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CONCERNING n-MUTUAL APOSYNDESIS IN HYPERSPACES

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Abstract. Given a metric continuum X, let 2^X denote the hyperspace of nonempty closed subsets of X. We prove that 2^X is n-mutually aposyndetic for each $n \geq 1$. That is, given n distinct elements of 2^X, there are n disjoint subcontinua of 2^X, each containing one of the elements in its interior.

1. Introduction

A *continuum* is a compact connected metric space with more than one point. A continuum X is said to be *mutually aposyndetic* provided that for every $p, q \in X$, with $p \neq q$, there exist disjoint subcontinua M and N of X such that $p \in \text{int}_X(M)$ and $q \in \text{int}_X(N)$; and X is said to be *n-mutually aposyndetic* provided that given n distinct points, there are n disjoint subcontinua of X, each containing one of the points in its interior. So, mutual aposyndesis means 2-mutual aposyndesis.

For the continuum X, we consider its hyperspaces:

$2^X = \{A \subset X : A \text{ is closed and nonemtpy}\}$,
$C_m(X) = \{A \in 2^X : A \text{ has at most } m \text{ components}\}$,
$C(X) = C_1(X)$, and
$F_m(X) = \{A \in 2^X : A \text{ has at most } m \text{ points}\}$.

The hyperspace 2^X is endowed with the Hausdorff metric H.

The concept of n-mutual aposyndesis was introduced by L. E. Rogers in [9] where he proved that the topological product of three continua is always n-mutually aposyndetic, for each n. Aposyndesis in hyperspaces has been studied by several authors; for recent results see [1–3, 7, 8]. Answering a

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question by H. Hosokawa ([2, p. 136]), J. M. Martínez-Montejano ([8, Theorem 3.1]) has recently proved that the hyperspace 2^X is mutally aposyndetic. In this paper we improve this result by proving the following theorem.

Theorem. For each continuum X and each positive integer n, the hyperspace 2^X is n-mutually aposyndetic.

Generalizing the main result of [6], recently, A. Illanes and J. M. Martínez-Montejano ([4]) have shown that for each $n \geq 2$ and $m \geq 3$, $F_m(X)$ is n-mutually aposyndetic. H. Hosokawa ([2, Theorem 2.4]) has proved that, for each $m \geq 2$, the hyperspace $C_m(X)$ is mutually aposyndetic, the following question arises naturally:

Question 1.1. *Let X be a continuum $n \geq 3$ and $m \geq 2$, is the hyperspace $C_m(X)$ n-mutually aposyndetic?*

2. Main result

Given a continuum X, with metric d, $\varepsilon > 0$, $p \in X$ and $A \subset X$, let $B(\varepsilon, p)$ be the ε-ball around p in X, $N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) : a \in A\}$ and $V(\varepsilon, A) = \{x \in X : \text{there exists } a \in A \text{ such that } d(x, a) \leq \varepsilon\}$. Given $A, B \in 2^X$ such that $A \subsetneq B$, an *order arc from A to B* is a continuous function $\alpha : [0,1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$ and, if $0 \leq s < t \leq 1$, then $\alpha(s) \subsetneq \alpha(t)$. It is well known (see [5, Theorem 15.3]) that, if $A, B \in 2^X$ and $A \subsetneq B$, then there exists an order arc from A to B if and only if each component of B intersects A.

Theorem 2.1. *For each continuum X and each positive integer n, the hyperspace 2^X is n-mutually aposyndetic.*

Proof. Let d be the metric for X. Let $A_1, \ldots, A_n$ be n different elements of 2^X, where $n > 1$. Let $\delta : 2^X \to [0, \infty)$ be the diameter map. We suppose that $\delta(X) = 1$. For each $t \in [0,1]$, let $\mathcal{N}(t) = \delta^{-1}(t) \cap C(X)$. By [5, Exercise 19.18], $\mathcal{N}(t)$ is a subcontinuum of 2^X. Let $\varepsilon > 0$ be such that $H(A_i, A_j) \geq 4\varepsilon$ for all $i \neq j$.

For each $A \in 2^X$ and $t \in [0,1]$, let $D(A,t) = \bigcup\{B \in \mathcal{N}(t) : A \cap B \neq \emptyset\}$. Clearly, $D(A,t) \in 2^X$, $A \subset D(A,t)$ and each component of $D(A,t)$ intersects A. Thus, in the case that $A \neq D(A,t)$, there exist order arcs from A to $D(A,t)$. Let $\mathcal{E}(A,t) = \bigcup\{\operatorname{Im} \alpha \subset 2^X : \alpha \text{ is an order arc in } 2^X \text{ from } A \text{ to } D(A,t)\}$, if $A \neq D(A,t)$, and $\mathcal{E}(A,t) = \{A\}$, if $A = D(A,t)$. By [4, 11.5 p. 91], $\mathcal{E}(A,t)$ is a subcontinuum of 2^X containing A and $D(A,t)$. Since $D(A,t) \subset N(t + \eta, A)$ for each $\eta > 0$, $H(A,B) \leq t$ for each element $B \in \mathcal{E}(A,t)$.

Given $i, j \in \{1, \ldots, n\}$, with $i \neq j$, we define a positive number $s(i,j)$ in the following way:

(a) if $A_j - A_i$ has at least two points, let $s(i,j) > 0$ be such that $A_j - N(4s(i,j), A_i)$ has at least two points and $4s(i,j) < \delta(A_j - N(4s(i,j), A_i))$.

(b) if $A_j - A_i$ is a one-point set, let $s(i,j) > 0$ be such that $N(2s(i,j), A_j - A_i) \cap N(2s(i,j), A_i) = \emptyset$.

(c) if $A_j \subset A_i$, by (a) and (b) the number $s(j,i)$ has been defined. Thus, we can define $s(i,j) = s(j,i)$.

Fix $r > 0$ such that $r < \min(\{s(i,j) : i,j \in \{1,\ldots,n\}$ and $i \neq j\} \cup \{\varepsilon\})$. For each $i \in \{1,\ldots,n\}$, let $m_i = \max\{k \in \{1,\ldots,n\}$: there exist elements $i_1,\ldots,i_k$ in $\{1,\ldots,n\}$ such that $A_i \subset A_{i_1} \subsetneq A_{i_2} \subsetneq \ldots \subsetneq A_{i_k}\}$. Note that m_i is well defined. Note also that if $A_i \subsetneq A_j$, then $m_i > m_j$. For each positive integer k, let $M_k = \{i \in \{1,\ldots,n\} : m_i = k\}$. Then $i \in M_1$ if and only if A_i is a maximal element of $\{A_1,\ldots,A_n\}$, with respect to the inclusion; $i \in M_2$ if and only if A_i is a maximal element of the set $\{A_j : j \in \{1,\ldots,n\} - M_1\}$; $i \in M_3$ if and only if A_i is a maximal element of the set $\{A_j : j \in \{1,\ldots,n\} - (M_1 \cup M_2)\}$; and so on. Notice also that $\emptyset = M_{n+1} = M_{n+2} = \ldots$.

For each $i \in \{1,\ldots,n\}$, we choose a positive number t_i as follows. For each $i \in M_1$, choose a number $v_i \in (0,1)$ in such a way that all the elements of $\{v_i : i \in M_1\}$ are different. If $M_2 \neq \emptyset$, for each $i \in M_2$, choose a number $v_i \in (5,6)$ in such a way that all the elements of $\{v_i : i \in M_2\}$ are different. In general, if $k \geq 1$ and $M_{k+1} \neq \emptyset$, for each $i \in M_{k+1}$ choose a number $v_i \in (5^k + \ldots + 5, 5^k + \ldots + 5 + 1)$ in such a way that all the elements of $\{v_i : i \in M_{k+1}\}$ are different. Notice that, if $A_i \subsetneq A_j$, since $m_j < m_i$, $5v_j < v_i$. Notice also that $v_i < 5^n$ for each $i \in \{1,\ldots,n\}$, and all the numbers $\{v_1,\ldots,v_n\}$ are different. For each $i \in \{1,\ldots,n\}$, define $t_i = r\frac{v_i}{5^n} < r$.

For each $i \in \{1,\ldots,n\}$, let $\mathcal{B}_i = \{A \in 2^X : H(A, A_i) \leq t_i\}$. For each $A \in \mathcal{B}_i$, let $\mathcal{F}(A) = \{D(A,t_i) \cup B : B \in \mathcal{N}(t_i)\}$. Note that $\mathcal{F}(A)$ is the image of $\mathcal{N}(t_i)$ under the map $B \to D(A,t_i) \cup B$, so, $\mathcal{F}(A)$ is a subcontinuum of 2^X. Fix a point $p \in A$, let $B \in \mathcal{N}(t_i)$ be such that $p \in B$. Then $B \subset D(A,t_i)$. Thus, $D(A,t_i) = D(A,t_i) \cup B \in \mathcal{F}(A)$. Define also $\mathcal{R}_i(t_i) = \bigcup\{\mathcal{E}(A,t_i) : A \in \mathcal{B}_i\}$. Then $\mathcal{R}_i(t_i)$ is a closed subset of 2^X such that $A_i \in \operatorname{int}_{2^X}(\mathcal{B}_i) \subset \operatorname{int}_{2^X}(\mathcal{R}_i(t_i)) \subset \mathcal{R}_i(t_i)$.

Define
$$\mathcal{M}_i = \mathcal{R}_i(t_i) \cup (\bigcup\{\mathcal{F}(A) : A \in \mathcal{B}_i\}).$$
Notice that $A_i \in \operatorname{int}_{2^X}(\mathcal{M}_i)$

CLAIM 1. *$\mathcal{M}_i$ is a subcontinuum of 2^X.*

In order to prove Claim 1, we check that $\mathcal{M}_i$ is connected. Let $A \in \mathcal{M}_i$ and let $\mathcal{C}$ be the component of $\mathcal{M}_i$ containing A. Let $E_0 = \bigcup\{E : E \in \mathcal{B}_i\}$. It is easy to see that $E_0 \in \mathcal{B}_i$. Let $D_0 = D(E_0, t_i)$. We show that $D_0 \in \operatorname{cl}_{2^X}(\mathcal{C}) = \mathcal{C}$. Take $\eta > 0$. Fix points $x_1,\ldots,x_k \in D_0$ such that $H(\{x_1,\ldots,x_k\}, D_0) < \eta$.

Then there exist points $e_1, \dots, e_k \in E_0$ and elements $B_1, \dots, B_k \in \mathcal{N}(t_i)$ such that $x_1, e_1 \in B_1, \dots, x_k, e_k \in B_k$.

In the case that $A \in \mathcal{R}_i(t_i)$, there exists $E \in \mathcal{B}_i$ such that $A \in \mathcal{E}(E, t_i)$. Since $\mathcal{E}(E, t_i)$ is connected and $D(E, t_i) \in \mathcal{E}(E, t_i)$, we may assume that $A = D(E, t_i)$. In the case that $A \in \mathcal{F}(E)$ for some $E \in \mathcal{B}_i$, since $\mathcal{F}(E)$ is connected and $D(E, t_i) \in \mathcal{F}(E)$, we may assume that $A = D(E, t_i)$. Therefore, in any case, we may assume that $A = D(E, t_i)$ for some $E \in \mathcal{B}_i$.

Note that the sets $E \cup \{e_1\}, E \cup \{e_1, e_2\}, \dots, E \cup \{e_1, \dots, e_k\}$ belong to $\mathcal{B}_i$. Since $A \cup B_1 = D(E, t_i) \cup B_1 \in \mathcal{F}(E) \subset \mathcal{M}_i$ and $\mathcal{F}(E)$ is connected, we have that $A \cup B_1 \in \mathcal{C}$. Since $E \cup \{e_1\} \subset A \cup B_1 \subset D(E \cup \{e_1\}, t_i)$ and each component of $A \cup B_1$ intersects $E \cup \{e_1\}$ and each component of $D(E \cup \{e_1\}, t_i)$ intersects $A \cup B_1$, in the case that $E \cup \{e_1\} \neq D(E \cup \{e_1\}, t_i)$, we have that there exists an order arc α from $E \cup \{e_1\}$ to $D(E \cup \{e_1\}, t_i)$ such that $A \cup B_1 \in \operatorname{Im} \alpha \subset \mathcal{E}(E \cup \{e_1\}, t_i) \subset \mathcal{R}_i(t_i) \subset \mathcal{M}_i$. Thus, $\operatorname{Im} \alpha \subset \mathcal{C}$. Hence, $D(E \cup \{e_1\}, t_i) \in \mathcal{C}$. In the case that $E \cup \{e_1\} = D(E \cup \{e_1\}, t_i)$, $D(E \cup \{e_1\}, t_i) = A \cup B_1$, thus, $D(E \cup \{e_1\}, t_i) \in \mathcal{C}$. In both cases, $D(E \cup \{e_1\}, t_i) \in \mathcal{C}$. Since $D(E \cup \{e_1\}, t_i) \cup B_2 \in \mathcal{F}(E \cup \{e_1\}) \subset \mathcal{M}_i$, we have $D(E \cup \{e_1\}, t_i) \cup B_2 \in \mathcal{C}$. Proceeding as before, we can conclude that $D(E \cup \{e_1, e_2\}, t_i) \in \mathcal{C}$. Repeating this procedure, we obtain that $D(E \cup \{e_1, \dots, e_k\}, t_i) \in \mathcal{C}$.

Notice that $\{e_1, \dots, e_k\} \subset D(E \cup \{e_1, \dots, e_k\}, t_i) \subset D(E_0, t_i) = D_0$. This implies that $H(D(E \cup \{e_1, \dots, e_k\}, t_i), D_0) < \eta$. This completes the proof that $D_0 \in \operatorname{cl}_{2^X}(\mathcal{C}) = \mathcal{C}$. Therefore, $\mathcal{M}_i$ is connected and Claim 1 is proved.

Claim 2. *The sets $\mathcal{M}_1, \dots, \mathcal{M}_n$ are pairwise disjoint.*

We prove Claim 2. Let $i, j \in \{1, \dots, n\}$ be such that $i \neq j$. Suppose that there exists an element $A \in \mathcal{M}_i \cap \mathcal{M}_j$. We consider four cases.

Case 1. $A \in \mathcal{R}_i(t_i) \cap \mathcal{R}_j(t_j)$.

In this case, there exist $E_i \in \mathcal{B}_i$ and $E_j \in \mathcal{B}_j$ such that $A \in \mathcal{E}(E_i, t_i) \cap \mathcal{E}(E_j, t_j)$. Thus, $H(A, E_i) \leq t_i < r < \varepsilon$ and $H(A, E_j) < \varepsilon$. Since $H(E_i, A_i) \leq \varepsilon$ and $H(E_j, A_j) \leq \varepsilon$, we have $H(A_i, A_j) < 4\varepsilon$. Contrary to the choice of ε. Hence, Case 1 is impossible.

Case 2. $A_j - A_i$ has at least two points.

Notice that $A \in \mathcal{M}_j$ implies that there exists an element $E_j \in \mathcal{B}_j$ such that $E_j \subset A$. By definition $s(i,j) > 0$ is such that $A_j - N(4s(i,j), A_i)$ is nondegenerate and $4s(i,j) < \delta(A_j - N(4s(i,j), A_i))$. Fix points $p, q \in A_j - N(4s(i,j), A_i)$ such that $d(p,q) > 4s(i,j) > \max\{4t_i, 4t_j\}$. Since $H(E_j, A_j) \leq t_j$, there exist points $u, v \in E_j$ such that $d(u,p) \leq t_j < r < s(i,j)$ and $d(v,q) \leq t_j < s(i,j)$. Thus, $u, v \notin N(3s(i,j), A_i)$. Since $A \in \mathcal{M}_i$, there exists $E_i \in \mathcal{B}_i$ such that either $E_i \subset A \subset D(E_i, t_i)$ or $A = D(E_i, t_i) \cup B$ for some $B \in \mathcal{N}(t_i)$. Since $t_i < r < s(i,j)$ and $H(D(E_i, t_i), A_i) \leq H(D(E_i, t_i), E_i) + H(E_i, A_i) \leq 2t_i$, we have $D(E_i, t_i) \subset N(2s(i,j), A_i)$ and $u, v \notin D(E_i, t_i)$. Since $u, v \in A$, we conclude that $A \not\subset D(E_i, t_i)$, so $A = D(E_i, t_i) \cup B$ for some

$B \in \mathcal{N}(t_i)$. Thus, $u, v \in B$. Hence, $d(u,v) \leq \delta(B) = t_i$. This implies that $d(p,q) \leq d(p,u) + d(u,v) + d(v,q) < 3s(i,j)$, which is a contradiction with the choice of p and q. We have proved that this case is impossible.

Similarly, it can be proven that the following case is also impossible.

CASE 3. $A_i - A_j$ has at least two points.

CASE 4. $A_j - A_i$ is a one-point set.

Let $p \in A_j - Ai$, by definition $s(i,j) > 0$ is such that $N(2s(i,j), \{p\}) \cap N(2s(i,j), A_i) = \emptyset$. Since $A \in \mathcal{M}_i$, there exists $E_i \in \mathcal{B}_i$ such that either $E_i \subset A \subset D(E_i, t_i)$ or $A = D(E_i, t_i) \cup B_i$ for some $B_i \in \mathcal{N}(t_i)$. Since $t_i < r < s(i,j)$ and $H(D(E_i, t_i), A_i) \leq 2t_i$, we have $D(E_i, t_i) \subset N(2s(i,j), A_i)$ and $N(2s(i,j), \{p\}) \cap D(E_i, t_i) = \emptyset$. Since $A \in \mathcal{M}_j$, there exists an element $E_j \in \mathcal{B}_j$ such that $E_j \subset A \subset D(E_j, t_j) \cup C_j$, for some $C_j \in \mathcal{N}(t_j)$ (even in the case that $A \in \mathcal{R}_j(t_j)$). Let $u \in E_j$ be such that $d(p,u) \leq t_j < s(i,j)$. Then $N(s(i,j), \{u\}) \subset N(2s(i,j), \{p\})$ and $N(s(i,j), \{u\}) \cap D(E_i, t_i) = \emptyset$. In particular, $u \in A - D(E_i, t_i)$. This implies that $A = D(E_i, t_i) \cup B_i$ for some $B_i \in \mathcal{N}(t_i)$, $u \in B_i$ and $D(E_i, t_i) \cap B_i = \emptyset$. We consider two subcases.

CASE 4.1. $A_i - A_j$ is a one-point set.

Proceeding as above, we have that A is of the form $A = D(E_j, t_j) \cup B_j$ for some $E_j \in \mathcal{B}_j$, $B_j \in \mathcal{N}(t_j)$. Notice that u was choosen in E_j. Let $R \in \mathcal{N}(t_j)$ be such that $u \in R$. Then $R \subset D(E_j, t_j) \subset A = D(E_i, t_i) \cup B_i$ and $R \subset N(s(i,j), \{u\})$. This implies that $R \subset B_i$. Hence, $t_j = \delta(R) \leq \delta(B_i) = t_i$. Since $t_j \neq t_i$, we conclude that $t_j < t_i$. Similarly, $t_i < t_j$. This contradiction shows that this subcase is impossible.

CASE 4.2. $A_i \subset A_j$.

In this case, $A_i \subsetneq A_j$, so $m_j < m_i$ and $5t_j < t_i$. Since $E_j \in \mathcal{B}_j$, and $H(E_j, D(E_j, t_j)) \leq t_j$, we have $D(E_j, t_j) \subset N(2t_j, A_j) = N(2t_j, A_i) \cup N(2t_j, \{p\})$. We also have $D(E_i, t_i) \subset N(2t_i, A_i)$ and $B_i \subset N(t_i, \{u\}) \subset N(t_i + t_j, \{p\})$. Given a point $x \in B_i \subset A \subset D(E_j, t_j) \cup C_j$ such that $x \notin C_j$, we have that $x \in D(E_j, t_j) \subset N(2t_j, A_i) \cup N(2t_j, \{p\})$. Since $N(2t_j, A_i) \cap N(t_i + t_j, \{p\}) \subset N(2s(i,j), A_i) \cap N(2s(i,j), \{p\}) = \emptyset$ and $B_i \subset N(t_i + t_j, \{p\})$, we obtain that $x \in N(2t_j, \{p\})$. We have shown that $B_i \subset V(2t_j, \{p\}) \cup C_j$. Since B_i is connected and the sets $V(2t_j, \{p\})$ and C_j are closed, we obtain that $t_i = \delta(B_i) \leq 5t_j$, a contradiction. This completes the proof that this subcase is also impossible.

Since we have considered all the possible cases, the proof of Claim 2 is complete. □

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