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WEIGHTED VARIABLE EXPONENT AMALGAM SPACES $W(L^{p(x)}, L^q_w)$

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Abstract. In the present paper a new family of Wiener amalgam spaces $W(L^{p(x)}, L^q_w)$ is defined, with local component which is a variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^n)$ and the global component is a weighted Lebesgue space $L^q_w(\mathbb{R}^n)$. We proceed to show that these Wiener amalgam spaces are Banach function spaces. We also present new Hölder-type inequalities and embeddings for these spaces. At the end of this paper we show that under some conditions the Hardy-Littlewood maximal function is not mapping the space $W(L^{p(x)}, L^q_w)$ into itself.

1. Introduction

A number of authors worked on amalgam spaces or some special cases of these spaces. The first appearance of amalgam spaces can be traced to N. Wiener ([22]). But the first systematic study of these spaces was undertaken by F. Holland ([17, 18]). The *amalgam* of L^p and l^q on the real line is the space $(L^p, l^q)(\mathbb{R})$ (or shortly (L^p, l^q)) consisting of functions f which are locally in L^p and have l^q behavior at infinity in the sense that the norms over $[n, n+1]$ form an l^q -sequence. For $1 \leq p, q \leq \infty$ the norm

$$\|f\|_{p,q} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty$$

makes (L^p, l^q) into a Banach space. If $p = q$ then (L^p, l^q) reduces to L^p. A generalization of Wiener's definition was given by H. G. Feichtinger in [9],

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describing certain Banach spaces of functions (or measures, distributions) on locally compact groups by global behaviour of certain local properties of their elements. C. Heil in [16] gave a good summary of results concerning amalgam spaces with global components being weighted $L^q(\mathbb{R})$ spaces. For a historical background of amalgams see [15].

Let $p : \mathbb{R}^n \to [1, \infty)$ be a measurable function (called the *variable exponent* on $\mathbb{R}^n$). We put

$$p_* = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p^* = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

The *variable exponent Lebesgue space* (or *generalized Lebesgue space*) $L^{p(x)}(\mathbb{R}^n)$ is defined to be the space of measurable functions (equivalence classes) f such that

$$\rho_p(\lambda f) = \int_{\mathbb{R}^n} |\lambda f(x)|^{p(x)}\, dx < \infty$$

for some $\lambda = \lambda(f) > 0$. The function ρ_p is called *modular* of the space $L^{p(x)}(\mathbb{R}^n)$. Then

$$\|f\|_{L^{p(x)}} = \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}$$

defines a norm (*Luxemburg norm*). This makes $L^{p(x)}(\mathbb{R}^n)$ a Banach space. If $p(x) = p$ is a constant function, then the variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^n)$ coincides with the classical Lebesgue space $L^p(\mathbb{R}^n)$, see [19]. Also there are recent many interesting and important papers appeared in variable exponent Lebesgue spaces (see [3–5, 7, 8]). In this paper we will assume that $p^* < \infty$.

The space $L^1_{loc}(\mathbb{R}^n)$ consists of all (classes of) measurable functions f on $\mathbb{R}^n$ such that $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$, where χ_K is the characteristic function of K. It is a topological vector space with the family of seminorms $f \mapsto \|f\chi_K\|_{L^1}$. A *Banach function space* (shortly *BF-space*) on $\mathbb{R}^n$ is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(\mathbb{R}^n)$, that is for any compact subset $K \subset \mathbb{R}^n$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_{L^1} \le C_K \|f\|_B$ for all $f \in B$ and two functions equal almost everywhere are identified as usual. We denote it by $B \hookrightarrow L^1_{loc}(\mathbb{R}^n)$. Obviously $L^{p(x)}(\mathbb{R}^n) \hookrightarrow L^1_{loc}(\mathbb{R}^n)$ and the space $L^{p(x)}(\mathbb{R}^n)$ is a *solid space*, that is , if $f \in L^{p(x)}(\mathbb{R}^n)$ is given and $g \in L^1_{loc}(\mathbb{R}^n)$ satisfies $|g(x)| \le |f(x)|$ a.e., then $g \in L^{p(x)}(\mathbb{R}^n)$ and $\|g\|_{L^{p(x)}} \le \|f\|_{L^{p(x)}}$ by [1, Lemma 1].

A positive, measurable and locally integrable function $\vartheta : \mathbb{R}^n \to (0, \infty)$ is called a *weight function.* We say that a weight function ϑ is *submultiplicative* if

$$\vartheta(x + y) \le \vartheta(x)\vartheta(y).$$

for any $x, y \in \mathbb{R}^n$. A weight function w is *moderate* with respect to a submultiplicative function ϑ (or ϑ-moderate) if

$$w(x+y) \leq w(x)\vartheta(y)$$

for any $x, y \in \mathbb{R}^n$. If the weight w is moderate than $1/w$ is also moderate. We say that $w_1 \prec w_2$ if there exists a constant $C > 0$ such that $Cw_1(x) \leq w_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called *equivalent* and written $w_1 \approx w_2$, if $w_1 \prec w_2$ and $w_2 \prec w_1$. The space $L^q_w(\mathbb{R}^n)$ (weighted $L^q(\mathbb{R}^n)$) is the space of all complex-valued measurable functions on $\mathbb{R}^n$ for which $fw \in L^q(\mathbb{R}^n)$. Obviously $\left(L^q_w(\mathbb{R}^n), \|.\|_{L^q_w}\right)$ is a Banach space with the norm

$$\|f\|_{L^q_w} = \|fw\|_{L^q} = \left\{ \int\limits_{\mathbb{R}^n} |f(x)w(x)|^q \, dx \right\}^{\frac{1}{q}}, \quad 1 \leq q < \infty,$$

or

$$\|f\|_{L^\infty_w} = \|fw\|_{L^\infty} = \underset{x\in\mathbb{R}^n}{ess\sup} |f(x)| \, w(x), \quad q = \infty.$$

Also the dual of the space $L^q_w(\mathbb{R}^n)$ is the space $L^s_{w^{-1}}(\mathbb{R}^n)$, where $1 \leq q < \infty$, $\frac{1}{q} + \frac{1}{s} = 1$ (see [12, 14, 16]).

Given a discrete family $X = (x_i)_{i\in I}$ in $\mathbb{R}^n$ and a weighted space $L^q_w(\mathbb{R}^n)$, the *associated weighted sequence space* over X is the appropriate weighted ℓ^q -space ℓ^q_w, the *discrete* w being given by $w(i) = w(x_i)$ for $i \in I$ (see [11, Lemma 3.5]).

2. The Wiener Amalgam Space $W\left(L^{p(x)}, L^q_w\right)$

Let $C_b(\mathbb{R}^n)$ be the the regular Banach algebra (with respect to pointwise multiplication) of complex-valued bounded, continuous functions on $\mathbb{R}^n$. Also let $C_0(\mathbb{R}^n)$, $C_c(\mathbb{R}^n)$ be the spaces of complex-valued continuous function $\mathbb{R}^n$ vanishing at infinity and the space of complex-valued continuous functions with compact support defined on $\mathbb{R}^n$ endowed with its natural inductive limit topology respectively. It is known that $(C_0, \|.\|_\infty) \hookrightarrow (C_b, \|.\|_\infty)$ and the dual space of $C_c(\mathbb{R}^n)$ (with respect to its natural inductive limit topology) is $M(\mathbb{R}^n)$, the space of regular Borel measures. For every $h \in C_c(\mathbb{R}^n)$ we define the semi-norm q_h on $M(\mathbb{R}^n)$ by $q_h(h') = h'(h)$. The locally convex topology on $M(\mathbb{R}^n)$ defined by the family $(q_h)_{h\in C_c(\mathbb{R}^n)}$ of seminorms is called the topology $\sigma(M(\mathbb{R}^n), C_c(\mathbb{R}^n))$ or *weak**-topology, also called *vague topology.* We define

$$L^{p(x)}_{loc}(\mathbb{R}^n) = \left\{ \sigma \in M(\mathbb{R}^n) : \phi\sigma \in L^{p(x)}(\mathbb{R}^n) \text{ for all } \phi \in C_c(\mathbb{R}^n) \right\}.$$

$L^{p(x)}_{loc}(\mathbb{R}^n)$ is a topological vector space with respect to the family of seminorms given by $\|\sigma\|_\phi = \|\phi\sigma\|_{L^{p(x)}}$, $\phi \in C_c(\mathbb{R}^n)$.

It is known by [1, Lemma 1] that $L^{p(x)}(\mathbb{R}^n)$ is continuously embedded into $L^1_{loc}(\mathbb{R}^n)$. Hence it is easily shown that $L^{p(x)}_{loc}(\mathbb{R}^n)$ is continuously embedded into $L^1_{loc}(\mathbb{R}^n)$. It is also obvious that $L^1_{loc}(\mathbb{R}^n)$ is continuously embedded into $M(\mathbb{R}^n)$ with the *weak**-topology. Therefore $L^{p(x)}_{loc}(\mathbb{R}^n)$ is continuously embedded into $M(\mathbb{R}^n)$.

Since the general hypotheses for the Wiener amalgam space denoted by $W\left(L^{p(x)}(\mathbb{R}^n), L^q_w(\mathbb{R}^n)\right)$ (shortly $W\left(L^{p(x)}, L^q_w\right)$) are satisfied, it is defined as follows as in [9].

Let fix an open set $Q \subset \mathbb{R}^n$ with compact closure. The *Wiener amalgam space* $W\left(L^{p(x)}, L^q_w\right)$ consists of all elements $f \in L^{p(x)}_{loc}(\mathbb{R}^n)$ such that $F_f(z) = \|f\|_{L^{p(x)}(z+Q)}$ belongs to $L^q_w(\mathbb{R}^n)$; the norm of $W\left(L^{p(x)}, L^q_w\right)$ is

$$\|f\|_{W\left(L^{p(x)},L^q_w\right)} = \|F_f\|_{L^q_w}.$$

In this definition $\|f\|_{L^{p(x)}(z+Q)}$ denotes the restriction norm of f to $z+Q$, that is

$$\|f\|_{L^{p(x)}(z+Q)} = \inf\left\{\|g\|_{L^{p(x)}} : \begin{array}{c} g \in L^{p(x)}(\mathbb{R}^n), g \text{ coincides with } f \text{ on } z+Q, \text{ i.e.,} \\ hf = hg \text{ for all } h \in C_C(\mathbb{R}^n) \text{ with } \operatorname{supp}(h) \subset z+Q \end{array}\right\}.$$

By the solidity of the BF-space the assumptions imply

$$\|f\|_{L^{p(x)}(z+Q)} = \|f\chi_{z+Q}\|_{L^{p(x)}}.$$

The following theorem, based on [9, Theorem 1], describes the basic properties of $W\left(L^{p(x)}, L^q_w\right)$.

THEOREM 2.1.

i) $W\left(L^{p(x)}, L^q_w\right)$ *is a Banach space with norm* $\|.\|_{W\left(L^{p(x)},L^q_w\right)}$.

ii) $W\left(L^{p(x)}, L^q_w\right)$ *is continuously embedded into* $L^{p(x)}_{loc}(\mathbb{R}^n)$.

iii) *The space*

$$\Lambda_0 = \left\{f \in L^{p(x)}(\mathbb{R}^n) : supp\,(f) \text{ is compact}\right\}$$

is continuously embedded into $W\left(L^{p(x)}, L^q_w\right)$.

iv) $W\left(L^{p(x)}, L^q_w\right)$ *does not depend on the particular choice of* Q, *i.e. different choices of* Q *define the same space with equivalent norms.*

By iii) and [1, Lemma 4] it is easy to see that $C_c(\mathbb{R}^n)$ is continuously embedded into $W\left(L^{p(x)}, L^q_w\right)$.

By using the techniques in [13], we prove the following proposition.

PROPOSITION 2.2. $W\left(L^{p(x)}, L^q_w\right)$ *is a solid BF-space on* $\mathbb{R}^n$.

Proof. Let $K \subset \mathbb{R}^n$ be a compact subset. Since $C_0(\mathbb{R}^n)$ is a regular Banach algebra with respect to pointwise multiplication one may choose a function $h_0 \in C_c(\mathbb{R}^n)$ with $0 \leq h_0(x) \leq 1$ and $h_0(x) = 1$ for all $x \in K$. Let $\mathrm{supp}(h_0) = K_0$. Then $\chi_K(x) \leq h_0(x)$ and hence $\chi_K(x)|f(x)| \leq h_0(x)|f(x)|$ for all $x \in \mathbb{R}^n$. Since $L^{p(x)}(\mathbb{R}^n) \hookrightarrow L^1_{loc}(\mathbb{R}^n)$, there exists $D_{K_0} > 0$ such that

$$\int_{K_0} |h_0(x)f(x)|\,dx \leq D_{K_0} \|h_0 f\|_{L^{p(x)}}. \tag{2.1}$$

Since $K \subset K_0$

$$\int_K |f(x)|\,dx \leq \int_{K_0} |h_0(x)f(x)|\,dx. \tag{2.2}$$

On the other hand by Theorem 2.1 ii), $W\left(L^{p(x)}, L^q_w\right) \hookrightarrow L^{p(x)}_{loc}(\mathbb{R}^n)$. Hence for this $h_0 \in C_c(\mathbb{R}^n)$ there exists a constant number $D_{h_0} > 0$ such that

$$p_{h_0}(f) = \|h_0 f\|_{L^{p(x)}} \leq D_{h_0} \|f\|_{W\left(L^{p(x)}, L^q_w\right)} \tag{2.3}$$

for all $f \in W\left(L^{p(x)}, L^q_w\right)$. Combining (2.1), (2.2) and (2.3) we obtain

$$\begin{aligned} \int_K |f(x)|\,dx &\leq \int_{K_0} |h_0(x)f(x)|\,dx \leq D_{K_0} \|h_0 f\|_{L^{p(x)}} \\ &\leq D_{K_0} D_{h_0} \|f\|_{W\left(L^{p(x)}, L^q_w\right)} = C_K \|f\|_{W\left(L^{p(x)}, L^q_w\right)}. \end{aligned}$$

It is easy to show that $W\left(L^{p(x)}, L^q_w\right)$ is solid. ∎

Proposition 2.3. *Let w_1, w_2 and w_3 be weight functions. Suppose that there exist constants $C_1, C_2 > 0$ such that*

$$\forall h \in L^{p_1(x)}(\mathbb{R}^n), \forall k \in L^{p_2(x)}(\mathbb{R}^n), \quad \|hk\|_{L^{p_3(x)}} \leq C_1 \|h\|_{L^{p_1(x)}} \|k\|_{L^{p_2(x)}}$$

and

$$\forall u \in L^{q_1}_{w_1}(\mathbb{R}^n), \forall v \in L^{q_2}_{w_2}(\mathbb{R}^n), \quad \|uv\|_{L^{q_3}_{w_3}} \leq C_2 \|u\|_{L^{q_1}_{w_1}} \|v\|_{L^{q_2}_{w_2}}.$$

Then there exists $C > 0$ such that for all $f \in W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)$ and $g \in W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right)$ we have

$$\|fg\|_{W\left(L^{p_3(x)}, L^{q_3}_{w_3}\right)} \leq C \|f\|_{W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)} \|g\|_{W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right)}.$$

In other words

$$W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right).W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right) \subset W\left(L^{p_3(x)}, L^{q_3}_{w_3}\right).$$

Proof. Let $f \in W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)$ and $g \in W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right)$. Then

$$\begin{aligned} \|fg\|_{W\left(L^{p_3(x)}, L^{q_3}_{w_3}\right)} &= \left\| \|fg\chi_{z+Q}\|_{L^{p_3(x)}} \right\|_{L^{q_3}_{w_3}} \\ &= \left\| \|(f\chi_{z+Q})(g\chi_{z+Q})\|_{L^{p_3(x)}} \right\|_{L^{q_3}_{w_3}} \\ &\le C_1 \left\| \|f\chi_{z+Q}\|_{L^{p_1(x)}} \|g\chi_{z+Q}\|_{L^{p_2(x)}} \right\|_{L^{q_3}_{w_3}} . \end{aligned} \tag{2.4}$$

If we put

$$F_f(z) = \|f\chi_{z+Q}\|_{L^{p_1(x)}} \text{ and } F_g(z) = \|g\chi_{z+Q}\|_{L^{p_2(x)}},$$

by (2.4) we obtain

$$\begin{aligned} \|fg\|_{W\left(L^{p_3(x)}, L^{q_3}_{w_3}\right)} &\le C_1 \|F_f F_g\|_{L^{q_3}_{w_3}} \le C_1 C_2 \|F_f\|_{L^{q_1}_{w_1}} \|F_g\|_{L^{q_2}_{w_2}} \\ &= C_1 C_2 \|f\|_{W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)} \|g\|_{W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right)} \\ &= C \|f\|_{W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)} \|g\|_{W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right)} . \end{aligned}$$

□

Corollary 2.4. *Define $k(x)$ by $\frac{1}{p(x)} + \frac{1}{r(x)} = \frac{1}{k(x)} \le 1$ and suppose $k^* < \infty$, $\frac{1}{q} + \frac{1}{s} = 1$. Then there exists a constant $C > 0$ such that*

$$\|fg\|_{W\left(L^{k(x)}, L^1\right)} \le C \|f\|_{W\left(L^{p(x)}, L^q_w\right)} \|g\|_{W\left(L^{r(x)}, L^s_{w^{-1}}\right)}$$

for all $f \in W\left(L^{p(x)}, L^q_w\right)$ and $g \in W\left(L^{r(x)}, L^s_{w^{-1}}\right)$. Thus

$$W\left(L^{p(x)}, L^q_w\right) W\left(L^{r(x)}, L^s_{w^{-1}}\right) \subset W\left(L^{k(x)}, L^1\right).$$

Proof. Let $f \in W\left(L^{p(x)}, L^q_w\right)$ and $g \in W\left(L^{r(x)}, L^s_{w^{-1}}\right)$. Then $f\chi_{z+Q} \in L^{p(x)}(\mathbb{R}^n)$ and $g\chi_{z+Q} \in L^{r(x)}(\mathbb{R}^n)$. Thus there exists $C(z) > 0$ such that

$$\|fg\chi_{z+Q}\|_{L^{k(x)}} \le C(z) \|f\chi_{z+Q}\|_{L^{p(x)}} \|g\chi_{z+Q}\|_{L^{r(x)}} \tag{2.5}$$

by [20, Lemma 2.18]. Also it is known by [20, Lemma 2.18] that $C(z) \le 2k^* = C < \infty$. Since $L^1(\mathbb{R}^n)$ is solid, then by (2.5)

$$\begin{aligned} \|fg\|_{W\left(L^{k(x)}, L^1\right)} &\le 2k^* \left\| \|f\chi_{z+Q}\|_{L^{p(x)}} \|g\chi_{z+Q}\|_{L^{r(x)}} \right\|_{L^1} \\ &= C \left\| \|f\chi_{z+Q}\|_{L^{p(x)}} \|g\chi_{z+Q}\|_{L^{r(x)}} \right\|_{L^1} . \end{aligned} \tag{2.6}$$

Finally since $\|f\chi_{z+Q}\|_{L^{p(x)}} \in L^q_w(\mathbb{R}^n)$, $\|g\chi_{z+Q}\|_{L^{r(x)}} \in L^s_{w^{-1}}(\mathbb{R}^n)$, by the Hölder inequality and (2.6) we obtain

$$\|fg\|_{W\left(L^{k(x)}, L^1\right)} \le C \|f\|_{W\left(L^{p(x)}, L^q_w\right)} \|g\|_{W\left(L^{r(x)}, L^s_{w^{-1}}\right)} .$$

□

Proposition 2.5. a) *If $p_1(x) \le p_2(x)$, $q_2 \le q_1$ and $w_1 \prec w_2$, then*

$$W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right) \subset W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right).$$

b) *If* $p_1(x) \le p_2(x)$, $q_2 \le q_1$ *and* $w_1 \prec w_2$, *then*

$$W\left(L^{p_1(x)} \cap L^{p_2(x)}, L^{q_2}_{w_2}\right) \subset W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right).$$

c) *If* $w_1 \prec w_2$, *then*

$$L^{p^*}_{w_2}(\mathbb{R}^n) \subset W\left(L^{p(x)}, L^{p^*}_{w_1}\right) \text{ and } W\left(L^{p(x)}, L^{p_*}_{w_2}\right) \subset L^{p_*}_{w_1}(\mathbb{R}^n).$$

PROOF. a) Let $f \in W\left(L^{p_2(x)}, L^{q_2}_{w}\right)$ be given. Since $p_1(x) \le p_2(x)$, then $L^{p_2(x)}(z+Q) \hookrightarrow L^{p_1(x)}(z+Q)$ and

$$\begin{aligned} \|f\chi_{z+Q}\|_{L^{p_1(x)}} &\le (\mu(z+Q)+1)\|f\chi_{z+Q}\|_{L^{p_2(x)}} \\ &= (\mu(Q)+1)\|f\chi_{z+Q}\|_{L^{p_2(x)}} \end{aligned} \tag{2.7}$$

for all $z \in \mathbb{R}^n$ by [19, Theorem 2.8], where μ is the Lebesgue measure. Hence by (2.7) and the solidity of $L^{q_2}_{w_2}(\mathbb{R}^n)$ we have

$$W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right) \subset W\left(L^{p_1(x)}, L^{q_2}_{w_2}\right).$$

It is known by [11, Proposition 3.7], that

$$W\left(L^{p_1(x)}, L^{q_2}_{w_2}\right) \subset W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)$$

if and only if $\ell^{q_2}_{w_2} \subset \ell^{q_1}_{w_1}$, where $\ell^{q_2}_{w_2}$ and $\ell^{q_1}_{w_1}$ are the associated sequence spaces of $L^{q_2}_{w_2}(\mathbb{R}^n)$ and $L^{q_1}_{w_1}(\mathbb{R}^n)$ respectively. Since $q_2 \le q_1$ and $w_1 \prec w_2$, then $\ell^{q_2}_{w_2} \subset \ell^{q_1}_{w_1}$ ([13]). This completes the proof.

b) The proof of this part is easy by a).

c) By using a) and [16, Proposition 11.5.2], we have

$$L^{p^*}_{w_2}(\mathbb{R}^n) = W\left(L^{p^*}, L^{p^*}_{w_2}\right) \subset W\left(L^{p(x)}, L^{p^*}_{w_2}\right).$$

Since $w_1 \prec w_2$, then $\ell^{p^*}_{w_2} \subset \ell^{p^*}_{w_1}$ ([12]). Hence

$$L^{p^*}_{w_2}(\mathbb{R}^n) \subset W\left(L^{p(x)}, L^{p^*}_{w_1}\right).$$

Similarly we can prove

$$W\left(L^{p(x)}, L^{p_*}_{w_2}\right) \subset L^{p_*}_{w_1}(\mathbb{R}^n).$$

□

The following lemma follows directly from the closed graph theorem.

LEMMA 2.6. *If* $p_1^*, p_2^* < \infty$, *then* $L^{p_1(x)}(\mathbb{R}^n) \subset L^{p_2(x)}(\mathbb{R}^n)$ *if and only if there exists a constant* $C > 0$ *such that* $\|f\|_{L^{p_2(x)}} \le C\|f\|_{L^{p_1(x)}}$ *for all* $f \in L^{p_1(x)}(\mathbb{R}^n)$.

PROPOSITION 2.7. *Let* B *be any solid space. If* $q_2 \le q_1$ *and* $w_1 \prec w_2$, *then we have*

$$W\left(B, L^{q_1}_{w_1} \cap L^{q_2}_{w_2}\right) = W\left(B, L^{q_2}_{w_2}\right).$$

PROOF. It is easy to see that the associated sequence space of $L^{q_1}_{w_1}(\mathbb{R}^n)\cap L^{q_2}_{w_2}(\mathbb{R}^n)$ is $\ell^{q_1}_{w_1}\cap\ell^{q_2}_{w_2}$. Since $q_2\leq q_1$ and $w_1\prec w_2$, thus the associated sequence space of $L^{q_1}_{w_1}(\mathbb{R}^n)\cap L^{q_2}_{w_2}(\mathbb{R}^n)$ is $\ell^{q_2}_{w_2}$. Then by [11, Proposition 3.7]

$$W\left(B, L^{q_1}_{w_1}\cap L^{q_2}_{w_2}\right)=W\left(B, L^{q_2}_{w_2}\right).$$

□

COROLLARY 2.8. *a) If $p_1^*, p_2^*<\infty$, $L^{p_1(x)}(\mathbb{R}^n)\subset L^{p_2(x)}(\mathbb{R}^n)$, $q_2\leq q_1$, $q_4\leq q_3$, $q_4\leq q_2$, $w_1\prec w_2$, $w_3\prec w_4$ and $w_2\prec w_4$, then*

$$W\left(L^{p_1(x)}, L^{q_3}_{w_3}\cap L^{q_4}_{w_4}\right)=W\left(L^{p_1(x)}, L^{q_4}_{w_4}\right)\subset W\left(L^{p_2(x)}, L^{q_1}_{w_1}\cap L^{q_2}_{w_2}\right)$$
$$=W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right).$$

b) If $p_1(x)\leq p_2(x)$, $q_1\leq q_2$ and $w_2\prec w_1$, then

$$W\left(L^{p_1(x)}\cap L^{p_2(x)}, L^{q_1}_{w1}\right)\subset W\left(L^{p_2(x)}, L^{q_2}_{w_2}\right).$$

A general interpolation theorem in Wiener Amalgam space has been given by H. Feichtinger (see [10, Theorem 2.2]). We will give a similar theorem for $W\left(L^{p(x)}, L^q_w\right)$ next:

PROPOSITION 2.9. *If $p_0(x)$ and $p_1(x)$ are variable exponents with $1<p_{j,*}\leq p_j^*<\infty$, $j=0,1$. Then, for $\theta\in(0,1)$, we have*

$$\left[W\left(L^{p_0(x)}, L^{q_0}_{w_0}\right), W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)\right]_{[\theta]}=W\left(\left[L^{p_0(x)}, L^{p_1(x)}\right]_{[\theta]}, L^{q_\theta}_{w}\right)$$
$$=W\left(L^{p_\theta(x)}, L^{q_\theta}_{w}\right)$$

where $\frac{1}{p_\theta(x)}=\frac{1-\theta}{p_0(x)}+\frac{\theta}{p_1(x)}$, $\frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$, $w=w_0^{1-\theta}w_1^{\theta}$.

PROOF. By [10, Theorem 2.2] the interpolation space

$$\left[W\left(L^{p_0(x)}, L^{q_0}_{w_0}\right), W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)\right]_{[\theta]}$$

for $\left(W\left(L^{p_0(x)}, L^{q_0}_{w_0}\right), W\left(L^{p_1(x)}, L^{q_1}_{w_1}\right)\right)$ is $W\left(\left[L^{p_0(x)}, L^{p_1(x)}\right]_{[\theta]}, \left[L^{q_0}_{w_0}, L^{q_1}_{w_1}\right]_{[\theta]}\right)$. We know that $\left[L^{q_0}_{w_0}, L^{q_1}_{w_1}\right]_{[\theta]}=L^{q_\theta}_{w}$, [2] and by [6, Corollary A.2] that $\left[L^{p_0(x)}, L^{p_1(x)}\right]_{[\theta]}=L^{p_\theta(x)}$. □

3. THE HARDY-LITTLEWOOD MAXIMAL FUNCTION ON $W\left(L^{p(x)}, L^q_w\right)(\mathbb{R}^n)$

We use the notation $B_r(x)$ to denote the open ball centered at x of radius r. For a locally integrable function f on $\mathbb{R}$, we define the (centered) Hardy-Littlewood maximal function Mf of f by

$$Mf(x)=\sup_{r>0}\frac{1}{\mu\left(B_r(x)\right)}\int_{B_r(x)}|f(y)|\,dy. \tag{3.1}$$

where the supremum is taken over all balls $B_r(x)$ and $\mu(B_r(x))$ denotes the Lebesgue measure of $B_r(x)$.

Although the local Hardy-Littlewood maximal function has been shown to be a bounded mapping on $L^{p(x)}$ over a bounded domain, it is not bounded on many of the amalgam spaces. We have the following result.

PROPOSITION 3.1. *Let* $p : \mathbb{R} \to [1, \infty)$, $1 \leq q \leq \infty$ *and* w *is a weight function. If* $\frac{1}{w} \in L^s(\mathbb{R})$ *and* $\frac{1}{q} + \frac{1}{s} = 1$ *then the Hardy-Littlewood maximal function* M *is not bounded on* $W\left(L^{p(x)}(\mathbb{R}), L^q_w(\mathbb{R})\right)$.

PROOF. Since $\frac{1}{w} \in L^s(\mathbb{R})$ and $\frac{1}{q} + \frac{1}{s} = 1$ then $L^q_w(\mathbb{R}) \subset L^1(\mathbb{R})$. Hence

$$W\left(L^{p(x)}(\mathbb{R}), L^q_w(\mathbb{R})\right) \subset W\left(L^{p(x)}(\mathbb{R}), L^1(\mathbb{R})\right) \subset L^1(\mathbb{R}). \tag{3.2}$$

Take the indicator function $\chi_{[-1,1]}$. It obvious by Theorem 2.1 iii) that $\chi_{[-1,1]} \in W\left(L^{p(x)}(\mathbb{R}), L^q_w(\mathbb{R})\right)$. By [21, Theorem 1] the Hardy-Littlewood maximal function $f \to M(f)$ is not bounded on $L^1(\mathbb{R})$. Also if $f \in L^1(\mathbb{R})$ is not identically zero then $M(f)$ is never integrable on $\mathbb{R}$. This implies that the Hardy-Littlewood maximal function $M\left(\chi_{[-1,1]}\right)$ is not in $L^1(\mathbb{R})$. Hence $M\left(\chi_{[-1,1]}\right) \notin W\left(L^{p(x)}(\mathbb{R}), L^q_w(\mathbb{R})\right)$. This completes the proof. ∎

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