F : R \to R$ and $f : R \to R$ be maps satisfying*

$$F(xy) = F(x)y + xf(y) \qquad \text{for all } x, y \in R.$$

Then the following holds.

(i) *If F is multiplicative on I then $f = 0$ and there exists an idempotent $c \in C(R)$ such that $F(x) = cx$ for all $x \in R$. Furthermore, F is multiplicative on R.*

(ii) *If F is anti-multiplicative on I then $f = 0$ and there exists an idempotent $c \in C(R)$ such that $c[R, R] = 0$ and $F(x) = cx$ for all $x \in R$. Furthermore, F is multiplicative and anti-multiplicative on R.*

Remark 3.6. Suppose that R is a prime ring. Then each nonzero ideal of R is essential and $C(R)$ is a field. Thus, Corollary 3.5 yields the result of Gusić [8] (see Theorem 1.1).

The following example (cf. [5]) shows that the assumptions that F is multiplicative or anti-multiplicative on I in Corollary 3.4 cannot be replaced by $F(x^2) = F(x)^2$ for all $x \in I$.

Example 3.7. Let $\mathcal{A} = \mathbb{F}\langle X, Y\rangle$ be the free algebra in noncommuting indeterminates X and Y over a field $\mathbb{F}$. Let $\mathcal{A}_1$ be a subalgebra of $\mathcal{A}$ generated by X and Y, that is, $\mathcal{A}_1 = X\mathcal{A} + Y\mathcal{A}$. We define $F : \mathcal{A}_1 \to \mathcal{A}_1$ by

$$F(p) = \begin{cases} p & \text{if } p \in X\mathcal{A} \\ 0 & \text{if } p \notin X\mathcal{A} \end{cases}.$$

Then $F(pq) = F(p)q$ for all $p, q \in \mathcal{A}_1$ and $F(p^2) = F(p)^2$ for all $p \in \mathcal{A}_1$. Suppose that there exists an idempotent $c \in C(\mathcal{A}_1)$ such that $F(p) = cp$ for all $p \in \mathcal{A}_1$. Then $0 = F(p) = cp$ for all $p \notin X\mathcal{A}$. In particular, $c(X + Y) = 0$ and $cY = 0$, which implies $cX = 0$. Then $X = F(X) = cX = 0$, a contradiction.

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