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ON CERTAIN FUNCTIONAL EQUATION ARISING FROM $(m,n)-$ JORDAN CENTRALIZERS IN PRIME RINGS

Nina Peršin and Joso Vukman
University of Maribor, Slovenia

Abstract. The purpose of this paper is to prove the following result. Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a prime ring with $char(R) = 0$ or $(m+n)^2 < char(R)$. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation $2(m+n)^2T(x^3) = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x)$ for all $x \in R$. In this case T is a two-sided centralizer.

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x,y]$. We shall use the commutator identities $[xy,z] = [x,z]\,y + x\,[y,z]$ and $[x,yz] = [x,y]\,z + y\,[x,z]$ for all $x,y,z \in R$. Recall that a ring R is prime if for $a,b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $char(R)$ the characteristic of a prime ring R. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x,y \in R$, and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = [a,x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([10]) asserts that any Jordan derivation on a prime ring with $char(R) \neq 2$ is a derivation. A brief proof of Herstein's result can be found in [3]. Cusack

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([8]) generalized Herstein's result to 2−torsion free semiprime rings (see also [4] for an alternative proof).

We denote by Q_r, Q_l, Q_s, C and RC the right, left, symmetric Martindale ring of quotients, extended centroid and central closure of a semiprime ring R, respectively. For the explanation of Q_r, Q_l, Q_s, C and RC we refer the reader to [1]. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$, where $a \in R$ is some fixed element. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where $q \in Q_r$ is some fixed element (see Chapter 2 in [1]). An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see Theorem 2.3.2 in [1]). Zalar ([20]) has proved that any left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer. For results concerning centralizers in rings and algebras we refer to [11, 15, 17–20] where further references can be found. A mapping F, which maps a ring R into itself, is called centralizing on R in case $[F(x), x] \in Z(R)$ holds for all $x \in R$. A classical result of Posner ([13]) (Posner's second theorem) states that the existence of a nonzero centralizing derivation on a prime ring R with $char(R) \neq 2$ forces the ring to be commutative. Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of Hahn-Banach theorem. Let $m \geq 0, n \geq 0$ be fixed integers with $m + n \neq 0$ and let R be an arbitrary ring. An additive mapping $T : R \to R$ is called an (m, n)−Jordan centralizer in case

$$(m + n)T(x^2) = mT(x)x + nxT(x) \tag{1}$$

holds for all $x \in R$. The concept of (m, n)−Jordan centralizer was introduced by Vukman ([19]). Obviously, $(1, 0)$-Jordan centralizer is a left Jordan centralizer, $(0, 1)$− Jordan centralizer is a right Jordan centralizer, and in case $(1, 1)$−Jordan centralizer we have the relation

$$2T(x^2) = T(x)x + xT(x), x \in R. \tag{2}$$

Vukman ([15]) has proved that in case there exists an additive mapping $T : R \to R$, where R is a 2−torsion free semiprime ring, satisfying the relation (2), then T is a two-sided centralizer. Vukman ([19]) conjectured that any $(m,n)-$Jordan centralizer on a semiprime ring with suitable torsion restrictions, where $m \geq 1, n \geq 1$ are some fixed integers, is a two-sided centralizer. Vukman ([19]) has proved the following result which proves a special case of the conjecture we have just mentioned above.

THEOREM 1. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a prime ring with $char(R) \neq 6mn(m+n)$. Suppose $T : R \to R$ is an $(m,n)-$Jordan centralizer. If $Z(R)$ is nonzero, then T is a two-sided centralizer.*

One can easily prove that any $(m,n)-$Jordan centralizer $T : R \to R$, where R is an arbitrary ring, satisfies the relation

$$2(m+n)^2T(x^3) = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x)$$

for all $x \in R$ (see [19] for the details). Recently, Vukman ([19]) considered the above relation in standard operator algebras on a real or complex Hilbert space. It is our aim in this paper to prove the following result.

THEOREM 2. *Let $m \geq 1, n \geq 1$ be some fixed integers and let R be a prime ring with $char(R) = 0$ or $(m+n)^2 < char(R)$ and let $T : R \to R$ be an additive mapping satisfying the relation*

$$2(m+n)^2T(x^3) = m(2m+n)T(x)x^2 + 2mnxT(x)x + n(2n+m)x^2T(x) \tag{3}$$

for all $x \in R$. In this case T is a two-sided centralizer.

For the proof of Theorem 2 we need Theorem 3 below, which might be of independent interest. As the main tool in this paper we use the theory of functional identities (Brešar-Beidar-Chebotar theory). We refer the reader to [6] for introductory account of functional identities and to [7] for full treatment of this theory. Let R be an algebra over a commutative ring ϕ. Further let

$$p(x_1,x_2,x_3) = \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \tag{4}$$

be a fixed multilinear polynomial in noncommuting indeterminates x_i over ϕ. Here S_3 stands for the symmetric group of order 3 and $e \in S_3$ for its identity element. Further, let $\mathcal{L}$ be a subset of R closed under p, i.e., $p(\bar{x}_3) \in \mathcal{L}$ for all $x_1, x_2, x_3 \in \mathcal{L}$, where $\bar{x}_3 = (x_1, x_2, x_3)$. We shall consider a mapping $T : \mathcal{L} \to R$ satisfying

$$\begin{aligned} 2(m+n)^2T(p(\bar{x}_3)) &= m(2m+n)\sum_{\pi \in S_3} T(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)} \\ &+ 2mn\sum_{\pi \in S_3} x_{\pi(1)}T(x_{\pi(2)})x_{\pi(3)} + n(2n+m)\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}T(x_{\pi(3)}) \end{aligned} \tag{5}$$

for all $x_1, x_2, x_3 \in \mathcal{L}$. In the first step of the proof of the following theorem we derive a functional identity from (5). Let us mention that the idea of considering the expression $[p(\bar{x}_3), p(\bar{y}_3)]$ in its proof is taken from [2].

Theorem 3. *Let R be an algebra over ϕ. Suppose that $\mathcal{L}$ is a 6-free Lie subalgebra of R closed under p. If $T : \mathcal{L} \to R$ is an additive map satisfying (3) for all $x \in \mathcal{L}$, then there exists $p \in C(\mathcal{L})$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$ such that $2m(2m+n)(m+n)^2T(x) = px + \lambda(x)$ for all $x \in \mathcal{L}$, where $C(\mathcal{L})$ is extended centroid of $\mathcal{L}$.*

Proof. A complete linearization of (3) gives us (5). Note that for any $a \in \mathcal{L}$ and $\bar{x}_3 \in \mathcal{L}^3$ we have

$$\begin{aligned}
&[p(\bar{x}_3), a] = \sum_{\pi \in S_3} \left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, a\right] \\
&= \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}a - a\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \\
&= \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}a - a\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)}ax_{\pi(2)}x_{\pi(3)} \\
&\quad - \sum_{\pi \in S_3} x_{\pi(1)}ax_{\pi(2)}x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}ax_{\pi(3)} - \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}ax_{\pi(3)} \\
&= \sum_{\pi \in S_3} x_{\pi(1)}ax_{\pi(2)}x_{\pi(3)} - a\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}ax_{\pi(3)} \\
&\quad - \sum_{\pi \in S_3} x_{\pi(1)}ax_{\pi(2)}x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}a - \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}ax_{\pi(3)} \\
&= \sum_{\pi \in S_3} \left[x_{\pi(1)}, a\right] x_{\pi(2)}x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)} \left[x_{\pi(2)}, a\right] x_{\pi(3)} + \sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)} \left[x_{\pi(3)}, a\right] \\
&= p(\left[x_1, a\right], x_2, x_3) + p(x_1, \left[x_2, a\right], x_3) + p(x_1, x_2, [x_3, a]).
\end{aligned}$$

Using this in (5) we obtain

$$\begin{aligned}
2(m+n)^2T([p(\bar{x}_3), a] &= 2(m+n)^2T(p(\left[x_1, a\right], x_2, x_3)) \\
&\quad + 2(m+n)^2T(p(x_1, \left[x_2, a\right], x_3)) + 2(m+n)^2T(p(x_1, x_2, [x_3, a])).
\end{aligned}$$

It follows that

$$\begin{aligned}
&2(m+n)^2T([p(\bar{x}_3), a] = \sum_{\pi \in S_3} m(2m+n)T(\left[x_{\pi(1)}, a\right])x_{\pi(2)}x_{\pi(3)} \\
&+ \sum_{\pi \in S_3} m(2m+n)T(x_{\pi(1)}) \left[x_{\pi(2)}x_{\pi(3)}, a\right] + \sum_{\pi \in S_3} 2mn \left[x_{\pi(1)}, a\right] T(x_{\pi(2)})x_{\pi(3)} \\
&+ \sum_{\pi \in S_3} 2mnx_{\pi(1)}T(\left[x_{\pi(2)}, a\right])x_{\pi(3)} + \sum_{\pi \in S_3} 2mnx_{\pi(1)}T(x_{\pi(2)}) \left[x_{\pi(3)}, a\right] \\
&+ \sum_{\pi \in S_3} n(2n+m) \left[x_{\pi(1)}x_{\pi(2)}, a\right] T(x_{\pi(3)}) \\
&+ \sum_{\pi \in S_3} n(2n+m)x_{\pi(1)}x_{\pi(2)}T(\left[x_{\pi(3)}, a\right]).
\end{aligned}$$

Further, let $s:\mathbb{Z}\to\mathbb{Z}$ be a mapping defined by $s(i)=i-3$. For each $\sigma\in S_3$ the mapping $s^{-1}\sigma s:\{4,5,6\}\to\{4,5,6\}$ will be denoted by $\overline{\sigma}$. Then we have in particular, where $a=p(x_4,x_5,x_6)$

$$
\begin{aligned}
&2(m+n)^2T([p(x_1,x_2,x_3),p(x_4,x_5,x_6)]\\
&\quad=\sum_{\pi\in S_3}m(2m+n)T(\left[x_{\pi(1)},p(x_4,x_5,x_6)\right])x_{\pi(2)}x_{\pi(3)}\\
&\quad\quad+\sum_{\pi\in S_3}m(2m+n)T(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)},p(x_4,x_5,x_6)\right]\\
&\quad\quad+\sum_{\pi\in S_3}2mn\left[x_{\pi(1)},p(x_4,x_5,x_6)\right]T(x_{\pi(2)})x_{\pi(3)}\\
&\quad\quad+\sum_{\pi\in S_3}2mnx_{\pi(1)}T(\left[x_{\pi(2)},p(x_4,x_5,x_6)\right]x_{\pi(3)}\\
&\quad\quad+\sum_{\pi\in S_3}2mnx_{\pi(1)}T(x_{\pi(2)})\left[x_{\pi(3)},p(x_4,x_5,x_6)\right]\\
&\quad\quad+\sum_{\pi\in S_3}n(2n+m)\left[x_{\pi(1)}x_{\pi(2)},p(x_4,x_5,x_6)\right]T(x_{\pi(3)})\\
&\quad\quad+\sum_{\pi\in S_3}n(2n+m)x_{\pi(1)}x_{\pi(2)}T(\left[x_{\pi(3)},p(x_4,x_5,x_6)\right])
\end{aligned}
\tag{6}
$$

and

$$
\begin{aligned}
&2(m+n)^2T(\left[x_{\pi(1)},p(x_4,x_5,x_6)\right])=-2(m+n)^2T(\left[p(x_4,x_5,x_6),x_{\pi(1)}\right])\\
&\quad=\sum_{\sigma\in S_3}m(2m+n)T(\left[x_{\pi(1)},x_{\bar\sigma(1)}\right]x_{\bar\sigma(2)}x_{\bar\sigma(3)}\\
&\quad\quad+\sum_{\sigma\in S_3}m(2m+n)T(x_{\bar\sigma(1)})\left[x_{\pi(1)},x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right]\\
&\quad\quad+\sum_{\sigma\in S_3}2mn\left[x_{\pi(1)},x_{\bar\sigma(1)}\right]T(x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\
&\quad\quad+\sum_{\sigma\in S_3}2mnx_{\bar\sigma(1)}T(x_{\bar\sigma(2)})\left[x_{\pi(1)},x_{\bar\sigma(3)}\right]\\
&\quad\quad+\sum_{\sigma\in S_3}2mnx_{\bar\sigma(1)}T(\left[x_{\pi(1)},x_{\bar\sigma(2)}\right])x_{\bar\sigma(3)}\\
&\quad\quad+\sum_{\sigma\in S_3}n(2n+m)\left[x_{\pi(1)},x_{\bar\sigma(1)}x_{\bar\sigma(2)}\right]T(x_{\bar\sigma(3)})\\
&\quad\quad+\sum_{\sigma\in S_3}n(2n+m)x_{\bar\sigma(1)}x_{\bar\sigma(2)}T(\left[x_{\pi(1)},x_{\bar\sigma(3)}\right],
\end{aligned}
$$

for all $x_1,...,x_6\in\mathcal{L}$. We shall write

$$\varphi(x_{\pi(1)})=2(m+n)^2T(\left[x_{\pi(1)},p(x_4,x_5,x_6)\right].$$

Similarly we define

$$\varphi(x_{\pi(2)}) = 2(m+n)^2 T(\left[x_{\pi(2)}, p(x_4, x_5, x_6)\right],$$

and

$$\varphi(x_{\pi(3)}) = 2(m+n)^2 T(\left[x_{\pi(3)}, p(x_4, x_5, x_6)\right].$$

Using this together with (6) we obtain

$$\begin{aligned}
&(2(m+n)^2)^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)] \\
&\quad = \sum_{\pi\in S_3} m(2m+n)\varphi(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3} 2m(2m+n)(m+n)^2 T(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)}, p(x_4, x_5, x_6)\right] \\
&\qquad + \sum_{\pi\in S_3} 4mn(m+n)^2\left[x_{\pi(1)}, p(x_4, x_5, x_6)\right] T(x_{\pi(2)})x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3} 2mnx_{\pi(1)}\varphi(x_{\pi(2)})x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3} 4mn(m+n)^2 x_{\pi(1)}T(x_{\pi(2)})\left[x_{\pi(3)}, p(x_4, x_5, x_6)\right] \\
&\qquad + \sum_{\pi\in S_3} 2n(2n+m)(m+n)^2\left[x_{\pi(1)}x_{\pi(2)}, p(x_4, x_5, x_6)\right] T(x_{\pi(3)}) \\
&\qquad + \sum_{\pi\in S_3} n(2n+m)x_{\pi(1)}x_{\pi(2)}\varphi(x_{\pi(3)}).
\end{aligned} \tag{7}$$

Since

$$\left[x_{\pi(1)}x_{\pi(2)}, p(x_4, x_5, x_6)\right] = \sum_{\sigma\in S_3}\left[x_{\pi(1)}x_{\pi(2)}, x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right],$$

then (7) reduces to

$$\begin{aligned}
&(2(m+n)^2)^2 T([p(x_1, x_2, x_3), p(x_4, x_5, x_6)] \\
&\quad = \sum_{\pi\in S_3}\sum_{\sigma\in S_3} m(2m+n)\varphi(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2m(2m+n)(m+n)^2 T(x_{\pi(1)})\left[x_{\pi(2)}x_{\pi(3)}, x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right] \\
&\qquad + \sum_{\pi\in S_3}\sum_{\sigma\in S_3} 4mn(m+n)^2\left[x_{\pi(1)}, x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right] T(x_{\pi(2)})x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2mnx_{\pi(1)}\varphi(x_{\pi(2)})x_{\pi(3)} \\
&\qquad + \sum_{\pi\in S_3}\sum_{\sigma\in S_3} 4mn(m+n)^2 x_{\pi(1)}T(x_{\pi(2)})\left[x_{\pi(3)}, x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right]
\end{aligned} \tag{8}$$

$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2n(2n+m)(m+n)^2\left[x_{\pi(1)}x_{\pi(2)},x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right]T(x_{\pi(3)})$$
$$+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} n(2n+m)x_{\pi(1)}x_{\pi(2)}\varphi(x_{\pi(3)}).$$

If we replace the roles of denotations π and σ, then from (8) we get

$$\begin{aligned}
&(2(m+n)^2)^2T([p(x_1,x_2,x_3),p(x_4,x_5,x_6)]\\
&=\sum_{\pi\in S_3}\sum_{\sigma\in S_3} m(2m+n)\bar\varphi(x_{\bar\sigma(1)})x_{\bar\sigma(2)}x_{\bar\sigma(3)}\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2m(2m+n)(m+n)^2T(x_{\bar\sigma(1)})\left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\bar\sigma(2)}x_{\bar\sigma(3)}\right]\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 4mn(m+n)^2\left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\bar\sigma(1)}\right]T(x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2mnx_{\bar\sigma(1)}\bar\varphi(x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 4mn(m+n)^2x_{\bar\sigma(1)}T(x_{\bar\sigma(2)})\left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\bar\sigma(3)}\right]\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} 2n(2n+m)(m+n)^2\left[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)},x_{\bar\sigma(1)}x_{\bar\sigma(2)}\right]T(x_{\bar\sigma(3)})\\
&\quad+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} n(2n+m)x_{\bar\sigma(1)}x_{\bar\sigma(2)}\bar\varphi(x_{\bar\sigma(3)}).
\end{aligned}\tag{9}$$

for all $x_1,...,x_6\in\mathcal{L}$, where

$$\bar\varphi(x_{\bar\sigma(i)})=2(m+n)^2T(\left[p(x_1,x_2,x_3),x_{\bar\sigma(i)}\right]$$

for $i=1,2,3$. We obtain that

$$\bar\varphi(x_{\pi(i)})=-\varphi(x_{\pi(i)})$$

for $i=1,2,3$. Comparing (8) and (9) we obtain the following identity

$$\begin{aligned}
0=&\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(m(2m+n)\varphi(x_{\pi(1)})x_{\pi(2)}+2mnx_{\pi(1)}\varphi(x_{\pi(2)})\\
&+4mn(m+n)^2x_{\bar\sigma(1)}T(x_{\bar\sigma(2)})x_{\bar\sigma(3)}x_{\pi(1)}x_{\pi(2)}\\
&-4mn(m+n)^2x_{\pi(1)}T(x_{\pi(2)})x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}\\
&+2m(2m+n)(m+n)^2T(x_{\bar\sigma(1)})x_{\bar\sigma(2)}x_{\bar\sigma(3)}x_{\pi(1)}x_{\pi(2)}\\
&-2m(2m+n)(m+n)^2T(x_{\pi(1)})x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}x_{\pi(2)}\Big)x_{\pi(3)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}\Big(m(2m+n)\varphi(x_{\bar\sigma(1)})x_{\bar\sigma(2)}+2mnx_{\bar\sigma(1)}\varphi(x_{\bar\sigma(2)})\\
&+4mn(m+n)^2x_{\pi(1)}T(x_{\pi(2)})x_{\pi(3)}x_{\bar\sigma(1)}x_{\bar\sigma(2)}\\
&-4mn(m+n)^2x_{\bar\sigma(1)}T(x_{\bar\sigma(2)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}\\
&+2m(2m+n)(m+n)^2T(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}x_{\bar\sigma(1)}x_{\bar\sigma(2)}\\
&-2m(2m+n)(m+n)^2T(x_{\bar\sigma(1)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\bar\sigma(2)}\Big)x_{\bar\sigma(3)}\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\pi(1)}\Big(n(2n+m)x_{\pi(2)}\varphi(x_{\pi(3)})\\
&-4mn(m+n)^2x_{\pi(2)}x_{\pi(3)}x_{\bar\sigma(1)}T(x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\
&+4mn(m+n)^2x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}T(x_{\pi(2)})x_{\pi(3)}x_{\pi(2)}x_{\pi(3)}\\
&-2n(2n+m)(m+n)^2x_{\pi(2)}x_{\pi(3)}x_{\bar\sigma(1)}x_{\bar\sigma(2)}T(x_{\bar\sigma(3)})\\
&+2n(2n+m)(m+n)^2x_{\pi(2)}x_{\bar\sigma(1)}x_{\bar\sigma(2)}x_{\bar\sigma(3)}T(x_{\pi(3)})\Big)\\
&+\sum_{\pi\in S_3}\sum_{\sigma\in S_3}x_{\bar\sigma(1)}\Big(n(2n+m)x_{\bar\sigma(2)}\varphi(x_{\bar\sigma(3)})\\
&-4mn(m+n)^2x_{\bar\sigma(2)}x_{\bar\sigma(3)}x_{\pi(1)}T(x_{\pi(2)})x_{\pi(3)}\\
&+4mn(m+n)^2x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}T(x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\
&-2n(2n+m)(m+n)^2x_{\bar\sigma(2)}x_{\bar\sigma(3)}x_{\pi(1)}x_{\pi(2)}T(x_{\pi(3)})\\
&+2n(2n+m)(m+n)^2x_{\bar\sigma(2)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}T(x_{\bar\sigma(3)})\Big)
\end{aligned}
\tag{10}
$$

for all $x_1, ..., x_6 \in \mathcal{L}$.

Define maps $E, F : \mathcal{L}^5 \to R$ by the rule

$$
\begin{aligned}
E(u_1,u_2,u_3,u_4,u_5) &= m(2m+n)\varphi(u_1)u_2 + 2mnu_1\varphi(u_2)\\
&+ 4mn(m+n)^2(u_3T(u_4)u_5u_1u_2 - u_1T(u_2)u_3u_4u_5)\\
&+ 2m(2m+n)(m+n)^2(T(u_3)u_4u_5u_1u_2 - T(u_1)u_3u_4u_5u_2)
\end{aligned}
$$

and

$$
\begin{aligned}
F(u_1,u_2,u_3,u_4,u_5) &= n(2n+m)u_1\varphi(u_2)\\
&- 4mn(m+n)^2(u_1u_2u_3T(u_4)u_5 - u_3u_4u_5T(u_1)u_2)\\
&- 2n(2n+m)(m+n)^2(u_1u_2u_3u_4T(u_5) - u_1u_3u_4u_5T(u_2))
\end{aligned}
$$

for all $\bar u_5 \in \mathcal{L}^5$.

Accordingly, (10) can be rewritten as

$$\begin{aligned}0 = &\sum_{\pi\in S_3}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},x_{\bar\sigma(1)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})x_{\pi(3)}\\ &+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(1)},x_{\bar\sigma(2)})x_{\bar\sigma(3)}\\ &+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} x_{\pi(1)}F(x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(1)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})\\ &+\sum_{\pi\in S_3}\sum_{\sigma\in S_3} x_{\bar\sigma(1)}F(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})\end{aligned}$$

and hence

$$\begin{aligned}0 = &\sum_{i=1}^{3}\left(\sum_{\substack{\pi\in S_3\\ \pi(3)=i}}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},x_{\bar\sigma(1)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})\right)x_i\\ &+\sum_{i=4}^{6}\left(\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \bar\sigma(3)=i}} E(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(1)},x_{\bar\sigma(2)})\right)x_i\\ &+\sum_{j=1}^{3}x_j\left(\sum_{\substack{\pi\in S_3\\ \pi(1)=j}}\sum_{\sigma\in S_3} F(x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(1)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})\right)\\ &+\sum_{j=4}^{6}x_j\left(\sum_{\pi\in S_3}\sum_{\substack{\sigma\in S_3\\ \bar\sigma(1)=j}} F(x_{\pi(1)},x_{\pi(2)},x_{\pi(3)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})\right)\end{aligned}$$

for all $x_1,..,x_6\in\mathcal{L}$. Then we have that

$$\sum_{i=1}^{6}E_i^i(\bar x_6)x_i+\sum_{j=1}^{6}F_j^j(\bar x_6)x_j=0$$

for all $\bar x_6\in\mathcal{L}^6$, where $E_i,F_j:\mathcal{L}^5\to R$ and $E^i,F^j:\mathcal{L}^6\to R$ are mappings

$$E^i(\bar x_6)=E(x_1,...,x_{i-1},x_i,...x_6)$$

and

$$F^j(\bar x_6)=F(x_1,...,x_{j-1},x_j,...x_6).$$

Now we simply apply the definition of 6-freeness $\mathcal{L}$. There exists maps $p_{6,j}:\mathcal{L}^4\to R, j=1,...,5$ and $\lambda_6:\mathcal{L}^5\to C(\mathcal{L})$ such that

$$\sum_{\substack{\pi\in S_3\\ \pi(3)=3}}\sum_{\sigma\in S_3} E(x_{\pi(1)},x_{\pi(2)},x_{\bar\sigma(1)},x_{\bar\sigma(2)},x_{\bar\sigma(3)})=\sum_{j=1}^{5}x_jp_{6,j}(\bar x_5^j)+\lambda_6(\bar x_5)$$

for all $\bar{x}_5 \in \mathcal{L}^5$. Recalling the definition of map E and after some steps we arrive at

$$2m(2m+n)(m+n)^2T(x) = xp + \lambda(x), \tag{11}$$

for all $x \in \mathcal{L}$, where $p \in \mathcal{L}$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$. The symmetric analogue in which maps F are involved, is clearly proved in the same way. Therefore

$$2n(2n+m)(m+n)^2T(x) = \bar{p}x + \bar{\lambda}(x)$$

for all $x \in \mathcal{L}$ and some $\bar{p} \in \mathcal{L}$ and $\bar{\lambda} : \mathcal{L} \to C(\mathcal{L})$. Therefore

$$2mn(2m+n)(2n+m)(m+n)^2T(x) = n(2n+m)(xp + \lambda(x)),$$
$$2mn(2m+n)(2n+m)(m+n)^2T(x) = m(2m+n)(\bar{p}x + \bar{\lambda}(x))$$

for all $x \in \mathcal{L}$. Comparing this two identities we arrive at

$$n(2n+m)xp - m(2m+n)\bar{p}x \in C(\mathcal{L})$$

for all $x \in \mathcal{L}$. It follows that $n(2n+m)p = m(2m+n)\bar{p} \in C(\mathcal{L})$, which yields $p, \bar{p} \in C(\mathcal{L})$. Therefore the proof is completed. ∎

We are now in a position to prove Theorem 2.

Proof of Theorem 2. The complete linearization of (3) gives us (5). Assume first that R is not a PI ring. According to Theorem 3 there exist $p \in C$ and $\lambda : R \to C$ such that

$$2m(2m+n)(m+n)^2T(x) = xp + \lambda(x).$$

Then we have

$$x^2((m+n)^2xp + 2(m+n)^2\lambda(x)) = (m+n)^2\lambda(x^3),$$

which yields

$$x^2(xp + 2\lambda(x)) = \lambda(x^3),$$

for all $x \in R$. A complete linearization of this identity leads to

$$\sum_{\pi \in S_3} x_{\pi(1)}x_{\pi(2)}(x_{\pi(3)}p + 2\lambda(x_{\pi(3)}) = \lambda(p(\bar{x}_3))$$

for all $x_1, x_2, x_3 \in R$. Since R is not a PI ring it follows that

$$xp + 2\lambda(x) = 0 \tag{12}$$

for all $x \in R$. Now our aim is to show that $\lambda = 0$. Thus $[xp, y] = 0$ for all $x, y \in R$. Then we have $[x, y]\,zp = 0$ for all $x, y, z \in R$. It follows that R is commutative or $p = 0$. If $p = 0$, then $\lambda(x) = 0$ for all $x \in R$ by (12). If $[x, y] = 0$, then from (12) follows that $\lambda(x)y - \lambda(y)x = 0$ for all $x, y \in R$. Consequently $\lambda = 0$.

Now suppose that R is a PI ring. It is well-known that in this case R has a nonzero center (see [14]). Let c be a nonzero central element. Pick any $x \in R$ and set $x_1 = x_2 = cx$ and $x_3 = x$ in (5) we get

$$\begin{aligned} &12(m+n)^2T(c^2x^3) = 2m(2m+n)(2T(cx)x^2 + T(x)x^2c)c \\ &\quad + 4mn(2xT(cx)x + xT(x)xc)c + 2n(2n+m)(2x^2T(cx) + x^2T(x)c)c. \end{aligned} \tag{13}$$

Next, setting $x_1 = x_2 = c$ and $x_3 = x^3$ in (5), we have

$$\begin{aligned} &12(m+n)^2T(c^2x^3) = 2m(2m+n)(2T(c)x^3 + T(x^3)c)c \\ &\quad + 4mn(x^3T(c) + T(c)x^3 + T(x^3)c)c + 2n(2n+m)(2x^3T(c) + T(x^3)c)c. \end{aligned} \tag{14}$$

Comparing both identities we obtain

$$\begin{aligned} &m(2m+n)T(cx)x^2 + 2mnxT(cx)x + n(2n+m)x^2T(cx) \\ &\qquad = 2m(m+n)T(c)x^3 + 2n(n+m)x^3T(c) \end{aligned} \tag{15}$$

for all $x \in R$. If $x = c$ we have

$$T(c^2) = T(c)c. \tag{16}$$

Setting $x_1 = x$ and $x_2 = x_3 = c$ in the complete linearization of (15) we get

$$(m+n)T(cx) = mT(c)x + nxT(c) \tag{17}$$

for all $x \in R$. Multiplying (17) by c^2 we get

$$(m+n)T(cx)c^2 = mT(c^2)xc + nxT(c^2)c$$

and substituting x by cx in (17) we get

$$(m+n)T(c^2x)c = mT(c^2)xc + nxT(c^2)c.$$

Comparing the last two identities we see that

$$T(c^2x) = T(cx)c. \tag{18}$$

Setting $x_1 = x$ and $x_2 = x_3 = c$ in (5) we have

$$\begin{aligned} 12(m+n)^2T(c^2x) = \; &2m(2m+n)(2T(x)c^2 + 4T(c)xc) \\ &+ 2mn(2T(x)c^2 + 2T(c)xc + 2xT(c)c) \\ &+ 2n(2n+m)(2c^2T(x) + 4xT(c)xc) \end{aligned} \tag{19}$$

and so

$$T(cx) = T(x)c = cT(x). \tag{20}$$

Setting $x_1 = x_2 = x$ and $x_3 = c$ in the complete linearization of (15) and using (20) we get

$$T(c)x^2 + x^2T(c) = 2xT(c)x.$$

This can be rewritten as

$$[[T(c), x], x] = 0$$

for all $x \in R$. From Posner's second theorem it follows that $[T(c), x] = 0$ for all $x \in R$. From (17) consequently we get

$$T(cx) = T(c)x = xT(c). \tag{21}$$

Next we replace x for xy in (17) we obtain

$$\begin{aligned}(m+n)T(xy)c &= (mT(c)x)y + x(nyT(c)) \\ &= (m+n)T(x)yc + (m+n)xT(y)c - (m+n)xT(c)y.\end{aligned} \tag{22}$$

Multiplying this identity on the left by z we get

$$\begin{aligned}(m+n)zT(xy)c = {} & (m+n)zT(x)yc \\ & + (m+n)zxT(y)c - (m+n)zxT(c)y.\end{aligned} \tag{23}$$

Then substituting x for zx in (22) we have

$$\begin{aligned}(m+n)T(zxy)c = {} & (m+n)T(zx)yc \\ & + (m+n)zxT(y)c - (m+n)zxT(c)y.\end{aligned} \tag{24}$$

From (23) and (24) we obtain

$$T(zxy) = zT(xy) + T(zx)y - zT(x)y. \tag{25}$$

We use the last identity and (21) to get

$$T(zcy) = zT(cy) + T(zc)y - zT(c)y.$$

Then by (21) we have

$$T(zyc) = zT(cy) + zT(c)y - zT(c)y,$$

and so

$$T(zy)c = zT(y)c$$

which yields $T(zy) = zT(y)$. Similarly we get $T(zy) = T(z)y$ for all $y, z \in R$ and T is two-sided centralizer. Therefore the proof is completed. ◻

The relation (25) leads to the following relation

$$F(xyx) = F(xy)x - xF(y)x + xF(yx)$$

for all $x, y \in R$, where F is an additive mapping which maps a ring R into itself. The question arises about the solution of the above equation. Let us consider some relations which are similar to the above relation. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a Jordan triple derivation in case

$$D(xyx) = D(x)yx + xD(y)x + xyD(x)$$

holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation on a $2-$torsion free ring is a Jordan triple derivation (see [3] for the details). Brešar ([5]) has proved the following result.

Theorem 4. *Let R be a $2-$ torsion free semiprime ring and let $D : R \to R$ be a Jordan triple derivation. In this case D is a derivation.*

Since, as we have mentioned above, any Jordan derivation on a 2−torsion free ring is a Jordan triple derivation, Theorem 4 generalizes Cusack's generalization of Herstein's theorem. Brešar's result above has been recently generalized by Liu and Shiue ([12]). Motivated by Theorem 4 Vukman, Eremita and Kosi-Ulbl ([16]) have proved the following result (see also [9]).

Theorem 5. *Let R be a 2− torsion free semiprime ring and let $T : R \to R$ be an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{26}$$

for all pairs $x, y \in R$. In this case T is of the form $2T(x) = qx + xq$ for all $x \in R$, where $q \in Q_S$ is some fixed element.

We proceed with the following result.

Proposition 6. *Let R be a 2−torsion free semiprime ring with the identity element e and let $F : R \to R$ be an additive mapping satisfying the relation*

$$F(xyx) = F(xy)x - xF(y)x + xF(yx) \tag{27}$$

for all pairs $x, y \in R$. In this case F is of the form

$$2F(x) = D(x) + F(e)x + xF(e),$$

where $D : R \to R$ is a derivation.

Proof. Putting in the relation (27) $y = e$ we obtain

$$F(x^2) = F(x)x - xF(e)x + xF(x) \tag{28}$$

for all $x \in R$. Let us denote $2F(x) - F(e)x - xF(e)$ by $D(x)$. Then applying the relation (28) a simple calculation shows that

$$D(x^2) = D(x)x + xD(x) \tag{29}$$

holds for all $x \in R$. We have an additive mapping $D : R \to R$ satisfying the relation (29) for all $x \in R$. In other words, D is a Jordan derivation on R. Applying Cusack's generalization of Herstein's theorem one concludes that D is a derivation, which completes the proof. ∎

Proposition 6 together with Theorem 5 leads to the following conjecture.

Conjecture 7. *Let R be a 2−torsion free semiprime ring and let $F : R \to R$ be an additive mapping satisfying the relation*

$$F(xyx) = F(xy)x - xF(y)x + xF(yx)$$

for all pairs $x, y \in R$. In this case F is of the form $2F(x) = D(x) + qx + xq$ for all $x \in R$, where $D : R \to R$ is a derivation and $q \in Q_S$ is some fixed element.

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N. Peršin
Prušnikova 48, 2000 Maribor
Slovenia
E-mail: `nina_persin@t-2.net`

J. Vukman
Department of Mathematics and Computer Science
University of Maribor
Koroška cesta 160, 2000 Maribor
Slovenia
E-mail: `joso.vukman@uni-mb.si`

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