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The relative upper bound for the third element in a $D(-1)$-quadruple

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Abstract. A set of m distinct positive integers is called a $D(-1)$-m-tuple if the product of any distinct two elements in the set decreased by one is a perfect square. In this paper, we show that if $\{1, b, c, d\}$ with $b < c < d$ is a $D(-1)$-quadruple, then $c < 9.6b^4$.
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1. Introduction

Let n be a nonzero integer. A set $\{a_1, \ldots, a_m\}$ of m distinct positive integers is called *a Diophantine m-tuple with the property* $D(n)$, or simply *a* $D(n)$*-m-tuple*, if $a_i a_j + n$ is a perfect square for each distinct i, j. Our concerns here are in the case of $n = -1$.

There is a conjecture ([3]) that there does not exist a $D(-1)$-quadruple. The biggest step toward this conjecture was taken by Dujella and Fuchs ([7]), who showed that if $\{a, b, c, d\}$ with $a < b < c < d$ is a $D(-1)$-quadruple, then $a = 1$. This immediately implies that there does not exist a $D(-1)$-quintuple. Moreover, it was shown by Dujella, Filipin and Fuchs ([6]) that there exist only finitely many $D(-1)$-quadruples and that if $\{1, b, c, d\}$ with $1 < b < c < d$ is a $D(-1)$-quadruple, then $c < \min\{11b^6, 10^{491}\}$. This bound was very recently improved by Bonciocat, Cipu and Mignotte ([2]) to $c < \min\{2.5b^6, 10^{146}\}$. Note that they also showed that the number of $D(-1)$-quadruples is less than 10^{71}, which improves the upper bound 10^{356} by the authors ([9]).

In this paper, we significantly improve the known upper bounds for c in terms of b.

Theorem 1. *If* $\{1, b, c, d\}$ *with* $b < c < d$ *is a* $D(-1)$*-quadruple, then* $c < 9.6b^4$.

The core of the proof is to improve Rickert's theorem ([14]) in our situation (see Theorem 2). The upper bound "$9.6b^4$" comes from "$\lambda < 2$" with $N = bc$ in Theorem 2, that is necessary in order to make the simultaneous approximation nontrivial. Theorem 1 is expected to take us one step closer to proving the conjecture.

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2. Preliminary results

Let $\{1, b, c\}$ be a $D(-1)$-triple with $b < c$ and let r, s, t be positive integers such that $b - 1 = r^2$, $c - 1 = s^2$, $bc - 1 = t^2$. Suppose that $\{1, b, c, d\}$ is a $D(-1)$-quadruple with $c < d$. Then, there exist positive integers x, y, z such that

$$d - 1 = x^2, \quad bd - 1 = y^2, \quad cd - 1 = z^2.$$

Eliminating d from these equations, we obtain the system of Diophantine equations

$$z^2 - cx^2 = c - 1, \tag{1}$$

$$bz^2 - cy^2 = c - b. \tag{2}$$

By Theorem 1 (i) in [6], we may assume that $c \le b^9$. Then, Lemmas 1 and 5 in [6] imply that the positive solutions (z, x) of (1) and (z, y) of (2) can be respectively expressed as follows:

$$z + x\sqrt{c} = s(s + \sqrt{c})^{2m} \qquad (m \ge 0),$$

$$z\sqrt{b} + y\sqrt{c} = (s\sqrt{b} \pm r\sqrt{c})(t + \sqrt{bc})^{2n} \qquad (n \ge 0).$$

Hence, we may write $z = v_m = w_n$, where

$$v_0 = s,\ v_1 = (2c - 1)s,\ v_{m+2} = (4c - 2)v_{m+1} - v_m,$$

$$w_0 = s,\ w_1 = (2bc - 1)s \pm 2rtc,\ w_{m+2} = (4bc - 2)w_{n+1} - w_n.$$

We conclude this section by quoting three lemmas from [6].

Lemma 1 (Lemma 2, [6]). *If $v_m = w_n$ with $n \ne 0$, then*

(i) $m \equiv n \pmod 2$;

(ii) $n \le m \le 2n$;

(iii) $(m^2 - bn^2)s \equiv \pm nrt \pmod{4c}$.

Lemma 2 (Lemma 3, [6]). *We have $w_n > (c - b)(4bc - 3)^{n-1}$ for $n \ge 1$.*

Lemma 3 (Lemma 6, [6]). *We have $v_1 \ne w_1$, $v_2 \ne w_2$ and $v_4 \ne w_2$.*

3. The proof of Theorem 1

Lemma 4. *Assume that $c \ge 9.6b^4$. If $v_m = w_n$ with $n \ge 3$, then $n > c^{1/8}$.*

Proof. Suppose that $n \le c^{1/8}$. Since

$$bn^2 s < c^{1/4}c^{1/4}c^{1/2} = c,$$

$$|m^2 s \mp nrt| < 4c^{1/4}c^{1/2} + c^{1/8}c^{1/8}c^{5/8} < 2c^{7/8} < c,$$

we have an equality in Lemma 1 (iii):

$$m^2 s = bn^2 s \pm nrt.$$

Now we have $m^2 s < m^2\sqrt{c}$ and

$$bn^2 s - nrt > rn(\sqrt{bc}n - \sqrt{bc-1}) > \sqrt{bc}rn(n-1),$$

which together imply $m^2 > \sqrt{b}rn(n-1)$. Hence, we see that

$$8 > \frac{4n^2}{n(n-1)} > \frac{m^2}{n(n-1)} > \sqrt{b}r,$$

which contradicts $b > 8$. □

Remark 1. It is known that for $b < 101$ the $D(-1)$-pair $\{1, b\}$ cannot be extended to a $D(-1)$-quadruple (see [4, 8, 12, 15]). Recently, it has been checked that the same is true for $b < 10^{10}$ (see the sentence before the last one of Section 5 in [11]). Note that this result is now extended to $b < 1.024 \cdot 10^{13}$ in [2].

Theorem 2. *Let b and N be integers with $b \geq 10$ and $N \geq 9.5b^2(b-1)^3$. Assume that N is divisible by b. Then the numbers $\theta_1 = \sqrt{1 - b/N}$ and $\theta_2 = \sqrt{1 - 1/N}$ satisfy*

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > (32.01b(b-1)N)^{-1}q^{-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda = 1 + \frac{\log(16.01b(b-1)N)}{\log(1.687b^{-1}(b-1)^{-2}N^2)} < 2.$$

Proof. Note that the assumption $N \geq 9.5b^2(b-1)^3$ immediately implies $\lambda < 2$. It suffices to find real numbers satisfying the conditions in the following lemma.

Lemma 5 (Lemma 22, [13]; Lemma 3.1, [14]; Lemma 2.1 [1])**.** *Let $\theta_1, \ldots, \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer k, we can find rational numbers p_{ijk} $(0 \leq i, j \leq m)$ with nonzero determinant such that $f^{-1}D^k p_{ijk}$ $(0 \leq i, j \leq m)$ are integers and*

$$|p_{ijk}| \leq pP^k \ \ (0 \leq i, j \leq m), \qquad \left|\sum_{j=0}^{m} p_{ijk}\theta_j\right| \leq lL^{-k} \ \ (0 \leq i \leq m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-\lambda}$$

holds for all integers $p_1, \ldots, p_m, q$ with $q > 0$, where

$$\lambda = 1 + \frac{\log(DP)}{\log(L/D)} \quad \textit{and} \quad c^{-1} = 2mf^{-1}pDP\left(\max\{1, 2f^{-1}l\}\right)^{\lambda}.$$

In our situation, we take $m = 2$, $a_0 = -b$, $a_1 = -1$, $a_2 = 0$ and θ_1, θ_2 as in Theorem 2. A contour integral has the form $\sum_{j=0}^{2} p_{ijk}\theta_j$, and estimating the integral and the relevant integrals we obtain the following (see the arguments following Lemma 3.1 in [1]).

$$\left|\sum_{j=0}^{2} p_{ijk}\theta_j\right| < \frac{27}{64}\left(1-\frac{b}{N}\right)^{-1}\left\{\frac{27}{4}\left(1-\frac{b}{N}\right)^{2} N^3\right\}^{-k} \tag{3}$$

and

$$|p_{ijk}|\theta_j \le \max_{z\in\Gamma_j}\left|\frac{(1+z/N)^{k+1/2}}{|A(z)|^k}\right| \quad (0 \le j \le 2), \tag{4}$$

where $A(z) = \prod_{i=0}^{2}(z-a_i)$ and the contours Γ_j $(0 \le j \le 2)$ are defined by

$$|z-a_j| = \min_{i\ne j}\left\{\frac{|a_j-a_i|}{2}\right\}.$$

Inequality (3) shows that we may take

$$l = \frac{27}{64}\left(1-\frac{b}{N}\right)^{-1}, \; L = \frac{27}{4}\left(1-\frac{b}{N}\right)^{2} N^3.$$

Since

$$\left|1+\frac{z}{N}\right| \le \begin{cases} 1+\dfrac{3b-1}{2N} & \text{on } \Gamma_0, \\ 1-\dfrac{1}{2N} & \text{on } \Gamma_1, \\ 1+\dfrac{1}{2N} & \text{on } \Gamma_2, \end{cases}$$

and $\min_{z\in\Gamma_j}|A(z)| \ge (2b-1)/8$ for all j, we see from (4) that

$$\begin{aligned} |p_{ijk}| &\le \frac{\max_{z\in\Gamma_j}|1+z/N|^{k+1/2}}{\theta_j\cdot\min_{z\in\Gamma_j}|A(z)|^k} \\ &\le \left(1+\frac{3b+1}{2(N-1)}\right)^{1/2}\left(\frac{8\left(1+\frac{3b-1}{2N}\right)}{2b-1}\right)^{k}. \end{aligned}$$

Therefore, we may take

$$p = \left(1+\frac{3b+1}{2(N-1)}\right)^{1/2}, \; P = \frac{8\left(1+\frac{3b-1}{2N}\right)}{2b-1}.$$

It remains to consider how to take D and f. The way of choosing them is similar to the one of the proof of Theorem 2.5 in [10]. By the proof of Lemma 3.3 in [14], we may express $p_{ijk} = p_{ij}(1/N)$ as

$$p_{ijk} = \sum_{ij}\binom{k+\frac{1}{2}}{h_j} C_{ij}^{-1}\prod_{l\ne j}\binom{-k_{il}}{h_l},$$

where

$$C_{ij}=\frac{N^k}{(N+a_j)^{k-h_j}}\prod_{l\neq j}(a_j-a_l)^{k_{il}+h_l},$$

$k_{il}=k+\delta_{il}$ with δ_{il} the Kronecker delta, $\sum_{ij}$ denotes the sum over all non-negative integers h_0, h_1, h_2 satisfying $h_0+h_1+h_2=k_{ij}-1$, and $\prod_{l\neq j}$ denotes the product from $l=0$ to $l=2$ omitting $l=j$ (which is expression (3.7) in [14] with $\nu=1/2$). Let $N=bN_0$ for some integer N_0. If $j=0$, then

$$|C_{i0}|=\frac{b^{h_0}N_0^k(b-1)^{k_{i1}+h_1}b^{k_{i2}+h_2}}{(N_0-1)^{k-h_0}}=\frac{b^{k_{i2}+h_0+h_2-k}(b-1)^{k_{i1}+h_1}N^k}{(N_0-1)^{k-h_0}}.$$

Since $k_{il}+h_j+h_l-k\le k_{il}+k_{ij}-1-k\le k$ and $k_{il}+h_l\le k_{il}+k_{ij}-1\le 2k$, we have $b^k(b-1)^{2k}N^kC_{i0}^{-1}\in\mathbb{Z}$ for all i. If $j=1$, then $|C_{i1}|=N^k(b-1)^{k_{i0}+h_0}/(N-1)^{k-h_1}$ and we have $(b-1)^{2k}N^kC_{i1}^{-1}\in\mathbb{Z}$ for all i. If $j=2$, then $|C_{i2}|=N^{h_2}b^{k_{i0}+h_0}=b^{k_{i0}+h_0+h_2-k}N^k/N_0^{k-h_2}$ and we have $b^kN^kC_{i2}^{-1}\in\mathbb{Z}$ for all i. It follows that $b^k(b-1)^{2k}N^kC_{ij}^{-1}\in\mathbb{Z}$ for all i, j. Since

$$2^{h_j+h'_j}\binom{k+\frac{1}{2}}{h_j}\in\mathbb{Z}$$

for all j (see the proof of Lemma 4.3 in [14]), we obtain $2^{-1}\{4b(b-1)^2N\}^kp_{ijk}\in\mathbb{Z}$ for all i, j, which means that we may take $f=2$ and $D=4b(b-1)^2N$.

To sum up, we see from the assumptions that

$$DP<16.01b(b-1)N,\qquad \frac{L}{D}>\frac{1.687N^2}{b(b-1)^2},\qquad c^{-1}<32.01a'bN,$$

which together with Lemma 5 completes the proof of Theorem 2. □

Lemma 6. *All positive integer solutions x,y,z of* (1) *and* (2) *satisfy*

$$\max\left\{\left|\theta_1-\frac{sbx}{bz}\right|,\left|\theta_2-\frac{ty}{bz}\right|\right\}<cz^{-2},$$

where $\theta_1=\sqrt{1-1/c}$ and $\theta_2=\sqrt{1-1/(bc)}$.

Proof. This is a special case of Lemma 1 in [5]. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Suppose that $c\ge 9.6b^4$. Since $bc>9.5b^2(b-1)^3$, we can apply Theorem 2 and Lemma 6 with $N=bc$, $p_1=sbx$, $p_2=ty$ and $q=bz$, and we have

$$(32.01b(b-1)bc)^{-1}(bz)^{-\lambda}<cz^{-2},$$

which together with $\lambda<2$ yields

$$z^{2-\lambda}<32.01b^4(b-1)c^2.$$

Now we have

$$\frac{1}{2-\lambda} < \frac{\log\left(\frac{1.687bc^2}{(b-1)^2}\right)}{\log\left(\frac{0.1053c}{b(b-1)^3}\right)} < \frac{2\log(1.299b^{-1/2}c)}{\log(0.1053b^{-4}c)},$$

where we used the inequality $b > 10^{10}$ (see Remark 1). Hence,

$$\log z < \frac{4\log(5.658b^{5/2}c)\log(1.299b^{-1/2}c)}{\log(0.1053b^{-4}c)}.$$

Moreover, by Lemma 2 we have

$$\log z > (n-1)\log(4bc-3) > (n-1)\log(3.999bc).$$

It follows from Lemma 4 that

$$c^{1/8} - 1 < \frac{4\log(5.658b^{5/2}c)\log(1.299b^{-1/2}c)}{\log(3.999bc)\log(0.1053b^{-4}c)}.$$

Since $c \geq 9.6b^4$, we obtain

$$9.6^{1/8}b^{1/2} - 1 < \frac{4\log(54.32b^{13/2})\log(12.48b^{7/2})}{\log(38.39b^5)\log(1.01)}.$$

Putting $g(b) = 1830\log(1.85b)\log(2.06b) - (1.32b^{1/2} - 1)\log(2.07b)$, we must have $g(b) > 0$. However, $g(b)$ is decreasing for $b \geq 10^8$ and $g(10^{10}) < 0$, which contradict $b > 10^{10}$. This completes the proof of Theorem 1. □

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