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ON DETERMINANTS OF RECTANGULAR MATRICES WHICH HAVE LAPLACE'S EXPANSION ALONG ROWS

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Abstract. Let A be any given $m \times n$ $(m \leq n)$ matrix over some field and let $\det A$ be the determinant of A calculated by Definition 1 given in [1]. Let $\det^* A$ denote determinant of A calculated by any other definition which possess Laplace's expansion along rows. Then there exists constant α such that $\det^* A = \alpha \det A$.

1. Introduction

In [1] we have the following definition of determinant of a rectangular matrix. Let A be a $m \times n$ matrix with $m \leq n$. Then

$$\det A = \sum_{1 \leq j_1 < \cdots < j_m \leq n} (-1)^{1+\cdots+m+j_1+\cdots+j_m} \begin{vmatrix} a_{1j_1} & \dots & a_{1j_m} \\ \dots & \dots & \dots \\ a_{mj_1} & \dots & a_{mj_m} \end{vmatrix}. \tag{1.1}$$

We show that this determinant possesses Laplace's expansion along rows, that is, for each $1 \leq i \leq m$ it is valid

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_j^i, \tag{1.2}$$

where A_j^i is the minor of the element a_{ij}.

The general Laplace's expansion along rows and many other interesting properties of this determinant are also established. Very interesting properties refer to its geometrical interpretation (see references from [2] to [5]).

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2. On determinants of rectangular matrices which possess Laplace's expansion along rows

We denote by $\mathbb{M}_{m\times n}$ the set of all $m \times n$ real matrices with $m \leq n$. For brevity we shall often write $(a_{ij})_{m\times n}$ instead of

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

Theorem 2.1. *Let $\mathbb{F}_{m,n}$ denote the set of all functionals defined on the set $\mathbb{M}_{m\times n}$ such that the following is valid:*

($\mathbf{i}_1$) *Every functional $f_{m,n}$ from the set $\mathbb{F}_{m,n}$ is linear with respect to the rows. If $m = 1$, then for every functional $f_{1,n}$ from the set $\mathbb{F}_{1,n}$ there are real numbers $\alpha^1_{1,n}, \dots, \alpha^n_{1,n}$ such that*

$$f_{1,n}(a_1, \dots, a_n) = \alpha^1_{1,n} a_1 + \dots + \alpha^n_{1,n} a_n \tag{2.1}$$

for every $(a_1, \dots, a_n) \in \mathbb{M}_{1\times n}$.

($\mathbf{i}_2$) *For every real matrix $A = (a_{ij})_{(m+1)\times(n+1)}$ and positive integer i ($1 \leq i \leq m+1$)*

$$f_{m+1,n+1}(A) = \sum_{j=1}^{n+1} (-1)^{i+j} a_{ij} f_{m,n}(A^i_j), \tag{2.2}$$

where A^i_j denotes the minor of a_{ij}.

Then there are real numbers $\alpha_1, \alpha_2, \dots, \alpha_{n-m+1}$ such that

$$f_{m,n} = \alpha_{n-m+1} \det_{m,n},$$

that is, $f_{m,n}(X) = \alpha_{n-m+1} \det_{m,n}(X)$, for every matrix $X \in \mathbb{M}_{m\times n}$. In other words, $\det_{m,n}$ are (up to factor proportionality) only functionals with the properties ($\mathbf{i_1}$) and ($\mathbf{i_2}$).

Proof. By induction on m, first let $m = 1$ and let $f_{1,n}$ be a functional given by (2.1). Let $A = (a_{ij})_{2\times(n+1)}$ be the matrix

$$A = \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \dots & 0 \\ 0 & \dots & 0 & \dots & 1 \dots & 0 \end{pmatrix},$$

where $a_{1r} = a_{2s} = 1$ ($1 \leq r < s \leq n+1$), and the all other a_{ij} ($i \neq r$ and $j \neq s$) are equal zero. If by (2.2) we expand the matrix A along first row ($i = 1$) we get

$$\begin{aligned} f_{2,n+1}(A) &= (-1)^{1+r} f_{1,n}(0, \dots, 1, 0, \dots, 0) \\ &= (-1)^{1+r} \alpha^{s-1}_{1,n}, \end{aligned} \tag{2.3}$$

where in the matrix $(0,\dots,0,1,0,\dots,0)$ we have $a_{s-1}=1$. But if we expand along the second row ($i=2$) we get

$$\begin{aligned} f_{2,n+1}(A) &= (-1)^{2+s} f_{1,n}(0,\dots,0,1,0,\dots,0) \\ &= (-1)^{2+s}\alpha_{1,n}^{r}, \end{aligned} \tag{2.4}$$

where in the matrix $(0,\dots,0,1,0,\dots,0)$ we have $a_r=1$.

Comparing (2.3) with (2.4) we have

$$\alpha_{1,n}^{r} = (-1)^{r+s+1}\alpha_{1,n}^{s-1}, \quad 1\le r<s\le n+1.$$

Taking $r=1$ and denoting $\alpha_{1,n}^{1}$ by α_n we have the following notation

$$(-1)^{2+s}\alpha_{1,n}^{s-1} = \alpha_n, \quad s=3,4,\dots,n+1$$

from which it follows

$$\alpha_{1,n}^{2} = -\alpha_n,\ \alpha_{1,n}^{3} = +\alpha_n,\ \alpha_{1,n}^{4} = -\alpha_n$$

and so on (alternatively).

Thus, the expansion for $f_{1,n}$ can be written as

$$\begin{aligned} f_{1,n}(a_1,\dots,a_n) &= \alpha_n\left(a_1-a_2+\cdots+(-1)^{1+n}a_n\right) \\ &= \alpha_n \mathrm{det}_{1,n}(a_1,\dots,a_n). \end{aligned} \tag{2.5}$$

Now, since $f_{1,n}=\alpha_n\mathrm{det}_{1,n}$, we suppose that

$$f_{m,n} = \alpha_{n-m+1}\mathrm{det}_{m,n}$$

for some positive integer m. Then for $m+1$ we can write

$$\begin{aligned} f_{m+1,n+1}(A) &= \sum_{j=1}^{n+1}(-1)^{1+j}a_{1j}f_{m,n}(A_j^1) \\ &= \sum_{j=1}^{n+1}(-1)^{1+j}a_{1j}\alpha_{n-m+1}\mathrm{det}_{m,n}(A_j^1) \\ &= \alpha_{n-m+1}\mathrm{det}_{m+1,n+1}(A) \end{aligned}$$

or, since $(n+1)-(m+1)+1=n-m+1$,

$$f_{m+1,n+1} = \alpha_{n-m+1}\mathrm{det}_{m+1,n+1}. \tag{2.6}$$

The induction on m is complete and Theorem 2.1 is proved ◻

We now show how other determinants of rectangular matrices which have Laplace's expansion along rows can be defined. Namely, we have the following theorem.

Theorem 2.2. *Let $A \in \mathbb{M}_{m\times n}$ be given and let s be any given integer such that $m < s \leq n$. Let $\det^* A$ be defined as*

$$(2.7)\qquad \det{}^* A = \sum_{1\leq j_1<\cdots<j_s\leq n} (-1)^{1+\cdots+m+j_1+\cdots+j_s} \begin{vmatrix} a_{1j_1} & \dots & a_{1j_s} \\ \dots & \dots & \dots \\ a_{mj_1} & \dots & a_{mj_s} \end{vmatrix},$$

where the right-hand side refers to the determinant calculated by (1.1)*. Then* $\det^*$ *has Laplace's expansion along rows.*

Proof. The proof is going in exactly the same way as the proof that $\det A$ has Laplace's expansion along rows. ∎

Here let us remark that $s = m$ in the determinant given by (1.1), and that in the determinant given by (2.7) we take $s > m$.

In this connection, notice that

$$(2.8)\qquad \det{}^* A = \alpha^m_{n,s} \det A,$$

where $\alpha^m_{n,s}$ is an integer given by

$$(2.9)\qquad \alpha^m_{n,s} = \begin{cases} 0, & n-m \text{ even, } s-m \text{ odd} \\ (-1)^{\left[\frac{m+1}{2}\right]+\left[\frac{s+1}{2}\right]} \begin{pmatrix} \left[\frac{n-m}{2}\right] \\ \left[\frac{s-m}{2}\right] \end{pmatrix}, & \text{in all other cases} \end{cases}.$$

Here $[x]$ denotes the largest integer which does not exceed x.

To prove that holds (2.9) we first prove the following lemma.

Lemma 2.3. *Let n and s be any given positive integer such that $1 \leq s \leq n$. Then*

$$(2.10)\qquad \sum_{1\leq j_1<\cdots<j_s\leq n} (-1)^{j_1+\cdots+j_s} = \begin{cases} 0, & n \text{ even and } s \text{ odd} \\ (-1)^{\left[\frac{s+1}{2}\right]} \begin{pmatrix} \left[\frac{n}{2}\right] \\ \left[\frac{s}{2}\right] \end{pmatrix}, & \text{in all other cases} \end{cases}.$$

Proof. Let by i be denoted the number of all odd integers in the set $\{j_1, \dots, j_s\}$. Then

$$(-1)^{j_1+\cdots+j_s} = (-1)^i.$$

Since between integers $1, \dots, n$ there are $\left[\frac{n+1}{2}\right]$ odd and $\left[\frac{n}{2}\right]$ even, it is clear that

$$\begin{pmatrix} \left[\frac{n}{2}\right] \\ s-i \end{pmatrix} \begin{pmatrix} \left[\frac{n+1}{2}\right] \\ i \end{pmatrix}$$

is the number of all s-tuples $j_1, \dots, j_s$, $(1 \leq j_1 < \cdots < j_s \leq n)$ such that there are i odd and $s-i$ even integers. Thus it holds

$$\sum_{1\leq j_1<\cdots<j_s\leq n} (-1)^{j_1+\cdots+j_s} = \sum_{i=0}^{s} (-1)^i \begin{pmatrix} \left[\frac{n}{2}\right] \\ s-i \end{pmatrix} \begin{pmatrix} \left[\frac{n+1}{2}\right] \\ i \end{pmatrix}.$$

It is easy to see that the right-hand side of the above relation is the coefficient of x^s in the polynomial

$$f(x) = (1-x)^{\left[\frac{n+1}{2}\right]}(1+x)^{\left[\frac{n}{2}\right]}.$$

This polynomial can also be written as

$$f(x) = \begin{cases} (1-x^2)^{\left[\frac{n}{2}\right]}, & n \text{ even} \\ (1-x^2)^{\left[\frac{n}{2}\right]}(1-x), & n \text{ odd} \end{cases}$$

or, using binomial formula,

$$f(x) = \begin{cases} \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]} (-1)^i \binom{\left[\frac{n}{2}\right]}{i} x^{2i}, & n \text{ even}, \\ \displaystyle\sum_{i=0}^{\left[\frac{n}{2}\right]} \left((-1)^i \binom{\left[\frac{n}{2}\right]}{i} x^{2i} + (-1)^{i+1} \binom{\left[\frac{n}{2}\right]}{i} x^{2i+1} \right), & n \text{ odd}. \end{cases}$$

From the above relations it can be seen that for the coefficient of x^s, depending on n, the following holds.

If n is even then coefficient of x^s is given by

$$\begin{cases} 0, & s \text{ odd}, \\ (-1)^{\frac{s}{2}} \binom{\left[\frac{n}{2}\right]}{\frac{s}{2}}, & s \text{ even}, \end{cases}$$

but if n is odd, then the coefficient of x^s is given by

$$\begin{cases} (-1)^{\frac{s+1}{2}} \binom{\left[\frac{n}{2}\right]}{\frac{s-1}{2}}, & s \text{ odd}, \\ (-1)^{\frac{s}{2}} \binom{\left[\frac{n}{2}\right]}{\frac{s}{2}}, & s \text{ even}. \end{cases}$$

Hence, since for even s we have $\left[\frac{s+1}{2}\right] = \frac{s}{2} = \left[\frac{s}{2}\right]$, and for odd s we have $\frac{s+1}{2} = \left[\frac{s+1}{2}\right]$ and $\frac{s-1}{2} = \left[\frac{s}{2}\right]$, it is clear that holds (2.10).

Now it is not difficult to show that (2.8) and (2.9) hold for every $1 \times n$ real matrix and $m \le s \le n$. Also, it is not difficult to show that (2.8) and (2.9) hold for every real $m \times n$ matrix and $m \le s \le n$.

Here is an example. Let $A \in \mathbb{M}_{2\times 6}$ and let $s = 4$. Then $\det^* A = -2 \det A$, since

$$\alpha_{6,4}^2 = (-1)^{1+2} \binom{2}{1} = -2.$$

□

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