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ORBIT PROJECTIONS AND G-ANR-RESOLUTIONS

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Abstract. We consider the orbit projection $p_E : E \to E/G$ of a G-space E with only one orbit type. We show that p_E admits a G-ANR-resolution consisting of G-fibrations if G is a compact metrizable group.

1. Introduction

The following fact is well known (see, for instance, [14, p. 54]): if G is a *compact Lie* group and E is a paracompact G-space with orbits only of one type, then the orbit projection $p_E : E \to E/G$ is a G-fibration. But what can one say about the orbit projection p_E when G is not necessarily a Lie group?

In this paper we try to answer this question for the case of any compact metrizable group G. Generally speaking, we show that, in this case, p_E can be approximated, in a good enough way, by G-fibrations of G-ANR's. More precisely, our main result (Theorem 4.11) states that the orbit projection p_E admits a G-ANR-resolution consisting of G-fibrations p_i. Moreover, this resolution can be chosen so that each p_i is an orbit projection $E_i \to E_i/G$, where E_i is a G-ANR.

The G-map p_E could be said to be a "shape G-fibration" because, in the non-equivariant case, continuous maps, which admit ANR-resolutions consisting of fibrations, are called shape fibrations. In fact, we use some notions of the shape theory; in particular, dealing with a given G-space E we consider ANR-resolutions of both G and E. We hope that the results of this paper will have applications in the equivariant shape theory whose foundations are given in [6] and [8].

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2. Preliminaries

The letter G will always denote a *compact Hausdorff* group; the unit element of G is denoted by e. In the most part of the paper we work in the category $\mathcal{M}_G$ of *metrizable* G-spaces and G-maps.

The basic definitions and facts of the theory of G-spaces (the theory of transformation groups) can be found in [9],[12] and [14]. We recall some of them below for the convenience of the reader.

By a *G-space* we mean a topological space X together with a fixed continuous action $\cdot : G \times X \to X$, $(g, x) \mapsto g \cdot x$, of G on X. It is used to write simply gx instead of $g \cdot x$.

Let X be a G-space. For $x \in X$, the subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *isotropy group* at x. For a subgroup H of G, the symbol X^H denotes the *H-fixed point set*; it can be described as follows: $X^H = \{x \in X \mid H \subseteq G_x\}$.

Let X and Y be G-spaces. A continuous map $f : X \to Y$ is said to be a *G-map* or an *equivariant map* if $f(gx) = gf(x)$ for every $(g, x) \in G \times X$. If a G-map f is a homeomorphism, it is called a *G-equivalence*. We write $X \approx Y$ and say that X and Y are *G-equivalent* if there exists some G-equivalence $X \to Y$. A homotopy $F : X \times I \to Y$, where $I = [0, 1]$, is called a *G-homotopy* if it is a G-map considering $X \times I$ with the action $g(x, t) = (gx, t)$. Therefore F satisfies $F(gx, t) = gF(x, t)$ for every $(x, t) \in X \times I$ and every $g \in G$.

The homogeneous space $G/H = \{gH \mid g \in G\}$, for a given closed subgroup H of G, is a G-space with the action $g \cdot g'H = gg'H$.

The conjugacy class (H) of any closed subgroup H of G will be called a *G-orbit type* (in [9] it is called G-isotropy type). The reason for this terminology is the following: if X is a G-space, $G(x)$ is the orbit of $x \in X$ and $H = G_x$ is the isotropy group at x, then

$$(H) = \{gHg^{-1} \mid g \in G\} = \{G_y \mid y \in G(x)\}$$

and the orbit $G(x)$ is G-equivalent to G/G_y for each $y \in G(x)$; in particular, $G(x) \approx G/H$. A G-space X has *only one orbit type* (H) if $(G_x) = (H)$ for each point $x \in X$; in this case every orbit is G-equivalent to G/H. A G-space (and the corresponding action of G) is called *free* if it has only one orbit type $(\{e\})$, so that every orbit is G-equivalent to G.

If H is a subgroup of G, then every G-space and every G-map can be regarded as an H-space and an H-map, respectively, by restricting the group action from G to H. Thus we get the *restriction* functor $\mathcal{M}_G \to \mathcal{M}_H$.

In general, every continuous group homomorphism $f : G' \to G$ induces the functor $\mathcal{F}(f) : \mathcal{M}_G \to \mathcal{M}_{G'}$: if X is a G-space, then the G'-action $\cdot$ on X is defined by $g \cdot x = f(g)x$. In particular, the restriction functor is induced by the natural inclusion $H \hookrightarrow G$. If N is a closed normal subgroup of a group G, then every G/N-space and every G/N-map can be considered as a G-space and a G-map respectively due to the functor $\mathcal{F}(q_N) : \mathcal{M}_{G/N} \to \mathcal{M}_G$ induced by the natural projection $q_N : G \to G/N$.

Given a G-space X, the set of its G-orbits, endowed with the quotient topology, is called the *G-orbit space of* X and is denoted by X/G. The natural projection $p_X : X \to X/G$ (defined by $p_X(x) = G(x)$) is called the *G-orbit projection* or, simply, *orbit projection* of X.

If $f : X \to Y$ is a G-map, then there exists a unique continuous map $f/G : X/G \to Y/G$, called the map *induced by* f, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X/G & \xrightarrow{f/G} & Y/G \end{array}$$

commutes. Clearly, f/G is defined by $(f/G)(G(x)) = G(f(x))$.

If N is a closed normal subgroup of G and X is a G-space, the N-orbit space X/N is a G/N-space with the action $(gN) \cdot (N(x)) = N(gx)$. Every G-map $f : X \to Y$ induces a G/N-map $f/N : X/N \to Y/N$ given by $(f/N)(N(x)) = N(f(x))$. The correspondence $f \mapsto f/N$ defines the N-orbit functor $-/N : \mathcal{M}_G \to \mathcal{M}_{G/N}$.

Note that in the above situation X/N, being a G/N-space, can be also regarded as a G-space (with the action $g \cdot N(x) = N(gx)$), and the N-orbit projection $X \to X/N$ can be considered as a G-map. In particular, $p_X : X \to X/G$ is a G-map if X/G is taken with the trivial action of G.

A G-map $p : E \to B$ is called a *G-fibration* if it has the G-equivariant covering homotopy property (ECHP), that is, if for every commutative diagram of G-maps

$$\begin{array}{ccccc} x & \quad & X & \xrightarrow{f} & E \\ \downarrow & & {\scriptstyle \partial_0}\downarrow & & \downarrow{\scriptstyle p} \\ (x,0) & & X \times I & \xrightarrow[F]{} & B \end{array}$$

there exists a G-homotopy $\widetilde{F} : X \times I \to E$ as a filler.

The special case of a G-fibration of main interest to us is given by the following statement (see [14, p. 54]) already mentioned in the Introduction.

PROPOSITION 2.1. *Let G be a compact Lie group. If a G-space E has only one orbit type, then the orbit projection $p_E : E \to E/G$ is a G-fibration.*

Proposition 2.1 can be easily obtained either from the Covering Homotopy Theorem of Palais ([9, II.7.3]) or from the fact that p_E is a G-bundle (in the sense of [10]) according to [9, II.5.8] and therefore has ECHP by [10, Corollary 2.12].

By G-ANR-space (resp. by G-AR-space) we mean, of course, a *G-equivariant absolute neighborhood retract* (resp. *G-equivariant absolute retract*) for the class of all metrizable G-spaces (see, for instance, [1]-[5] and [12] for the equivariant theory of retracts).

It is known ([1]) that a metrizable G-space Y is a G-ANR if and only if it is a G-ANE. In other words, it has the following extension property: for any G-map $f : A \to Y$, where A is a closed invariant subset of a metrizable G-space X, there exists a G-map $\overline{f} : U \to Y$ such that $\overline{f}|_A = f$, where U is some invariant neighborhood of A in X. In this paper we shall use also the next two propositions.

Proposition 2.2 ([1],[6]). *Let G be a compact Hausdorff group. For any metrizable G-space X there exists a G-equivariant closed embedding $X \hookrightarrow M$ into some normed linear G-AR-space M.*

Proposition 2.3 ([2],[5]). *Let G be a compact Hausdorff group and let N be a closed normal subgroup of G. If X is a G-$A(N)R$-space, then X/N is a G/N-$A(N)R$-space. In particular, X/G is an $A(N)R$-space.*

Some more facts concerning G-fibrations and G-ANRs are given in the next section.

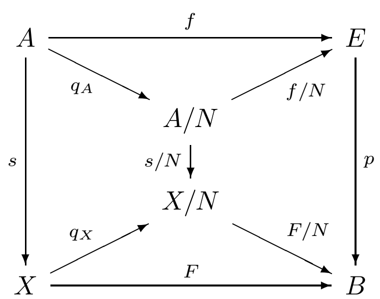
3. G-ANR-spaces and G-fibrations

We have already noticed in the previous section that, for a given closed normal subgroup N of a group G, every G/N-space X can be considered in a natural way as a G-space (if $*$ is the action of G/N on X, then the action $\cdot$ of G on X is defined by $g \cdot x = gN * x$) and hence every G/N-map of G/N-spaces can be regarded as a G-map.

Proposition 3.1. *Let N be a closed normal subgroup of a compact group G.*

(a) *If $p : E \to B$ is a G/N-fibration, then p is a G-fibration.*
(b) *If E is a G/N-ANR, then E is a G-ANR.*

Proof. Let $p : E \to B$ be a G/N-map and let $s : A \hookrightarrow X$ be a closed G-embedding. Consider the following commutative diagram of G-maps



and observe that the existence of a filler $\overline{F} : X/N \to E$ is equivalent to the existence of a filler $\widetilde{F} : X \to E$.

(a) Let $X = A \times I$ and $s(a) = (a, 0)$. Obviously, $(A \times I)/N$ can be identified with $(A/N) \times I$. If p is a G/N-fibration, then there is a filler $\overline{F} : (A/N) \times I \to E$. Consequently, the G-map $\widetilde{F} : A \times I \to E$, given by $\widetilde{F} = \overline{F} \circ q_X$, is also a filler of the diagram. This proves that p is a G-fibration.

(b) In order to prove that E is a G-ANR we need only the upper part of the diagram, so we may assume that B is a one-point set with the trivial action of G. Since E is a G/N-ANR-space, for some G/N-invariant neighborhood U of A/N in X/N there exists a G/N-map $\overline{f} : U \to E$ such that $\overline{f}|_{A/N} = f/N$. Then $V = q_X^{-1}(U)$ is a G-invariant neighborhood of A in X and $(\overline{f} \circ q_X)|_V : V \to E$ is a G-extension of f. It means that E is a G-ANE and hence it is a G-ANR. □

In the proof of Proposition 3.1, in fact, we have used that the functor $\mathcal{F}(q_N)$, induced by the projection $q_N : G \to G/N$, is a right adjoint to the N-orbit functor $-/N : \mathcal{M}_G \to \mathcal{M}_{G/N}$. The proof of the next statement is analogous and can be found, for instance, in [7]. It is based on the fact that the restriction functor, induced by the inclusion $i_H : H \hookrightarrow G$, is a right adjoint to the functor of the twisted product $G \times_H -$ (see [14, Ch. I, Proposition 4.3]).

Proposition 3.2. *Let H be a closed subgroup of a compact group G.*

(a) *If $p : E \to B$ is a G-fibration, then p is an H-fibration.*

(b) *If E is a G-ANR and G/H is metrizable, then E is an H-ANR.*

Remark 3.3. The condition "G/H is metrizable" is added in the assertion (b) in order to get a *metrizable* G-space $G \times_H X$ for any metrizable H-space X (see [4, Proposition 3]) and, therefore, to have a well defined functor $G \times_H - : \mathcal{M}_H \to \mathcal{M}_G$. Note that this restriction is unnecessary for (a).

Proposition 3.4 ([12, Corollary 1.6.7]). *Let H be a closed subgroup of a compact Lie group G. Then G/H is a G-ANR.*

Corollary 3.5. *Let X be a G-space such that G is a compact Lie group. Then for every $x \in X$, the orbit $G(x)$ is a G-retract of some open invariant neighborhood of $G(x)$ in X.*

Indeed, it is sufficient to notice that $G(x)$ is G-equivalent to the G-space G/G_x and therefore $G(x)$ is a G-ANR.

The property of the homogeneous space G/H to be a G-ANR deserves special attention; it is characterized in [3] as follows:

Proposition 3.6. *Let H be a closed subgroup of a compact group G. Then the following conditions are equivalent:*

(1) *G/H is a G-ANR-space,*

(2) *G/H is locally contractible,*
(3) *G/H is a smooth manifold,*
(4) *there exists a closed normal subgroup N of G such that $N \subseteq H$ and G/N is a Lie group.*

DEFINITION 3.7 ([4]). *A closed subgroup H of a compact group G is called large if it satisfies the equivalent conditions (1)-(4) of Proposition 3.6.*

REMARK 3.8. Though the conditions of Proposition 3.6 are equivalent, in the original definition of a large subgroup, given in [3], namely the condition (4) was used; in this form the definition can be easily generalized for the case of non-compact groups as it was done in [4].

We shall need the following simple proposition.

PROPOSITION 3.9. *Let G be a compact group.*

(a) *If H is a large subgroup of G, then the quotient group $N(H)/H$ is a compact Lie group, where $N(H)$ is the normalizer of H in G. In particular, a closed normal subgroup N is large if and only if G/N is a Lie group.*
(b) *If N is a large normal subgroup of G, then for each closed subgroup H of G, the subgroup NH of G is also large.*

PROOF. (a) If N is a closed normal subgroup of G such that $N \subseteq H$ and G/N is a Lie group, then $N(H)/N$ is a Lie group as a closed subgroup of G/N. The inclusion $N \hookrightarrow H$ induces a continuous epimorphism $N(H)/N \to N(H)/H$. Thus, $N(H)/H$ is isomorphic to a quotient group of the Lie group $N(H)/N$, and therefore it is a Lie group too.

(b) It follows immediately from Proposition 3.6(4), because $N \subseteq NH$. □

The notion of a large subgroup is used in the next statement which is a slight generalization of Proposition 2.1.

PROPOSITION 3.10. *Let H be a large subgroup of a compact group G. If a G-space E has only one orbit type (H), then $p_E : E \to E/G$ is a G-fibration.*

It is not hard to prove this proposition by slightly modifying the proof of Proposition 2.1 which, in fact, is based on the existence of tubes of the form $G/H \times A$ about the orbits (see the proof of Theorem II.5.8. in [9]). But we prefer to obtain Proposition 3.10 as a direct corollary of the statement of Proposition 2.1 and the following proposition of general interest:

PROPOSITION 3.11. *Let Y be an H-space, where H is a closed subgroup of a group G. If the orbit projection $q_Y : Y \to Y/H$ is an H-fibration, then the map $p : G \times_H Y \to Y/H$, given by $p([g, y]) = q_Y(y)$, is a G-fibration.*

PROOF. The map p is well defined and continuous being the map (of H-orbit spaces) induced by the projection $G \times Y \to Y$.

Suppose that it is given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & G \times_H Y \\ \Big\downarrow{\scriptstyle \partial_0} & & \Big\downarrow{\scriptstyle p} \\ X \times I & \xrightarrow{F} & Y/H \end{array}$$

of G-maps. We have to show that there exists a filler $\widetilde{F} : X \times I \to G \times_H Y$.

Let $S = f^{-1}(i_Y(Y))$, where $i_Y : Y \hookrightarrow G \times_H Y$ is the closed H-embedding given by $i_Y(y) = [e, y]$. Let $f' : S \to Y$ be an H-map defined by $f'(s) = (i_Y)^{-1}(f(s))$. With no loss of generality we can assume that $X = G \times_H S$ (see [12, Corollary 1.7.8]) and $f = G \times_H f'$, i.e., $f([g, s]) = [g, f'(s)]$ for $[g, s] \in X$.

Define $F' : S \times I \to Y/H$ by $F'(s, t) = F([e, s], t)$. Clearly, F' is an H-map and $F'(s, 0) = F([e, s], 0) = (p \circ f)([e, s]) = p([e, f'(s)]) = q_Y(f'(s))$ for every $s \in S$. Since q_Y is an H-fibration, there exists an H-homotopy $\widehat{F} : S \times I \to Y$ such that diagram

$$\begin{array}{ccc} S & \xrightarrow{f'} & Y \\ \Big\downarrow{\scriptstyle \partial'_0} & \nearrow{\scriptstyle \widehat{F}} & \Big\downarrow{\scriptstyle q_Y} \\ S \times I & \xrightarrow{F'} & Y/H \end{array}$$

commutes.

Now we can define $\widetilde{F} : X \times I \to G \times_H Y$ as follows:

$$\widetilde{F}([g, s], t) = [g, \widehat{F}(s, t)], \quad [s, t] \in X, \quad g \in G.$$

It is easy to check that $\widetilde{F}$ is the required filler. ∎

PROOF OF PROPOSITION 3.10. Let $N = N(H)$ be the normalizer of H in G. Since all orbits of E have type (H), there is a commutative diagram (see [9, II.5.9, II.5.10])

$$\begin{array}{ccc} G \times_N E^H & \xrightarrow{\eta} & E \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle p_E} \\ E^H/N & \xrightarrow{\zeta} & E/G \end{array}$$

where η is a G-equivalence, defined by $[g, y] \mapsto gy$, ζ is a homeomorphism taking $N(y) \in E^H/N$ to $G(y) \in E/G$ and $p([g, y]) = N(y)$. Therefore we must show only that $p_{E^H} : E^H \to E^H/N$ is an N-fibration because in this case p is a G-fibration by Proposition 3.11.

Note that the N-space E^H can be regarded as a *free* N/H-space. By Proposition 2.1, the orbit projection $q : E^H \to E^H/(N/H)$ is an N/H-fibration, because N/H is a Lie group. Hence q is also an N-fibration by Proposition 3.1 (a). Since we have the natural homeomorphism $E^H/N \approx E^H/(N/H)$, the orbit projection p_{E^H} is also an N-fibration. ∎

4. Resolutions of orbit projections

In this paper we are interested only in the G-ANR-resolutions of compact G-spaces. Therefore the general definition of a G-resolution, given in [2] and [6], can be reduced to a simpler one which looks as follows:

Definition 4.1. Let X be a compact G-space. An inverse sequence of G-ANR-spaces and G-maps $\{X_i, q_i^j\}$ is called *G-ANR-resolution of X* if:

(1) $X = \varprojlim \{X_i, q_i^j\}$,
(2) the family of natural projections $\{q_i : X \to X_i\}$ satisfies the following condition: for every i and any invariant open neighborhood U of $q_i(X)$ in X_i there exists $j \geq i$ such that $q_i^j(X_j) \subseteq U$.

It is easy to see that every compact G-space admits a G-ANR-resolution (for the general case see [6]). Indeed, by Proposition 2.2, we can consider a given compact G-space X as a closed invariant subset of some G-AR-space M. Since the G-space X is compact it has a countable neighborhood basis $\{U_i\}_{i\in\mathbb{N}}$ in M consisting of invariant open subsets U_i such that $U_{i+1} \subseteq U_i$ for every i. Then the inverse sequence $\{U_i, u_i^j\}$, where u_i^j are the inclusions $U_j \hookrightarrow U_i$ for $j \geq i$, is a G-ANR-resolution of X.

Proposition 4.2. *Let H be a large subgroup of a compact group G. If a compact G-space X has only one orbit type (H), then X admits a G-ANR-resolution $\{X_i, u_i^j\}$ such that all orbits of every G-ANR-space X_i have the same orbit type (H).*

Clearly, Proposition 4.2 is an immediate consequence of the following lemma.

Lemma 4.3. *Let H be a large subgroup of a compact group G. If a G-space X has only one orbit type (H), then there is a closed G-embedding $X \hookrightarrow U$ such that U is a G-ANR-space with all orbits of the same type (H).*

Proof. First we consider the particular case when $H = \{e\}$. In other words, we suppose that G is a Lie group and X is a free G-space. According to Proposition 2.2 the G-space X can be considered as an invariant closed subspace of some G-AR-space M. For every $x \in X$ the orbit $G(x)$ is a G-retract of some invariant open neighborhood U_x of $G(x)$ in M. If $r : U_x \to G(x)$ is a G-retraction, then $G_y \subseteq G_{r(y)} = \{e\}$, that is, $G_y = \{e\}$ for all $y \in U_x$. Thus the action of G on U_x is free for each $x \in X$ and therefore the action of G on $U = \bigcup_{x \in X} U_x$ is free. Obviously, U is an invariant neighborhood of X and $X \hookrightarrow U$ is the required embedding because U is G-ANR-space as an open invariant subset of the G-AR-space M.

Now let us return to the general case. Since X has orbits only of type (H), there exists a canonical G-equivalence $X \approx G \times_N X^H$, where X^H is the set of H-fixed points and $N = N(H)$ is the normalizer of H (see [9, II.5.9]); therefore we can assume that $X = G \times_N X^H$. Note that the quotient group N/H acts freely on X^H according to the rule: $nH \cdot x = nx$. Moreover, N/H is a Lie group (see Proposition 3.9 (a)). Hence, by the initial part of the proof, there is a closed N/H-embedding $i : X^H \hookrightarrow V$ in some *free* N/H-ANR-space V. Clearly, i can be regarded as an N-embedding of the N-space X^H in the N-ANR-space V (see Proposition 3.1(b)).

The induced closed G-embedding $G \times_N i : G \times_N X^H \hookrightarrow G \times_N V$ has the desired properties. Indeed, since N is obviously a large subgroup of G and V is an N-ANR, $U = G \times_N V$ is a G-ANR-space by Proposition 8 of [4]. It is easy to see that U has only orbits of type (H). □

The following fact is well known (see, for instance, [13, §46]):

PROPOSITION 4.4. *If G is a compact metrizable group, then there exists a decreasing sequence*

$$N_1 \supseteq N_2 \supseteq ... \supseteq N_i \supseteq N_{i+1} \supseteq ...,$$

of large normal subgroups of G such that $\bigcap_{i \in \mathbb{N}} N_i = \{e\}$. Therefore

$$\varprojlim \{G/N_i, q_i^j\} = G,$$

where $q_i^j : G/N_j \longrightarrow G/N_i$, $j \geq i$, are natural projections.

DEFINITION 4.5. *If a sequence $\{N_i\}_{i \in \mathbb{N}}$ of subgroups of a group G satisfies the conditions of Proposition 4.4, we say that it is a pro-Lie sequence.*

PROPOSITION 4.6. *Let $\{N_i\}_{i \in \mathbb{N}}$ be a pro-Lie sequence of subgroups of a compact group G. If X is a G-space then*

$$X = \varprojlim \{X/N_i, p_i^j\}$$

where $p_i^j : X/N_j \longrightarrow X/N_i$, $j \geq i$, are natural projections.

Note that here the spaces X/N_i are G/N_i-spaces, but we also can regard them as G-spaces and consider the maps p_i^j as G-maps. The proof of Proposition 4.6 is given in [7] (see also [1, Corollary 9]).

THEOREM 4.7. *Let G be a compact metrizable group and let $\{N_i\}_{i\in\mathbb{N}}$ be any pro-Lie sequence of subgroups of G. If all orbits of a compact G-space E have type (H), then E admits a G-ANR-resolution $\{E_i, q_i^j\}$ such that each E_i has orbits only of type (HN_i).*

PROOF. According to Proposition 4.6, we can represent E as

$$E = \varprojlim\{E/N_i, p_i^j\}$$

where $p_i^j : E/N_j \longrightarrow E/N_i$, $j \geq i$, are the natural projections. Since, for each i, E/N_i is a G-space with all orbits of type (HN_i) and HN_i is a large subgroup of G, we can consider, by Lemma 4.3, each E/N_i as a closed invariant subset of some G-ANR-space U_i with all orbits of the same orbit type (HN_i).

For each i we shall find an open invariant neighborhood V_i of E/N_i in U_i by induction as follows. Put $V_1 = U_1$ and suppose that V_i is given. Let V_{i+1} be an open invariant neighborhood of E/N_{i+1} in U_{i+1} for which there is a G-equivariant extension $f_i^{i+1} : V_{i+1} \to V_i$ of the composition

$$E/N_{i+1} \to E/N_i \hookrightarrow V_i.$$

This extension exists because V_i is a G-ANE.

Now, by obvious induction on i and j, we choose a collection of G-spaces $\{W_i^{(j)}\}_{i,j\in\mathbb{N}}$ which satisfies the following conditions:

(1) For each i the family $\{W_i^{(j)}\}_{j\in\mathbb{N}}$ is a basis of open invariant neighborhoods of E/N_i in V_i such that $W_i^{(j+1)} \subset W_i^{(j)}$ for all j.

(2) $W_{i+1}^{(j)} \subseteq (f_i^{i+1})^{-1}(W_i^{(j)})$ for all i and j.

Finally, we put $E_i = W_i^{(i)}$ for each i, and define $q_i^{i+1} : E_{i+1} \to E_i$ as the restriction of f_i^{i+1} to E_{i+1} (note that $f_i^{i+1}(W_{i+1}^{(i+1)}) \subseteq W_i^{(i+1)} \subseteq W_i^{(i)}$). Of course, $q_i^j : E_j \to E_i$ for $j > i$ is defined by $q_i^j = q_i^{i+1} \circ q_{i+1}^{i+2} \circ \ldots \circ q_{j-1}^j$.

It is easily checked that $\{E_i, q_i^j\}$ is the required G-ANR-resolution of E. ∎

DEFINITION 4.8. *Let $f : X \to Y$ be a G-map of compact G-spaces. An inverse sequence $\{f_i, \beta_i^j\}$ consisting of G-maps f_i and pairs $\beta_i^j = (q_i^j, r_i^j)$ of G-maps such that the diagram*

$$\begin{array}{ccc} X_i & \xleftarrow{q_i^{i+1}} & X_{i+1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\ Y_i & \xleftarrow[r_i^{i+1}]{} & Y_{i+1} \end{array}$$

commutes, for every i, is called G-ANR-resolution of f if

1. *$\{X_i, q_i^j\}$ is a G-ANR-resolution of X,*

2. $\{Y_i, r_i^j\}$ *is a* G-ANR*-resolution of* Y,
3. $f = \varprojlim \{f_i, \beta_i^j\}$.

Of course, if the group G is trivial this definition (as well as Definition 4.1) reduces to the definition of an ANR-resolution for non-equivariant case (see [11, Definition 4]).

PROPOSITION 4.9. *Let* N *be a closed normal subgroup of a compact group* G *and let* $\{X_i, q_i^j\}$ *be a* G-ANR*-resolution of a compact* G*-space* X. *Then*

(i) $\{X_i/N, q_i^j/N\}$ *is a* G/N-ANR*-resolution of* X/N.
(ii) *If, for each* i, $p_{X_i} : X_i \to X_i/N$ *is the orbit projection, then* $\{p_{X_i}, \beta_i^j\}$ *is a* G-ANR*-resolution of the* N*-orbit projection* $p_X : X \to X/N$, *where* $\beta_i^j = (q_i^j, q_i^j/N)$ *for* $i \leq j$.

The most part of the proof of Proposition 4.9 is covered by the following general lemma. This lemma seems to be well known, but we shall give its proof because we could not find an adequate reference.

LEMMA 4.10. *Let* N *be a closed normal subgroup of a compact group* G *and let* $\{X_\lambda, q_\lambda^\mu, \Lambda\}$ *be an inverse system of Hausdorff* G*-spaces. If* $X = \varprojlim \{X_\lambda, q_\lambda^\mu, \Lambda\}$ *with the projections* $q_\lambda : X \to X_\lambda$, *then*

$$X/N = \varprojlim \{X_\lambda/N, q_\lambda^\mu/N, \Lambda\}$$

with the projections $q_\lambda/N : X/N \to X_\lambda/N$.

PROOF. First note that for any N-orbit $N(x)$ in X

$$N(x) = \bigcap_{\lambda \in \Lambda} q_\lambda^{-1}(q_\lambda(N(x)))$$

because $N(x)$ is a closed subset of X.

Let

$$Y = \varprojlim \{X_\lambda/N, q_\lambda^\mu/N, \Lambda\}$$

with the projections $\widehat{q}_\lambda : Y \to X_\lambda/N$. Since, for $\lambda \leq \mu$, $q_\lambda/N = q_\lambda^\mu/N \circ q_\mu/N$, there is a unique G/N-map $h : X/N \to Y$ such that $\widehat{q}_\lambda \circ h = q_\lambda/N$ for each $\lambda \in \Lambda$. We must show that h is a homeomorphism.

"*h is injective*". Suppose that $N(x), N(x') \in X/N$ are such that $h(N(x)) = h(N(x'))$. Then, for each λ, $(q_\lambda/N)(N(x)) = (q_\lambda/N)(N(x'))$, which means that $q_\lambda(N(x)) = q_\lambda(N(x'))$. Therefore

$$N(x) = \bigcap_{\lambda \in \Lambda} q_\lambda^{-1}(q_\lambda(N(x))) = \bigcap_{\lambda \in \Lambda} q_\lambda^{-1}(q_\lambda(N(x'))) = N(x').$$

"*h is surjective*". Let $y \in Y$ and let $N(x_\lambda) = \widehat{q}_\lambda(y)$ for every λ, where $x_\lambda \in X_\lambda$. For $\lambda \leq \mu$ we have $(q_\lambda^\mu/N)(N(x_\mu)) = N(x_\lambda)$ and hence $q_\lambda^\mu(N(x_\mu)) =$

$N(x_\lambda)$. So we get the inverse system $\{N(x_\lambda), \widetilde{q}^\mu_\lambda, \Lambda\}$, where $\widetilde{q}^\mu_\lambda(z) = q^\mu_\lambda(z)$ for $z \in N(x_\mu)$. Since all the spaces $N(x_\lambda)$ are compact and non-empty,

$$F = \varprojlim \{N(x_\lambda), \widetilde{q}^\mu_\lambda, \Lambda\} \neq \emptyset.$$

Clearly F can be regarded as a closed subset of X such that $F = \bigcap_{\lambda\in\Lambda} q_\lambda^{-1}(N(x_\lambda))$. Now let $x \in F$. Then $q_\lambda(x) \in N(x_\lambda)$ and therefore $q_\lambda(N(x)) = N(x_\lambda)$ (recall that q_λ takes any N-orbit onto some N-orbit) for each λ. This implies that $h(N(x)) = y$ because $(q_\lambda/N)(N(x)) = q_\lambda(N(x)) = N(x_\lambda) = \widehat{q_\lambda}(y)$ for every $\lambda \in \Lambda$.

"*h is an open map*". Let $y \in h(U)$, where U is an open subset of X/N. For $x \in X$ satisfying $h(N(x)) = y$, we can find λ and an open neighborhood V of $q_\lambda(x)$ in X_λ such that $q_\lambda^{-1}(V) \subseteq (p_X)^{-1}(U)$, where $p_X : X \to X/N$ is the N-orbit projection. Since the N-orbit projection $p_{X_\lambda} : X_\lambda \to X_\lambda/N$ is open, $\widehat{q_\lambda}^{-1}(p_{X_\lambda}(V))$ is an open subset of Y. It is easy to see that

$$y \in \widehat{q_\lambda}^{-1}(p_{X_\lambda}(V)) \subseteq h(U).$$

Indeed,

$$\widehat{q_\lambda}(y) = \widehat{q_\lambda} h(p_X(x)) = (q_\lambda/N)(p_X(x)) = p_{X_\lambda} q_\lambda(x)$$

and therefore $y \in \widehat{q_\lambda}^{-1}(p_{X_\lambda}(V))$, because $q_\lambda(x) \in V$. If $y' \in \widehat{q_\lambda}^{-1}(p_{X_\lambda}(V))$, then for $x' \in X$, satisfying $y' = h(N(x'))$, we have $p_{X_\lambda} q_\lambda(x') = \widehat{q_\lambda}(y') \in p_{X_\lambda}(V)$. Hence we can find $v \in V$ and $n \in N$ such that $nq_\lambda(x') = v$. This implies that $nx' \in q_\lambda^{-1}(V) \subseteq (p_X)^{-1}(U)$. Thus $N(x') \in U$ and $y' = h(N(x')) \in h(U)$. □

Proof of Proposition 4.9. We shall prove only the assertion (i) because (ii) is its obvious consequence.

By lemma 4.10 we already have that $X/N = \varprojlim \{X_i/N, q_i^j/N\}$.

According to Proposition 2.3, every X_i/N is a G/N-ANR. To finish the proof we must show that $\{X_i/N, q_i^j/N\}$ satisfies the second condition of Definition 4.1. Let U be a neighborhood of $(q_i/N)(X/N)$ in X_i/N. Since $\{X_i, q_i^j\}$ is a G-ANR-resolution and $q_i(X) \subseteq p_{X_i}^{-1}(U)$, there is $j \geq i$ such that $q_i^j(X_j) \subseteq p_{X_i}^{-1}(U)$. Then $(q_i^j/N)(X_j/N) \subset U$. □

Finally, combining Theorem 4.7 and Proposition 4.9, we obtain the main result of this paper.

Theorem 4.11. *Let G be a compact metrizable group and let $\{N_i\}_{i\in\mathbb{N}}$ be any pro-Lie sequence of subgroups of G. If all orbits of a compact G-space E have type (H), then the orbit projection*

$$p_E : E \to E/G$$

admits a G-ANR-resolution $\{p_i, \beta_i^j\}$ consisting of the orbit projections $p_i : E_i \to E_i/G$ such that each E_i has only one orbit type (HN_i) and therefore each p_i is a G-fibration.

Proof. It is sufficient to apply the orbit projection functor $-/G$ to the G-ANR-resolution $\{E_i, q_i^j\}$ given by Theorem 4.7. Hence, by Proposition 4.9, we obtain a G-ANR-resolution $\{p_i, \beta_i^j\}$ of p_E where $p_i = p_{E_i} : E_i \to E_i/G$ for each i. By Proposition 3.10 every p_i is a G-fibration. ∎

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