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Repdigits as sums of three Fibonacci numbers

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Abstract. In this paper, we find all base 10 repdigits which are sums of three Fibonacci numbers.

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1. The main result and its proof

It has been shown by Senge and Straus [8] that if a and b are multiplicatively independent positive integers, then every large positive integer N has either many nonzero digits in base a, or many nonzero digits in base b. This was made effective by Stewart [9] using Baker's theory of lower bounds for linear forms in logarithms of algebraic numbers (see also [4]). There are also variants of these results involving for example either the Zeckendorf representation instead of just representations in integer bases, or asking that the digits of a number N in a fixed integer base b to be distinct from a fixed given one, instead of only asking that they be distinct from 0 (see, for example, [4] and [9]). Recall that the Zeckendorf representation [10] of a positive integer N is the representation

$$N = F_{m_1} + F_{m_2} + \cdots + F_{m_t}, \quad \text{with} \quad m_i - m_{i+1} \geq 2 \quad \text{for} \quad i = 1, \ldots, t-1,$$

where $\{F_n\}_{n\geq 1}$ is the Fibonacci sequence $F_1 = F_2 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 1$. We also set $F_0 := 0$. In particular, large repdigits in a base b, that is numbers with identical digits in base b, must have many terms in their Zeckendorf representation. In [5], it was shown in an elementary way that the largest repdigit in base 10 which is a Fibonacci number is 55.

Here, we find all repdigits in base 10 which are the sums of at most three Fibonacci numbers. Similar problems were recently investigated. For example, Fibonacci numbers which are sums of three factorials were found in [1], while factorials which are sums of at most three Fibonacci numbers were found in [6].

We follow the general method described in [4].

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Theorem 1. *All nonnegative integer solutions* (m_1, m_2, m_3, n) *of the equation*

$$N = F_{m_1} + F_{m_2} + F_{m_3} = d\left(\frac{10^n - 1}{9}\right), \quad \textit{with} \;\; d \in \{1, \ldots, 9\} \tag{1}$$

have

$$N \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 44, 55, 66, 77, 99, 111, 555, 666, 11111\}.$$

Proof. To fix ideas, we assume that $m_1 \geq m_2 \geq m_3$. A brute force search with Mathematica in the range $0 \leq m_1 \leq 50$ turned up only the solutions shown in the statement of Theorem 1. This took a few minutes. When $m_1 \geq 51$, we have that $N \geq F_{m_1} \geq F_{51} > 10^{10}$, so all solutions of equation (1) must have

$$F_{m_1} + F_{m_2} + F_{m_3} \equiv d\left(\frac{10^{10} - 1}{9}\right) \pmod{10^{10}}, \quad \text{with some} \;\; d \in \{1, \ldots, 9\}.$$

We generated the list of residues

$$\mathcal{F} := \{F_{m_1} \pmod{10^{10}} : 51 \leq m_1 \leq 1000\}.$$

Then we tested, again with Mathematica, whether for some $m_2,\ m_3 \in [0, 1000]$ and $d \in \{1, \ldots, 9\}$, we can have

$$d\left(\frac{10^{10} - 1}{9}\right) - F_{m_2} - F_{m_3} \pmod{10^{10}} \in \mathcal{F}.$$

This took a few minutes and no new solution turned up.

A faster and more clever way of testing this range was pointed out to me by Juan José Alba Gonzalez. Namely, one first shows by using only elementary manipulations with the recurrence defining the Fibonacci numbers that if a number N is a sum of at most three Fibonacci numbers, then its Zeckendorf representation contains at most three terms. Next, if $m_1 \leq 1000$, then $10^{n-1} \leq (10^n - 1)/9 \leq 3F_{m_1} \leq 3F_{1000}$, so $n \leq 210$. For each $d \in \{1, \ldots, 9\}$ and each $n \in [1, 210]$, we generated the Zeckendorf representations of $N = d(10^n - 1)/9$, and selected only the instances for which such representation has at most three terms. This computation took a few seconds and returned only the numbers N appearing in the statement of the theorem.

So, from now on, we may assume that $m_1 \geq 1001$, therefore $n \geq 208$. We use the fact that

$$F_m = \frac{\alpha^m - \beta^m}{\sqrt{5}} \quad \text{for} \;\; m \geq 0, \quad \text{where} \;\; (\alpha, \beta) := \left(\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\right).$$

We rewrite equation (1) in three different ways as

$$\begin{aligned} \frac{\alpha^{m_1}}{\sqrt{5}} - \frac{d \times 10^n}{9} &= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} - \frac{\alpha^{m_2}}{\sqrt{5}} + \frac{\beta^{m_2}}{\sqrt{5}} - \frac{\alpha^{m_3}}{\sqrt{5}} + \frac{\beta^{m_3}}{\sqrt{5}}, \\ \frac{\alpha^{m_1}}{\sqrt{5}}(1 + \alpha^{m_2 - m_1}) - \frac{d \times 10^n}{9} &= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} + \frac{\beta^{m_2}}{\sqrt{5}} - \frac{\alpha^{m_3}}{\sqrt{5}} + \frac{\beta^{m_3}}{\sqrt{5}}, \\ \frac{\alpha^{m_1}}{\sqrt{5}}(1 + \alpha^{m_2 - m_1} + \alpha^{m_3 - m_1}) - \frac{d \times 10^n}{9} &= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} + \frac{\beta^{m_2}}{\sqrt{5}} + \frac{\beta^{m_3}}{\sqrt{5}}. \end{aligned} \tag{2}$$

We take absolute values in each one of the three equations (2) obtaining

$$\begin{aligned}\left|\frac{\alpha^{m_1}}{\sqrt{5}}-\frac{d\times 10^n}{9}\right| &\le \frac{d}{9}+\frac{|\beta|^{m_1}}{\sqrt{5}}+\frac{\alpha^{m_2}}{\sqrt{5}}+\frac{|\beta|^{m_2}}{\sqrt{5}}\\ &\quad+\frac{\alpha^{m_3}}{\sqrt{5}}+\frac{|\beta|^{m_3}}{\sqrt{5}}\le 4+\frac{2\alpha^{m_2}}{\sqrt{5}}\\ &\le \frac{11\alpha^{m_2}}{\sqrt{5}}<\frac{\alpha^{m_2+5}}{\sqrt{5}},\\ \left|\frac{\alpha^{m_1}}{\sqrt{5}}(1+\alpha^{m_2-m_1})-\frac{d\times 10^n}{9}\right| &\le \frac{d}{9}+\frac{|\beta|^{m_1}}{\sqrt{5}}+\frac{|\beta|^{m_2}}{\sqrt{5}}+\frac{\alpha^{m_3}}{\sqrt{5}}+\frac{|\beta|^{m_3}}{\sqrt{5}}\\ &\le 4+\frac{\alpha^{m_3}}{\sqrt{5}}<\frac{10\alpha^{m_3}}{\sqrt{5}}<\frac{\alpha^{m_3+5}}{\sqrt{5}},\\ \left|\frac{\alpha^{m_1}}{\sqrt{5}}(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1})-\frac{d\times 10^n}{9}\right| &\le \frac{d}{9}+\frac{|\beta|^{m_1}}{\sqrt{5}}+\frac{|\beta|^{m_2}}{\sqrt{5}}+\frac{|\beta|^{m_3}}{\sqrt{5}}\\ &\le 4<\frac{9}{\sqrt{5}}<\frac{\alpha^5}{\sqrt{5}}.\end{aligned} \tag{3}$$

Dividing the left–hand sides of the three inequalities (3) by

$$\frac{\alpha^{m_1}}{\sqrt{5}},\qquad \frac{\alpha^{m_1}}{\sqrt{5}}(1+\alpha^{m_2-m_1})\quad\text{and}\quad \frac{\alpha^{m_1}}{\sqrt{5}}(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1}),$$

respectively, we get

$$\begin{aligned}\left|1-\alpha^{-m_1}10^n\left(\frac{d\sqrt{5}}{9}\right)\right| &< \frac{1}{\alpha^{m_1-m_2-5}},\\ \left|1-\alpha^{-m_1}10^n\left(\frac{d\sqrt{5}\alpha^{m_1-m_2}}{9(\alpha^{m_1-m_2}+1)}\right)\right| &< \frac{(1+\alpha^{m_2-m_1})^{-1}}{\alpha^{m_1-m_3-5}}<\frac{1}{\alpha^{m_1-m_3-5}},\\ \left|1-\alpha^{-m_1}10^n\left(\frac{d\sqrt{5}\alpha^{m_1-m_3}}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}\right)\right| &< \frac{(1+\alpha^{m_2-m_1}+\alpha^{m_3-m_1})^{-1}}{\alpha^{m_1-5}}\\ &< \frac{1}{\alpha^{m_1-5}},\end{aligned} \tag{4}$$

respectively.

We use a result of Matveev (see [7], or Theorem 9.4 in [2]), which asserts that if $\alpha_1,\ \alpha_2,\ \alpha_3$ are positive real algebraic numbers in an algebraic number field of degree D and $b_1,\ b_2,\ b_3$ are rational integers, then

$$|1-\alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3}|>\exp\left(-1.4\times 30^6\times 3^{4.5}D^2(1+\log D)(1+\log B)A_1A_2A_3\right) \tag{5}$$

(assuming that the left–hand side above is nonzero), where

$$B:=\max\{|b_1|,|b_2|,|b_3|\},$$

and

$$A_i\ge \max\{Dh(\alpha_i),|\log\alpha_i|,0.16\},\quad \text{for}\quad i=1,2,3.$$

Here, for an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X]$$

being the minimal polynomial over the integers having η as a root.

We shall apply this to the left–hand sides of (4). In all three cases, we take $\alpha_1 := \alpha,\ \alpha_2 := 10,\ b_1 := -m_1,\ b_2 := n,\ b_3 := 1$. Only the number α_3 is different in each of the three instances, namely it is

$$\frac{d\sqrt{5}}{9}, \quad \frac{d\sqrt{5}\alpha^{m_1-m_2}}{9(\alpha^{m_1-m_2}+1)} \quad \text{and} \quad \frac{d\sqrt{5}\alpha^{m_1-m_3}}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)},$$

respectively, according to whether we work with the first, second, or third of the inequalities (4). In all cases, the three numbers $\alpha_1,\ \alpha_2,\ \alpha_3$ are real, positive and belong to $\mathbb{Q}(\sqrt{5})$, so we can take $D := 2$.

We next justify that the amounts on the left–hand sides of (4) are not zero.

Indeed, if the left–hand side of the first of inequalities (4) is zero, we then get $\alpha^{m_1} = 10^n d\sqrt{5}/9$, so $\alpha^{2m_1} \in \mathbb{Q}$ which is false.

If the left–hand side of the second of inequalities (4) is zero, we then get that

$$\alpha^{m_1-m_2} + 1 = 10^n \left(\frac{d\sqrt{5}}{9}\right)\alpha^{-m_2},$$

or

$$\alpha^{m_1} + \alpha^{m_2} = 10^n \left(\frac{d\sqrt{5}}{9}\right).$$

Conjugating the above relation in $\mathbb{Q}(\sqrt{5})$, we get

$$\beta^{m_1} + \beta^{m_2} = -10^n \left(\frac{d\sqrt{5}}{9}\right).$$

Hence,

$$\alpha^{m_1} < \alpha^{m_1} + \alpha^{m_2} = |\beta^{m_1} + \beta^{m_2}| \le |\beta|^{m_1} + |\beta|^{m_2} < 2,$$

which is impossible for $m_1 \ge 1001$. A similar argument deals with the situation when the left–hand side of the third of inequalities (4) were zero. Indeed, if this were so, we would then get that

$$\alpha^{m_1-m_3} + \alpha^{m_2-m_3} + 1 = 10^n \left(\frac{d\sqrt{5}}{9}\right)\alpha^{-m_3},$$

or

$$\alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3} = 10^n \left(\frac{d\sqrt{5}}{9}\right).$$

Conjugation in $\mathbb{Q}(\sqrt{5})$ above replaces α by β on the left–hand side but only changes the sign of the right–hand side. Thus, we get

$$\alpha^{m_1} < \alpha^{m_1} + \alpha^{m_2} + \alpha^{m_3} = |\beta^{m_1} + \beta^{m_2} + \beta^{m_3}| \le 3,$$

which is false for $m_1 \ge 1001$.

Hence, indeed the left–hand sides of all three inequalities (4) are nonzero.

Next observe that

$$\begin{aligned} 10^{n-1} &\le 10^{n-1} + 10^{n-2} + \cdots + 1 \le d\left(\frac{10^n - 1}{9}\right) = F_{m_1} + F_{m_2} + F_{m_3} \\ &\le 3F_{m_1} < 3\alpha^{m_1 - 1} < \alpha^{m_1 + 2}, \end{aligned}$$

therefore

$$m_1 + 2 > \left(\frac{\log 10}{\log \alpha}\right)(n-1) > 4.78(n-1).$$

Hence, $m_1 > 4.78n - 6.78 > n$ since $n \ge 208$. So, with the notation $B := \max\{|b_1|, |b_2|, |b_3|\}$, we definitely have $B = m_1$. We can choose $A_1 := 0.5 > 2h(\alpha_1)$, $A_2 := 4.7 > 2\log \alpha_2$. We now put

$$C_1 := 3 \times 10^{12} > 1.4 \times 30^6 \times 3^{4.5} \times D^2 \times (1 + \log D) \times A_1 \times A_2. \tag{6}$$

We have $C_1 > 2.27889 \times 10^{12}$. We apply inequality (5) iteratively on the left–hand sides of inequalities (4). For the first one, we can take

$$A_3 := 6.1 > (\log 81 + 2\log\sqrt{5}) \ge 2h(\alpha_3),$$

and get that

$$\exp\left(-6.1C_1(1 + \log m_1)\right) < \left|1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9}\right)\right| < \frac{1}{\alpha^{m_1 - m_2 - 5}},$$

implying

$$7 + (m_1 - m_2)\log\alpha < (7 + 5\log\alpha) + 6.1C_1(1 + \log m_1) < 7C_1(1 + \log m_1). \tag{7}$$

We now apply inequality (5) to the second of inequalities (4). We need some statistics for the corresponding α_3. Observe first that in this case

$$\alpha_3 = \frac{d\sqrt{5}\alpha^{m_1 - m_2}}{9(\alpha^{m_1 - m_2} + 1)} \le \sqrt{5}, \quad \alpha_3^{-1} = \frac{9(\alpha^{m_1 - m_2} + 1)}{d\sqrt{5}\alpha^{m_1 - m_2}} \le \frac{18}{\sqrt{5}},$$

so that $|\log\alpha_3| < 2.1$. Next, observe that

$$a_0 \mid 9^2(\alpha^{m_1-m_2}+1)(\beta^{m_1-m_2}+1); \quad \text{hence,} \quad a_0 \le 324\alpha^{m_1-m_2}.$$

The conjugate of α_3 is in absolute value at most

$$\frac{d\sqrt{5}|\beta|^{m_1-m_2}}{9(\beta^{m_1-m_2}+1)} \le \frac{\sqrt{5}}{1-|\beta|} = \frac{2\sqrt{5}}{1+\sqrt{5}}.$$

Hence, we can take

$$A_3 := 7+(m_1-m_2)\log\alpha > \log(324)+(m_1-m_2)\log\alpha+\log(\sqrt{5})+\log\left(\frac{2\sqrt{5}}{1+\sqrt{5}}\right)$$
$$> \max\{2h(\alpha_3), |\log\alpha_3|, 0.16\}.$$

We then get, by applying inequality (5) to the second inequality (4),

$$\exp\left(-(7+(m_1-m_2)\log\alpha)C_1(1+\log m_1)\right) < \frac{1}{\alpha^{m_1-m_3-5}},$$

so

$$11+(m_1-m_3)\log\alpha < (11+5\log\alpha)+(7+(m_1-m_2)\log\alpha)C_1(1+\log m_1).$$

Using also inequality (7), we get that

$$11+(m_1-m_3)\log\alpha < (11+5\log\alpha)+7C_1^2(1+\log m_1)^2 < 8C_1^2(1+\log m_1)^2. \quad (8)$$

We now move to the third inequality (4). We need some statistics on the current α_3. Observe that

$$\alpha_3 = \frac{d\sqrt{5}\alpha^{m_1-m_3}}{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)} \le \sqrt{5}, \quad \alpha_3^{-1} = \frac{9(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)}{d\sqrt{5}\alpha^{m_1-m_3}} \le \frac{27}{\sqrt{5}},$$

so that $|\log\alpha_3| < 2.5$. Next, observe that

$$a_0 \mid 9^2(\alpha^{m_1-m_3}+\alpha^{m_2-m_3}+1)(\beta^{m_1-m_3}+\beta^{m_2-m_3}+1); \quad \text{hence,} \quad a_0 \le 729\alpha^{m_1-m_3}.$$

The conjugate of α_3 is in absolute value at most

$$\frac{d\sqrt{5}|\beta|^{m_1-m_3}}{9|\beta^{m_1-m_3}+\beta^{m_2-m_3}+1|} \le \frac{\sqrt{5}}{|\beta^{m_1-m_3}+\beta^{m_2-m_3}+1|}.$$

We need a lower bound for $|\beta^{m_1-m_3}+\beta^{m_2-m_3}+1|$. We distinguish a few cases.

If $m_2-m_3=0$, then

$$|\beta^{m_1-m_3}+\beta^{m_2-m_3}+1| = |\beta^{m_1-m_3}+2| \ge 2-1 = 1.$$

Assume next that $m_2-m_3=1$. If $m_1-m_3=1$, we get

$$|\beta^{m_1-m_3}+\beta^{m_2-m_3}+1| = |2\beta+1| = \sqrt{5}-2.$$

If $m_2 - m_3 = 1$ and $m_1 - m_3 = 2$, then

$$|\beta^{m_1-m_3} + \beta^{m_2-m_3} + 1| = |\beta^2 + \beta + 1| = 3 - \sqrt{5}.$$

If $m_2 - m_3 = 1$ and $m_1 - m_3 \geq 3$, then

$$|\beta^{m_1-m_3} + \beta^{m_2-m_3} + 1| \geq 1 - |\beta| - |\beta|^3 = \frac{7-3\sqrt{5}}{2}.$$

Finally, if $m_2 - m_3 \geq 2$, then

$$|\beta^{m_1-m_3} + \beta^{m_2-m_3} + 1| \geq 1 - 2\beta^2 = \sqrt{5} - 2.$$

The above calculations show that $|\beta^{m_1-m_3} + \beta^{m_2-m_3} + 1| \geq (7-3\sqrt{5})/2$, therefore the conjugate of α_3 has absolute value at most

$$\frac{\sqrt{5}}{(7-3\sqrt{5})/2} = \frac{2\sqrt{5}}{7-3\sqrt{5}}.$$

Hence, we can take

$$\begin{aligned} A_3 := 11 + (m_1 - m_2)\log\alpha &> \log(729) + (m_1 - m_3)\log\alpha + \log(\sqrt{5}) + \log\left(\frac{2\sqrt{5}}{7-3\sqrt{5}}\right) \\ &> \max\{2h(\alpha_3), |\log\alpha_3|, 0.16\}. \end{aligned}$$

We then get, by applying inequality (5) to the third inequality (4),

$$\exp\left(-(11 + (m_1 - m_3)\log\alpha)C_1(1 + \log m_1)\right) < \frac{1}{\alpha^{m_1-5}},$$

so

$$m_1 \log\alpha < 5\log\alpha + (11 + (m_1 - m_3)\log\alpha)C_1(1 + \log m_1).$$

Combining this with inequality (8), we get

$$m_1\log\alpha < 5\log\alpha + 8C_1^3(1+\log m_1)^3 < 9C_1^3(1+\log m_1)^3 < 9(3\times 10^{12})^3(1+\log m_1)^3,$$

so $m_1 < 6 \times 10^{44}$.

Now we only need to lower the bound.

Let

$$\Lambda_1 := -m_1\log\alpha + n\log 10 + \log(d\sqrt{5}/9).$$

Observe that the first equation (2) is

$$\begin{aligned} \frac{\alpha^{m_1}}{\sqrt{5}} - \frac{d\times 10^n}{9} &= \frac{\alpha^{m_1}}{\sqrt{5}}\left(1 - \alpha^{-m_1}10^n\left(\frac{d\sqrt{5}}{9}\right)\right) = \frac{\alpha^{m_1}}{\sqrt{5}}\left(1 - e^{\Lambda_1}\right) \\ &= -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} - F_{m_2} - F_{m_3} \leq -\frac{1}{9} + \frac{|\beta|^{1001}}{\sqrt{5}} < 0, \end{aligned}$$

where the last inequality holds because $m_1 \geq 1001$. This implies that $\Lambda_1 > 0$. Hence,

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = \left| 1 - \alpha^{-m_1} 10^n \left(\frac{d\sqrt{5}}{9} \right) \right| < \frac{1}{\alpha^{m_1 - m_2 - 5}},$$

by the first of inequalities (4). Thus, we get that

$$0 < \Lambda_1 = n \log 10 - m_1 \log \alpha + \log(d\sqrt{5}/9) < \frac{1}{\alpha^{m_1 - m_2 - 5}},$$

giving

$$0 < n \left(\frac{\log 10}{\log \alpha} \right) - m_1 + \left(\frac{\log(d\sqrt{5}/9}{\log \alpha} \right) < \frac{1}{(\log \alpha)\alpha^{m_1 - m_2 - 5}} < \frac{1}{\alpha^{m_1 - m_2 - 7}}.$$

We put $\gamma := (\log 10)/(\log \alpha)$, $\mu := (\log(d\sqrt{5}/9))/(\log \alpha)$. We also put $M := 6 \times 10^{44}$. Thus, we have the inequality

$$0 < n\gamma - m_1 + \mu < \frac{1}{\alpha^{m_1 - m_2 - 7}}, \tag{9}$$

where $n < m_1 \leq M$. By the standard Baker-Davenport reduction lemma (see Lemma 5 in [3]), it follows that

$$m_1 - m_2 \leq 7 + \frac{\log(q/\varepsilon)}{\log \alpha},$$

where $q > 4 \times 10^{45} > 6M$ is the denominator of a convergent to γ and $\varepsilon := \|\mu q\| - M\|\gamma q\| > 0$. We let $[a_0, a_1, \ldots] = [0, 4, 1, 3, 1, 1, 1, \ldots]$ be the continued fraction of γ and p_k/q_k the kth convergent to γ for all $k \geq 0$.

We took $q := q_{108}$, which is a number with 52 base 10 digits. Then $q > 6M$ and $\varepsilon > 0.08$ for all choices $d \in \{1, \ldots, 9\}$, giving

$$m_1 - m_2 \leq 7 + \frac{\log(q/0.08)}{\log \alpha} < 263,$$

so that $m_1 - m_2 \leq 262$. In particular, $m_2 \geq 739$.

Now we go back to the second equation (2) and use the same argument as the one we used for the first equation (2). Namely, we fix $m_1 - m_2 \leq 262$, and we put

$$\Lambda_2 := -m_1 \log \alpha + n \log 10 + \log \left(\frac{d\sqrt{5}\alpha^{m_1 - m_2}}{9(\alpha^{m_1 - m_2} + 1)} \right).$$

By an argument similar to the previous one, the second equation (2) is

$$\frac{\alpha^{m_1}}{\sqrt{5}} (1 + \alpha^{m_2 - m_1}) \left(1 - e^{\Lambda_2}\right) = -\frac{d}{9} + \frac{\beta^{m_1}}{\sqrt{5}} + \frac{\beta^{m_2}}{\sqrt{5}} - F_{m_3} \leq -\frac{1}{9} + \frac{|\beta|^{1001}}{\sqrt{5}} + \frac{|\beta|^{739}}{\sqrt{5}} < 0,$$

since $m_1 \geq 1001$ and $m_2 \geq 739$. So, as before, we get that $\Lambda_2 > 0$. We get, as before, that

$$0 < n\left(\frac{\log 10}{\log \alpha}\right) - m_1 + \left(\frac{1}{\log \alpha}\right) \log\left(\frac{d\sqrt{5}\alpha^{m_1-m_2}}{9(\alpha^{m_1-m_2}+1)}\right) < \frac{1}{\alpha^{m_1-m_3-7}}. \quad (10)$$

We keep the value of γ and of q, replace μ by

$$\mu := \left(\frac{1}{\log \alpha}\right) \log\left(\frac{d\sqrt{5}\alpha^{m_1-m_2}}{9(\alpha^{m_1-m_2}+1)}\right),$$

and recognize that inequality (10) is of the same type as inequality (9), except that the exponent $m_1 - m_2$ on α in the right–hand side has been replaced by $m_1 - m_3$. We next compute the lower bound $\varepsilon > 10^{-4}$ valid for all choices $d \in \{1, \ldots, 9\}$ and $0 \leq m_1 - m_2 \leq 262$, except for the pair $(d, m_1 - m_2) = (9, 2)$ for which actually one gets that $\mu = 0$, so the amount $\|\mu q\| - M\|q\gamma\| = -M\|q\gamma\|$ is negative. So, except for this pair, we have

$$m_1 - m_3 < 7 + \frac{\log(10^4 q)}{\log \alpha} < 275,$$

therefore $m_1 - m_3 \leq 274$. When $(d, m_1 - m_2) = (9, 2)$, we then have that

$$0 < n\gamma - m_1 < \frac{1}{\alpha^{m_1-m_3-7}}.$$

The largest partial quotient a_k for $0 \leq k \leq 109$ is $a_{35} = 106$. We now use the fact that if ζ is a positive irrational number with continued fraction $[b_0, b_1, \ldots, b_k, \ldots]$, partial quotients $P_k/Q_k = [b_0, \ldots, b_k]$ for $k \geq 0$, and we put $\zeta_k := [b_k, b_{k+1}, \ldots]$ for $k \geq 0$, then whenever x and y are integers with $|y| \leq Q_k$, we have

$$|x - \zeta y| \geq |P_k - \zeta Q_k| = \frac{1}{Q_k(\zeta_{k+1}Q_k + Q_{k-1})} \geq \frac{1}{Q_k^2(\zeta_{k+1}+1)} > \frac{1}{Q_k^2(b_{k+1}+2)}.$$

Applying the above inequality with $\zeta := \gamma$ and $k := 108$, we conclude, from the fact that $m_1 < q = q_{108}$,

$$\frac{1}{108q} \leq \frac{1}{(a_{109}+2)q} < |q_{108}\gamma - p_{108}| \leq n\gamma - m_1 < \frac{1}{\alpha^{m_1-m_3-7}}.$$

Hence,

$$m_1 - m_3 < 7 + \frac{\log(108q)}{\log \alpha} < 266,$$

so we get the same conclusion as before, namely that $m_1 - m_3 \leq 274$. Thus, $m_3 \geq 737$.

Next, we fix $d \in \{1, \ldots, 9\}$, $m_1 - m_2 \leq 262$, and $m_1 - m_2 \leq m_1 - m_3 \leq 274$. The same argument as the one we did before ensures that

$$0 < n\gamma - m_1 + \mu < \frac{1}{\alpha^{m_1-7}},$$

where

$$\mu := \left(\frac{1}{\log\alpha}\right)\log\left(\frac{d\sqrt{5}\alpha^{m_3-m_1}}{9(\alpha^{m_1-m_3}+\alpha^{m_1-m_2}+1)}\right).$$

We computed that $\varepsilon > 10^{-7}$ for all the above instances except when $(d, m_1-m_2, m_1-m_3) = (9,0,1),\ (9,3,3),\ (9,4,1)$ for which we have that $\mu = 0$ for the first two triples and $\mu = 1$ for the last triple. In fact, except for these three triples, the minimum value of ε is $> 7\times 10^{-7}$ and it is achieved in the triple $(d, m_1-m_2, m_1-m_3) = (3,168,2)$. Therefore, except for the above three triples, we have

$$m_1 < 7 + \frac{\log(10^7 q)}{\log\alpha} < 300,$$

which is false since $m_1 > 1000$. However, in the case of the three exceptional triples, the previous argument based solely on the continued fraction of γ shows, as previously, that

$$m_1 < 7 + \frac{\log(108q)}{\log\alpha} < 266,$$

which is impossible again. This finishes the proof. □

2. Comments

A few words about the computations. They were carried out with Mathematica, and the largest loop, which consisted of computing a lower bound for ε over all triples (d, m_2-m_2, m_1-m_3) with components in $[1,9]\times[0,262]\times[0,274]$ with the three exceptions $(9,0,1),\ (9,3,3),\ (9,4,1)$ took a few minutes. It is not unreasonable to conjecture that the same method may be applied to compute all solutions of the equation

$$d\left(\frac{10^n-1}{9}\right) = F_{m_1}+F_{m_2}+F_{m_3}+F_{m_4} \quad \text{with} \quad m_1 \ge m_2 \ge m_3 \ge m_4 \ge 0.$$

We leave this as a problem for other researchers.

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