GLASNIK MATEMATIČKI
Vol. 47(67)(2012), 149 – 152

ALTERNATE PROOF OF THE REINHOLD BAER THEOREM ON 2-GROUPS WITH NONABELIAN NORM

Yakov Berkovich
University of Haifa, Israel

Abstract. We present a new easy proof of the classical theorem due to Reinhold Baer asserting that the nonabelian norm of a 2-group G coincides with G, i.e., G is Dedekindian. Our proof is independent of all papers devoted to this theme.

According to R. Baer ([Bae1]), the norm $\mathcal{N}(G)$ of a group G is the intersection of normalizers of all subgroups of G. Clearly, the subgroup $\mathcal{N}(G)$ is characteristic in G and Dedekindian, i.e., $\mathcal{N}(G)$ is either abelian or $\mathcal{N}(G) = Q \times E \times A$, where Q_8 is the ordinary quaternion group, $\exp(E) \leq 2$ and all elements of the abelian subgroup A have odd order ([H, Theorem 12.5.4]) (it is possible that $\mathcal{N}(G)$ has subgroups that are not normal in G). Some additional information on the norm is contained in [Bae1, Bae2, Bae3, BHN, Sch, W] and in a number of other papers. For example, it is proved in [Sch] that, for an arbitrary group G, $G/\mathrm{C}_G(\mathcal{N}(G))$ is abelian and $\mathcal{N}(G) \leq \mathrm{Z}_2(G)$, where $\mathrm{Z}_2(G)$ is the second member of the upper central series of G; this was obtained as a result of intricate computations (for finite groups, see [BJ, Theorem 140.9]). It follows ([Bae3]) that if $\mathrm{Z}(G) = \{1\}$, then $\mathcal{N}(G) = \{1\}$ (for finite groups this is easily to prove using the same argument as in the proof of Theorem 3, below).

We use the same standard notation as in [Ber].

We offer a new proof of the following remarkable result due to R. Baer.

Theorem 1 ([Bae2]). *If the norm $\mathcal{N}(G)$ of a 2-group G is nonabelian, then $\mathcal{N}(G) = G$.*

We do not assume, in Theorem 1, that G is finite.

2010 *Mathematics Subject Classification.* 20D15.
Key words and phrases. Norm, ordinary quaternion group, Dedekindian groups, 2-groups of maximal class, 2-groups with nonabelian norm.

All prerequisites are collected in the following

LEMMA 2. *Let G be a finite nonabelian 2-group.*

(a) *(Dedekind; see [Ber, Theorem 1.20] and [H, Theorem 12.5.4]) If all subgroups of G are normal, then $G = Q \times E$, where $Q \in \{\{1\}, \mathrm{Q}_8\}$, $\exp(E) \leq 2$ (this is also true if G is an infinite 2-group).*

(b) *(Burnside; see [Ber, Theorem 1.2]) If G has a cyclic subgroup of index 2, then G is one of the following groups: dihedral, semidihedral, generalized quaternion or minimal nonabelian of order $> 2^3$ with cyclic center of index 4 (in the last case, $\mathcal{N}(G) \leq \Phi(G)$).*

(c) *(see [Ber, Propositions 10.17 and 1.6]) If $B < G$ is nonabelian of order 8 and $\mathrm{C}_G(B) < B$, then G is of maximal class. If G is of maximal class, then it has a cyclic subgroup of index 2 (see (b)).*

(d) *(see [Ber, Appendix 16]) If $G = B * C$ (central product) has order 16, where B is nonabelian of order 8, C is cyclic of order 4, then G has exactly 7 subgroups of order 2 and only one of them is normal in G.*

(e) *If G is of maximal class and order > 8 then $\mathcal{N}(G)$ is of order 2 unless G is generalized quaternion group with cyclic $\mathcal{N}(G)$ of order 4.*

(f) *[Ber, Theorem 10.28] G is generated by minimal nonabelian subgroups.*

If $M < \mathcal{N}(G)$ and M is normal in G, then $\mathcal{N}(G)/M \leq \mathcal{N}(G/M)$.

PROOF OF THEOREM 1. Assume, to the contrary, that $\mathcal{N}(G) < G$.

By Lemma 2(a), there is in $\mathcal{N}(G)$ a subgroup $Q \cong \mathrm{Q}_8$. By assumption, $Q < G$ so that $|G| \geq 16$. Let $A = \langle a \rangle$ and $B = \langle b \rangle$ be distinct cyclic subgroups of Q of order 4.

(i) If $Q \leq H \leq G$, then $Q \leq \mathcal{N}(H)$. This is obvious.

(ii) If $Q < H \leq G$ and $|H : Q| = 2$, then $H = Q \times C$ is Dedekindian. Assume that this is false. Then H has a nonnormal cyclic subgroup L of order ≤ 4 such that $L \not\leq Q$. Since $QL = H$, it follows that Q does not normalize L, contrary to the hypothesis. Thus, H is Dedekindian, and our claim follows in view of Lemma 2(a).

(iii) $\mathrm{C}_G(Q)$ has no cyclic subgroup of order 4. Assume, on the contrary, that $L = \langle x \rangle \leq \mathrm{C}_G(Q)$ is cyclic of order 4. Set $H = Q*L$; then $16 \leq |H| \leq 32$. However, $|H| \neq 16$, by (ii). Now let $|H| = 32$; then $H = Q \times L$ and $\langle ax \rangle$ is not b-invariant, a contradiction.

(iv) By (ii), $|G : Q| > 2$.

(v) It follows from hypothesis and (iii) that if $D = \langle d \rangle < G$ is cyclic of order 4, then $Q \cap D > \{1\}$. Assume, on the contrary, that $Q \cap D = \{1\}$. Then, by (iii), Q is not normal in $F = QD$ (otherwise, $F = Q \times D$) so $F/Q_F \cong \mathrm{D}_8$. But the norm of D_8 coincides with its center, and this is a contradiction since $Q/Q_F \leq \mathcal{N}(F/Q_F)$ and $Q/Q_F \not\leq \mathrm{Z}(G/Q_F)$.

(vi) We claim that $\exp(G) = 4$. Assume, on the contrary, that $T < G$ is cyclic of order 8. Since Q normalizes T, we get $H = QT \leq G$. Taking in mind our aim, one may assume that $G = QT$. By (v), one has $Q \cap T > \{1\}$ so that $16 \le |G| \le 32$. By (iv), $|G| > 16$. Let $|G| = 32$; then $Q \cap T = \mathrm{Z}(Q)$. Write $H_1 = AT$ and $H_2 = BT$. Since, by (i), $A \le \mathcal{N}(H_1)$, it follows that H_1 is not of maximal class in view of $A \not\le \Phi(H_1)$ (Lemma 2(b)). Similarly, H_2 is not of maximal class. Then, by Lemma 2(b), $\mho_1(T) = \Phi(H_i) \le \mathrm{Z}(H_i)$, $i = 1, 2$. Thus, $\mho_1(T)$, a cyclic subgroup of order 4, centralizes $\langle A, B\rangle = Q$, contrary to (iii).

Now we are ready to complete the proof.

By (v) and (vi), $\mho_1(G) = \mho_1(Q)$ so that $G/\mho_1(G)$, being of exponent 2, is abelian, and we conclude that $G' = \mho_1(G)$ has order 2 since G is nonabelian. The quotient group $G/\mathrm{C}_G(Q)$ is isomorphic to a subgroup of $\mathrm{D}_8 \in \mathrm{Syl}_2(\mathrm{Aut}(Q))$, and $G/\mathrm{C}_G(Q)$ contains a four-subgroup $\cong Q/\mathrm{Z}(Q)$. Since $G' = \mho_1(Q) \le \mathrm{C}_G(Q)$, we get $G/\mathrm{C}_G(Q) \cong Q/\mathrm{Z}(Q)$ since $Q \cap \mathrm{C}_G(Q) = \mathrm{Z}(Q)$, and we conclude that $G = Q\mathrm{C}_G(Q)$. Since $\exp(\mathrm{C}_G(Q)) = 2$, by (iii), we get $\mathrm{C}_G(Q) = \mathrm{Z}(Q) \times E$, where $E < \mathrm{C}_G(Q)$. In that case, $G = Q\mathrm{C}_G(Q) = Q(\mathrm{Z}(Q) \times E) = QE$, where $Q \cap E = \{1\}$. It follows that $G = Q \times E$ hence G is Dedekindian since $\exp(E) = 2$. □

The following theorem is a partial case of Schenkman's result [Sch] mentioned above.

THEOREM 3. *Let G be an arbitrary finite group such that $\mathcal{N}(G)$ is nonabelian. Then $P \in \mathrm{Syl}_2(\mathcal{N}(G))$ centralizes all elements of G of odd order and $P \le \mathrm{Z}_2(G)$.*

PROOF. Let $\mathrm{Q}_8 \cong Q \le \mathcal{N}(G)$, let $a \in Q^{\#}$ and let $x \in G$ be of order p^k, where a prime $p > 2$. Set $H = \langle a, x\rangle$. Let $y \in \langle x\rangle$ be of order p and $F = \langle a, y\rangle$. Assume that F is nonabelian. If $o(a) = 2$, then F is a nonabelian group of order $2p$ so its norm equals $\{1\}$, a contradiction since $a \in \mathcal{N}(G)$. If $o(a) = 4$, then F is minimal nonabelian with norm of order 2, a contradiction again. Thus, F is abelian. Thus, Q centralizes all subgroups of G of odd order. Since, by Lemma 2(f), $P \in \mathrm{Syl}_2(\mathcal{N}(G))$ is generated by its minimal nonabelian subgroups all of which are $\cong \mathrm{Q}_8$, it follows that P centralizes all subgroups of G of odd order. Note that P is normal in G.

Now let $P \le P_1 \in \mathrm{Syl}_2(G)$. By Theorem 1, the subgroup P_1 is Dedekindian since $P \le \mathcal{N}(P_1)$. Therefore, $\mathrm{Z}(P) \le \mathrm{Z}(P_1)$ (Lemma 2(a)). Since, by the above, $\mathrm{Z}(P)$ centralizes all elements of G of odd order, we get $\mathrm{Z}(P) \le \mathrm{Z}(G)$. Since $P/\mathrm{Z}(P) \le P_1/\mathrm{Z}(P)$, $P_1/\mathrm{Z}(P)$ is abelian and $P/\mathrm{Z}(P)$ centralizes all elements of $G/\mathrm{Z}(P)$ of odd order, we get $P/\mathrm{Z}(P) \le \mathrm{Z}(G/\mathrm{Z}(P))$ so that $P \le \mathrm{Z}_2(G)$, proving the last assertion. □

Let $Q \cong \mathrm{Q}_8$ be a subgroup of a 2-group G not necessarily finite. If Q normalizes all cyclic subgroups of G of order ≤ 8, then G is Dedekindian, as follows immediately from the proof of Theorem 1.

Problems

1. Classify the p-groups G satisfying $\mathcal{N}(H) \le \mathcal{N}(G)$ for all nonabelian $H \le G$. In particular, classify the p-groups G such that $\mathcal{N}(H) = H \cap \mathcal{N}(G)$ for all nonabelian $H \le G$.
2. Classify the finite groups H such that $\mathcal{N}(\mathrm{Q}_8 \times H)$ is nonabelian (if H is a 2-group of exponent > 2, then $\mathcal{N}(\mathrm{Q}_8 \times H)$ is abelian, by Theorem 1).
3. Classify the p-groups G such that $\mathcal{N}(G)$ is maximal in G (if $G = \langle a, b \mid a^{p^n} = b^p = 1,\ a^b = a^{1+p}\rangle$, where $n > 1$ and $n > 2$ provided $p = 2$, then $\mathcal{N}(G) = \langle a^p, b\rangle$ is maximal in G).
4. Classify the p-groups G satisfying $\mathcal{N}(H) = \mathrm{Z}(H)$ for all nonabelian $H \le G$.
5. Study the finite groups G such that, whenever $H < G$ is either minimal nonabelian or minimal nonnilpotent, then $H \cap \mathcal{N}(G) = \{1\}$.
6. Describe $\mathcal{N}(A \times B)$ and $\mathcal{N}(A * B)$ in the terms of A and B (if $A \cong \mathrm{Q}_8 \cong B$ and $G = A * B$, the central product of order 32, then $\mathcal{N}(G) = \mathrm{Z}(G)$ is of order 2 and $A = \mathcal{N}(A) \cong \mathrm{Q}_8 \cong \mathcal{N}(B) = B$).
7. Study the pairs of p-groups $H < G$ such that H normalizes all $C < G$ with $C \not\le H$ (for each H this is a separate problem. Note that $H \not\cong \mathrm{D}_8$).

Acknowledgements.

I am indebted to Zvonimir Janko drawing my attention to Theorem 1, for discussion of this note and a number of useful remarks. Note that he also produced simultaneously a new proof of Theorem 1 based on entirely other ideas; see [BJ, Theorem 140.9].

References

[Bae1] R. Baer, *Der Kern, eine charakteristische Untergruppe*, Compositio Math. **1** (1935), 254–283.

[Bae2] R. Baer, *Gruppen mit hamiltonschem Kern*, Compositio Math. **2** (1935), 241–246.

[Bae3] R. Baer, *Norm and hypernorm*, Publ. Math. Debrecen **4** (1956), 347–350.

[BHN] C. Beidleman, H. Heineken and M. Newell, *Centre and norm*, Bull. Austral. Math. Soc. **69** (2004), 457–464.

[Ber] Y. Berkovich, Groups of prime power order. Vol. 1, Walter de Gruyter, Berlin, 2008.

[BJ] Y. Berkovich and Z. Janko, Groups of prime power order. Vol. 3, Walter de Gruyter, Berlin, 2011.

[H] M. Hall, Jr., The theory of groups, Macmillan, New York, 1959.

[Sch] E. Schenkman, *On the norm of a group*, Illinois J. Math. **4** (1960), 150–152.

[W] L. T. Wos, *On commutative prime power subgroups of the norm*, Illinois J. Math. **2** (1958), 271–284.

Y. Berkovich
Department of Mathematics
University of Haifa
Mount Carmel, Haifa 31905
Israel
Received: 3.2.2011.
Revised: 18.7.2011.