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THE SPECIAL HYPERSURFACES OF MINKOWSKI SPACE

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Abstract. Let $x : (M^n, F) \hookrightarrow (V^{n+1}, \overline{F})$ be a simply connected hypersurface in a Minkowski space $(V^{n+1}, \overline{F})$. In this paper, using the Gauss formula of Chern connection on Finsler submanifolds, we shall prove that if $x(p)$ is normal to $T_p(M)(\forall p \in M)$, then M with the induced metric is isometric to the standard Euclidean sphere.

1. Introduction

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M, Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM \backslash 0$;
(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;
(iii) Strong convexity: The fundamental quadratic form $g = g_{ij}(x, Y)dx^i \otimes dx^j$ is positively definite, where $g_{ij} = [\frac{1}{2}F^2]_{Y^iY^j} = \frac{\partial^2}{\partial Y_i \partial Y_j}(\frac{1}{2}F^2)$.

The simplest Finsler manifolds are Minkowski spaces. Let V^{n+1} be a real vector space. A Finsler metric $\overline{F} : TV^{n+1} \to [0, \infty)$ is called Minkowski if $\overline{F}$ is a function of $\overline{Y} \in V^{n+1}$ only and $(V^{n+1}, \overline{F})$ is called Minkowski space.

Let $x : M^n \hookrightarrow R^{n+1}$ be a simply connected Riemannian hypersurface of the Euclidean spaces. It is well-known that if the position vector x_p is normal to $T_p(M)(\forall p \in M)$, then M is an Euclidean sphere. The main purpose of this paper is to generalize the result from the Riemannian hypersurfaces of the Euclidean spaces to the Finsler hypersurfaces of the Minkowski space. To

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the author's knowledge, there is no one using the induced Chern conection in studying Finsler submanifolds. In this paper, by the Gauss formula of Chern connection on Finsler submanifolds, we shall prove the following

MAIN THEOREM. *Let $x : (M^n, F) \hookrightarrow (V^{n+1}, \overline{F})$ be a simply connected hypersurface in a Minkowski space $(V^{n+1}, \overline{F})$. If $x(p)$ is normal to $T_p(M)(\forall p \in M)$, then M with the induced metric is isometric to the standard Euclidean sphere.*

2. The Gauss formula

Let (M^n, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form and the *Cartan* tensor as follows:

$$\omega = \frac{\partial F}{\partial Y^i} dx^i, \qquad A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \qquad A_{ijk} := \frac{F \partial g_{ij}}{2 \partial Y^k}.$$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. We have[5]

$$\begin{aligned} g_Y(U,V) &= \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)), \\ A_Y(U,V,W) &= \overline{A}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V), \varphi_*(W)), \end{aligned} \tag{2.1}$$

where $Y, U, V, W \in TM, \overline{g}$ and $\overline{A}$ are the fundamental tensor and the *Cartan* tensor of $\overline{M}$, respectively.

It can be seen from (2.1) that $\varphi^*(\overline{\omega}) = \omega$, where $\overline{\omega}$ is the *Hilbert* form of $\overline{M}$.

We shall make use of the following convention:

$$\varphi_*(U) := U, \quad \forall U \in TM,$$

and

$$\begin{aligned} &1 \le i, j, \cdots \le n, \qquad n+1 \le \alpha, \beta, \cdots \le n+p, \\ &1 \le a, b, \cdots \le n+p, \qquad 1 \le \lambda, \mu, \cdots \le n-1. \end{aligned}$$

Take a $\overline{g}$-orthonormal frame form $\{e_a\}$ for $\pi^* T\overline{M}$ and $\{\omega^a\}$ to be its local dual coframe, such that $\{e_i\}$ is a frame field of $\pi^* TM$ and ω^n is the *Hilbert* form. Let θ^a_b and ω^i_j denote the Chern connection 1-form of $\overline{F}$ and F, respectively, i.e., $\overline{\nabla} e_a = \theta^b_a e_b$ and $\nabla e_i = \omega^j_i e_j$, where $\overline{\nabla}$ and ∇ are the Chern connection of $\overline{M}$ and M, respectively. We obtain

$$A(e_i, e_j, e_n) = \overline{A}(e_a, e_b, e_n) = 0, \ e_n = \frac{Y^i}{F} \frac{\partial}{\partial x^i} \quad \forall i, j, a, b. \tag{2.2}$$

The structure equations of $\overline{M}$ are given by

$$\begin{cases} d\omega^a = -\theta^a_b \wedge \omega^b, \\ \theta^a_b + \theta^b_a = -2\overline{A}_{abc} \theta^c_n, \\ \theta^a_n + \theta^n_a = 0, \ \theta^n_n = 0, \\ d\theta^a_b = -\theta^a_c \wedge \theta^c_b + \frac{1}{2} \overline{R}^a_{bcd} \omega^c \wedge \omega^d + \overline{P}^a_{bcd} \omega^c \wedge \theta^d_n. \end{cases} \tag{2.3}$$

By $\omega^\alpha = 0$ on M and the structure equations of $\overline{M}$, we have

$$\theta_j^\alpha \wedge \omega^j = 0. \tag{2.4}$$

It follows from (2.4) that

$$\theta_j^\alpha = h_{ij}^\alpha \omega^i, \ h_{ij}^\alpha = h_{ji}^\alpha \tag{2.5}$$

We have[4]

THEOREM 2.1 (The Gauss formula). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold and $\overline{\nabla}$ be the Chern connection of $\overline{M}$. If*

$$\begin{aligned}\overline{\nabla} V = \nabla V + B(V, e_i)\omega^i + \sum_i \Big\{ & \overline{A}(V, e_j, B(e_i, e_n)) - \overline{A}(e_j, e_i, B(V, e_n)) \\ & - \overline{A}(V, e_i, B(e_j, e_n)) - \sum_\lambda \overline{A}(e_j, V, e_\lambda)\overline{A}(e_\lambda, e_i, B(e_n, e_n)) \\ & + \sum_\lambda \overline{A}(e_j, e_i, e_\lambda)\overline{A}(e_\lambda, V, B(e_n, e_n)) \\ & + \sum_\lambda \overline{A}(V, e_i, e_\lambda)\overline{A}(e_\lambda, e_j, B(e_n, e_n)) \Big\} \omega^j \otimes e_i,\end{aligned} \tag{2.6}$$

where $V = v^i e_i \in \Gamma(\pi^* TM), B(V, e_i) = \theta_i^\alpha(V) e_\alpha = v^j h_{ij}^\alpha e_\alpha$, then ∇ is the Chern connection of M.

It can be seen from (2.6) that

$$\omega_i^j = \theta_i^j - \Psi_{jik}\omega^k, \tag{2.7}$$

where

$$\begin{aligned}\Psi_{jik} = & h_{jn}^\alpha \overline{A}_{ki\alpha} - h_{kn}^\alpha \overline{A}_{ji\alpha} - h_{in}^\alpha \overline{A}_{kj\alpha} - h_{nn}^\alpha \overline{A}_{iks}\overline{A}_{sj\alpha} \\ & + h_{nn}^\alpha \overline{A}_{ijs}\overline{A}_{sk\alpha} + h_{nn}^\alpha \overline{A}_{jks}\overline{A}_{si\alpha}.\end{aligned} \tag{2.8}$$

From (2.8), we get that

$$\theta_n^j = \omega_n^j - h_{nn}^\alpha \overline{A}_{j\lambda\alpha}\omega^\lambda, \ \theta_j^n = \omega_j^n + h_{nn}^\alpha \overline{A}_{j\lambda\alpha}\omega^\lambda. \tag{2.9}$$

By (2.9) and the almost $\overline{g}$-compatibility , we have

$$\theta_\alpha^j = (-h_{ij}^\alpha - 2h_{ni}^\beta \overline{A}_{j\alpha\beta} + 2h_{nn}^\beta \overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta})\omega^i - 2\overline{A}_{j\alpha\lambda}\omega_n^\lambda. \tag{2.10}$$

In particular,

$$\theta_\alpha^n = -h_{ni}^\alpha \omega^i. \tag{2.11}$$

3. The hypersurface in Minkowski space

Let $(V^{n+1}, \overline{F})$ be a Minkowski space and $x = \overline{x}^a \frac{\partial}{\partial \overline{x}^a}$ be the position vector field of V^{n+1} with respect to the origin. By a direct simple computation, we get

Lemma 3.1. *For any vector field $Z = z^a \frac{\partial}{\partial \overline{x}^a}$ on V^{n+1},*

$$\overline{\nabla}_{Z^H} x = Z, \tag{3.1}$$

where $Z^H = z^a \frac{\delta}{\delta \overline{x}^a} = z^a (\frac{\partial}{\partial \overline{x}^a} - \overline{N}^b_a \frac{\partial}{\partial \overline{Y}^b})$ denotes the horizontal part of Z.

For the Minkowski space $(V^{n+1}, \overline{F})$, we have that the formal Christoffel symbols $\overline{\gamma}^a_{bc}$ of the second kind for $\overline{g}_{ab}$ must vanish and so $\overline{N}^a_{bc} = \overline{\gamma}^a_{bc} \overline{Y}^c - \frac{1}{\overline{F}} \overline{A}^a_{bc} \overline{\gamma}^c_{df} \overline{Y}^d \overline{Y}^f = 0$, then the horizontal part e^H_i of $e_i = u^j_i \frac{\partial}{\partial x^j}$ can be written as

$$e^H_i = u^j_i \frac{\delta}{\delta x^j} = u^j_i (\frac{\partial}{\partial x^j} - \overline{N}^a_j \frac{\partial}{\partial \overline{Y}^a}) = u^j_i \frac{\partial}{\partial x^j} = e_i. \tag{3.2}$$

Definition 3.2. *Let $x : M^n \hookrightarrow (V^{n+1}, \overline{F})$ be a simply connected hypersurface in a Minkowski space. If x_p satisfies $\langle x_p, X \rangle_{\overline{g}} = 0, \forall X \in T_pM$ at $p \in M$, we call that x_p is normal to $T_p(M)$.*

Now we can prove the following

Main theorem. *Let $x : M^n \hookrightarrow (V^{n+1}, \overline{F})$ be a simply connected hypersurface in a Minkowski space $(V^{n+1}, \overline{F})$. If $x(p)$ is normal to $T_p(M) (\forall p \in M)$, then M with the induced metric is isometric to the standard Euclidean sphere.*

Proof. That the x_p is normal to $T_p(M), \forall p \in M$ means $x = f e_{n+1}, f \in C^\infty(\overline{M})$. We have that the vertical covariant differentials $f_;$ of f with respect to the Chern connection satisfies $f_; = 0$ and

$$e^H_i(f^2) = e^H_i \langle x, x \rangle = 2\langle \overline{\nabla}_{e^H_i} x, x \rangle + 2\overline{A}(x, x, \overline{\nabla}_{e^H_i} e_n) = 0. \tag{3.3}$$

Then f =constant. Let $f = c$, i.e., $x = c e_{n+1}$. We have

$$de_{n+1} = \frac{1}{c} dx = \frac{1}{c} \omega^i e_i. \tag{3.4}$$

On the other hand, it is easy to see that

$$de_{n+1} = \theta^i_{n+1} e_i + \theta^{n+1}_{n+1} e_{n+1}. \tag{3.5}$$

By (3.4) and (3.5), we get

$$\theta^i_{n+1} = \frac{1}{c} \omega^i, \ \theta^{n+1}_{n+1} = 0. \tag{3.6}$$

It can be seen from (2.10) that

$$\theta^i_{n+1} = (-h_{ij} - 2h_{nj} \overline{A}_{in+1n+1} + 2h_{nn} \overline{A}_{i\lambda n+1} \overline{A}_{j\lambda n+1}) \omega^j - 2\overline{A}_{i\lambda n+1} \omega^\lambda_n. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\left\{\begin{array}{rl} \overline{A}_{ijn+1} = & 0, \\ -h_{ij} - 2h_{nj}\overline{A}_{in+1n+1} = & \frac{1}{c}\delta_{ij}. \end{array}\right. \tag{3.8}$$

Substituting (3.8) into (2.8), we get

$$\Psi_{ijk} = 0. \tag{3.9}$$

It is easy to see from (2.7) and (3.9) that

$$\theta_i^j = \omega_i^j. \tag{3.10}$$

By (2.9), (3.6), (3.10) and $\theta_{n+1}^{n+1} = -\overline{A}_{n+1n+1\lambda}\theta_n^\lambda - \overline{A}_{n+1n+1n+1}\theta_n^{n+1}$, we have

$$\overline{A}_{n+1n+1i}\omega_n^i + h_{ni}\overline{A}_{n+1n+1n+1}\omega^i = 0. \tag{3.11}$$

Then we obtain immediately

$$\overline{A}_{n+1n+1i} = 0,\ h_{ni}\overline{A}_{n+1n+1n+1} = 0. \tag{3.12}$$

It follows from (3.8) and (3.12) that

$$\left\{\begin{array}{l} \overline{A}_{n+1ab} = 0, \\ -h_{ij} = \frac{1}{c}\delta_{ij}. \end{array}\right. \tag{3.13}$$

and hence we have $\theta_i^{n+1} = -\theta_{n+1}^i = -\frac{1}{c}\omega^i$. Exterior differentiate (3.6), by using (3.2) and (3.10), we obtain

$$d\omega^i = -\omega_j^i \wedge \omega_j. \tag{3.14}$$

Exterior differentiate (3.10), we get

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \frac{1}{c^2}\omega^i \wedge \omega^j. \tag{3.15}$$

(3.14) and (3.15) are the structure equations of M^n.
(3.15) is equivalent to

$$\frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l + P^i_{jk\lambda}\omega^k \wedge \omega_n^\lambda = \frac{1}{c^2}\omega^i \wedge \omega^j. \tag{3.16}$$

It can be seen from (3.16) that

$$\left\{\begin{array}{rl} P^i_{jkl} = & 0, \\ R^i_{jkl} = & \frac{1}{c^2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{array}\right. \tag{3.17}$$

From (3.17), we obtain immediately that M is a Berwald space with constant flag curvature $K = \frac{1}{c^2}$, then M is a Riemannian with constant curvature $K = \frac{1}{c^2}$, so we obtain the main theorem immediately. □

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