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A modified projective algorithm of common elements for equilibrium problems and fixed point problems in Banach spaces

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Abstract. In this paper, we consider a modified projective algorithm for finding common elements of the set of common fixed points of a finite family of quasi-ϕ-nonexpansive mappings and the set of solutions of an equilibrium problem in uniformly smooth and strictly convex Banach spaces with the property(K). Our results improve and extend the corresponding results announced by many others.

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1. Introduction

Throughout this paper, we assume that E is a real Banach space, E^* the dual space of E. Let C be a nonempty closed convex subset of E and f a bifunction from $C \times C$ to R, where R denotes the set of numbers. The equilibrium problem is to find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(f)$. Given a mapping $T : C \to E^*$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $p \in EP(f)$ if and only if $\langle Tp, y - p\rangle \geq 0$ for all $y \in C$, i.e., p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1). Some methods have been proposed to solve the equilibrium problem. See, for instance, [3, 8, 12].

Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

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Recently, many authors studied the problems of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and uniformly smooth and uniformly convex Banach spaces, respectively. See, for instance, [16, 18, 19] and the references therein.

Motivated and inspired by the research going on in this direction, we introduce a modified projective algorithm for finding common elements of the set of common fixed points of finite quasi-ϕ-nonexpansive mappings and the set of solutions of an equilibrium problem in uniformly smooth and strictly convex Banach spaces with the property(K).

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ the generalized duality pairing between E and E^*. It is well-known that E is uniformly smooth if and only if E^* is uniformly convex.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2}$$

Observe that in a Hilbert space H (2) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x). \tag{3}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 7, 17]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{4}$$

Remark 1. *If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$*

then $x = y$. From (4), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [7, 17] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [15] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive [4, 5, 11] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [4, 5, 11].
T is said to be ϕ-nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$.

Remark 2. *The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [4, 5, 13, 14] which requires the strong restriction: $F(T) = \widetilde{F(T)}$.*

Next, we give some examples that are closed quasi-ϕ-nonexpansive which are due to Zhou [20].

Example 1. *Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.*

Example 2. *Let Π_C be the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed and quasi-ϕ-nonexpansive mapping from E onto C with $F(\Pi_C) = C$.*

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well-known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Recall that a Banach space E has the Kadeč-Klee property (property(K) for brevity) if for any sequence $\{x_n\} \subset E$ and $x \in E$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning the property(K) the reader is referred to [9] and references cited there in. It is well-known that if E is a uniformly convex Banach space, then E has the property(K); Banach space E is uniformly smooth if and only if E^* is uniformly convex.

In order to establish our main results, we need the following lemmas.

Lemma 1 (See [2])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 2 (See [2])**.** *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 3. *Let E be a reflexive, strictly convex and smooth Banach space, C a closed convex subset of E, and T a quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we prove that $p \in F(T)$. From the definition of quasi$-\phi-$nonexpansive mappings, one has $\phi(p_n, Tp) \leq \phi(p_n, p)$, which implies that $\phi(p_n, Tp) \to 0$ as $n \to \infty$. Noticing that

$$\phi(p_n, Tp) = \|p_n\|^2 - 2\langle p_n, J(Tp)\rangle + \|Tp\|^2.$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \phi(p_n, Tp) = \|p\|^2 - 2\langle p, J(Tp)\rangle + \|Tp\|^2 = \phi(p, Tp).$$

Hence $\phi(p, Tp) = 0$. It implies that $p = Tp$. We next show that $F(T)$ is convex. To this end, for arbitrary $p_1, p_2 \in F(T), t \in (0, 1)$, putting $p_3 = tp_1 + (1 - t)p_2$, we prove that $Tp_3 = p_3$. Indeed, by using the definition of $\phi(x, y)$, we have

$$\begin{aligned}
\phi(p_3, Tp_3) &= \|p_3\|^2 - 2\langle p_3, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2\langle tp_1 + (1 - t)p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2t\langle p_1, J(Tp_3)\rangle - 2(1 - t)\langle p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 + t\phi(p_1, Tp_3) + (1 - t)\phi(p_2, Tp_3) - t\|p_1\|^2 - (1 - t)\|p_2\|^2 \\
&\leq \|p_3\|^2 + t\phi(p_1, p_3) + (1 - t)\phi(p_2, p_3) - t\|p_1\|^2 - (1 - t)\|p_2\|^2 \\
&= \|p_3\|^2 - 2\langle p_3, Jp_3\rangle + \|p_3\|^2 = 0.
\end{aligned}$$

This implies that $Tp_3 = p_3$. □

Lemma 4 (See [6])**.** *Let E be a uniformly convex Banach space, r a fixed positive real number and $B_r(0)$ a dosed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Remark 3. *Let E, $B_r(0)$ and $g : [0, \infty) \to [0, \infty)$ be the same as in Lemma 4. By a simple induction we have the following more general inequality:*

$$\|\sum_{i=1}^{n} \lambda_i x_i\|^2 \leq \sum_{i=1}^{n} \lambda_i\|x_i\|^2 - \lambda_i\lambda_j g(\|x_i - x_j\|)$$

for all $\lambda_i \in [0, 1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and all $x_i \in B_r(0) (i = 1, 2, \cdots, n, \forall n \geq 1)$.

In order to solve the equilibrium problem for a bifunction from $C \times C$ to R, we assume that f satisfies the following conditions:

(A_1) $f(x,x) = 0$ for all $x \in C$;

(A_2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A_3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x,y)$;

(A_4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semi-continuous.

Lemma 5 (See [3]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space. Let f be a bifunction from $C \times C$ to R satisfying $(A_1)-(A_4)$. Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 6 (See [3]). *Let C be a closed convex subset of a uniformly smooth, strictly convex Banach space E. Let f be a bifunction from $C \times C$ to R satisfying $(A_1)-(A_4)$. For $r > 0$ and $x \in E$. Define the mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(1) T_r is single-valued;

(2) T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,

$$\langle T_r(x) - T_r(y), JT_r(x) - JT_r(y)\rangle \leq \langle T_r(x) - T_r(y), Jx - Jy\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

Proof. (1)-(3) are due to Takahashi and Zembayashi [19]. We just show (4). From the proof of Lemma 2.8 of [25], one sees that T_r is a quasi-ϕ-nonexpansive mapping. It follows from Lemma 3 that $F(T_r)$ is closed and convex. This implies that $EP(f)$ is closed and convex. □

Lemma 7 (See [19]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to R satisfying $(A_1) - (A_4)$, and let $r > 0$. Then for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r(x)) + \phi(T_r(x), x) \leq \phi(q, x).$$

3. Main results

Theorem 1. *Let E be a uniformly smooth and strictly convex Banach space with the property(K), and C a nonempty closed convex subset of E. Let N be a fixed positive integer, $\{T_i\}_{i=1}^N : C \to C$ a finite family of closed quasi-ϕ-nonexpansive mappings and f a bifunction from $C \times C$ to R satisfying $(A_1) - (A_4)$ such that $F := \cap_{i=1}^N F(T_i) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \sum_{i=1}^N \alpha_{n,i} J T_i x_n), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{5}$$

where J is the duality mapping on E, $\{\alpha_{n,i}\}$ are $N+1$ sequences in [0,1] such that

(a) $\sum_{i=0}^N \alpha_{n,i} = 1$;

(b) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. First of all, we show that C_n is closed and convex for every $n \geq 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. For $z \in C_{k+1}$, one obtains that

$$\phi(z, u_k) \leq \phi(z, x_k)$$

is equivalent to

$$2\langle z, Jx_k - Ju_k \rangle \leq \|x_k\|^2 - \|u_k\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \geq 0$, C_n are closed and convex.

Noting that $u_n = T_{r_n} y_n$ for all $n \geq 0$. From Lemma 6, one has T_{r_n} is a quasi-ϕ-nonexpansive mapping. Next, we prove $F \subset C_n$ for all $n \geq 1$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some k. Then, for $\forall w \in F \subset C_k$, noting the fact that

$\|\cdot\|^2$ is convex, one has

$$\begin{aligned}\phi(w,u_k) &= \phi(w,T_{r_k}y_k)\\ &\le \phi(w,y_k)\\ &= \phi(w,J^{-1}(\alpha_k Jx_k+\beta_k JTx_k+\gamma_k JSx_k)\\ &= \|w\|^2-2\langle w,\alpha_{k,0}Jx_k+\sum_{i=1}^{N}\alpha_{k,i}JT_ix_k\rangle\\ &\quad +\|\alpha_{k,0}Jx_k+\sum_{i=1}^{N}\alpha_{k,i}JT_ix_k\|^2\\ &\le \|w\|^2-2\alpha_{k,0}\langle w,Jx_k\rangle-2\sum_{i=1}^{N}\alpha_{k,i}\langle w,JT_ix_k\rangle\rangle \qquad (6)\\ &\quad +\alpha_{k,0}\|x_k\|^2+\sum_{i=1}^{N}\alpha_{k,i}\|T_ix_k\|^2\\ &= \alpha_{k,0}\phi(w,x_k)+\sum_{i=1}^{N}\alpha_{k,i}\phi(w,T_ix_k)\\ &\le \alpha_{k,0}\phi(w,x_k)+\sum_{i=1}^{N}\alpha_{k,i}\phi(w,x_k)\\ &= \phi(w,x_k),\end{aligned}$$

which shows that $w\in C_{k+1}$. This implies that $F\subset C_n$ for all $n\ge 1$. By the assumption that F is nonempty, we have that C_n are nonempty closed and convex subsets of E, which in turn shows that $\Pi_{C_{n+1}}x_0$ is well defined.

Now we shall show that $\{x_n\}$ is bounded. From $x_n=\Pi_{C_n}x_0$, one sees

$$\langle x_n-u,Jx_0-Jx_n\rangle\ge 0,\quad \forall u\in C_n. \qquad (7)$$

Since $F\subset C_n$ for all $n\ge 1$, we arrive at

$$\langle x_n-w,Jx_0-Jx_n\rangle\ge 0,\quad \forall w\in F. \qquad (8)$$

From Lemma 2, one has

$$\phi(x_n,x_0)=\phi(\Pi_{C_n}x_0,x_0)\le\phi(w,x_0)-\phi(w,x_n)\le\phi(w,x_0),$$

for each $w\in F\subset C_n$. Therefore, the sequence $\{\phi(x_n,x_0)\}$ is bounded. On the other hand, noticing that $x_n=\Pi_{C_n}x_0$ and $x_{n+1}=\Pi_{C_{n+1}}x_0\in C_{n+1}\subset C_n$, one has

$$\phi(x_n,x_0)\le\phi(x_{n+1},x_0)$$

for all $n\ge 1$. Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n,x_0)\}$ exists.

By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It follows that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_0) \\ &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0) \end{aligned} \tag{9}$$

Letting $m, n \to \infty$ in (9), one has $\phi(x_m, x_n) \to 0$. Hence $|\|x_m\| - \|x_n\|| \to 0$. This implies that $\{x_n\}$ is bounded.

At this point, we are in a position to prove that $x_n \to p$ as $n \to \infty$. Since X is reflexive, without loss of generality, we can assume that $x_n \to p$ weakly as $n \to \infty$. Since $C_j \subset C_n$ for $j \geq n$, we have $x_j \in C_n$ for $j \geq n$. Since C_n is closed and convex, one has $p \in C_n$ for all $n \geq 1$. Hence $p \in \bigcap_{n=1}^{\infty} C_n = D$. Noticing that $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \phi(p, x_0)$, we have

$$\phi(p, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(p, x_0),$$

which implies that $\phi(x_n, x_0) \to \phi(p, x_0)$ as $n \to \infty$. Hence $\|x_n\| \to \|p\|$. By the property (K) of X, we have $x_n \to p$ as $n \to \infty$.

Next, we show $p \in \cap_{i=1}^{N} F(T_i)$. By taking $m = n + 1$ in (9), one arrives at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{10}$$

Since $x_n \to p$, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{11}$$

Noting that $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{12}$$

It follows from (10) that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{13}$$

Noting that $0 \leq (\|x_{n+1}\| - \|u_n\|)^2 \leq \phi(x_{n+1}, u_n)$. Hence $\|u_n\| \to \|p\|$ and consequently $\|Ju_n\| \to \|Jp\|$. This implies that $\{J(u_n)\}$ is bounded. Since E is reflexive, E^* is also reflexive. So we can assume that

$$J(u_n) \to f_0 \in X^*$$

weakly. On the other hand, in view of the reflexivity of E, one has $J(E) = E^*$, which means that for $f_0 \in E^*$, there exists $x \in E$, such that $Jx = f_0$. By using (13), $x_n \to p$ and the weak lower semi-continuity of the norm $\|\cdot\|$, we have that

$$\begin{aligned} 0 = \lim_{n\to\infty} \phi(x_{n+1}, u_n) &= \liminf_{n\to\infty} [\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2] \\ &= \liminf_{n\to\infty} [\|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2] \\ &\geq \|p\|^2 - 2\langle p, f_0\rangle + \|f_0\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \phi(p, x) \geq 0, \end{aligned}$$

which gives that $\phi(p,x)=0$ and hence $p=x$, which implies that $f_0=Jp$. Consequently,

$$Ju_n \to Jp \in E^*$$

weakly. Since $\|Ju_n\| \to \|Jp\|$ and E^* has the property(K), we have

$$\|Ju_n - Jp\| \to 0.$$

Noting that $J^{-1}: E^* \to E$ is demi-continuous, we have $u_n \to p$ weakly. Since $\|u_n\| \to \|p\|$ and E has the property(K), we obtain that

$$u_n \to p \quad as \quad n \to \infty. \tag{14}$$

Hence

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{15}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{16}$$

Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r=\sup_{n\geq 0}\{\|x_n\|, \max\{\|T_ix_n\| : i=1,2,\cdots,N\}\}$. By using Remark 3, we have

$$\begin{aligned}
\phi(w,u_n) &= \phi(w, T_{r_n}y_n) \\
&\leq \phi(w,y_n) \\
&= \phi(w, J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i}JT_ix_n) \\
&= \|w\|^2 - 2\langle w, \alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i}JT_ix_n + \rangle \\
&\quad + \|\alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i}JT_ix_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_{n,0}\langle w, Jx_n\rangle - 2\sum_{i=1}^{N} \alpha_{n,i}\langle w, JT_ix_n\rangle \qquad (17) \\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^{N} \alpha_{n,i}\|T_ix_n\|^2 - \alpha_{n,0}\alpha_{n,i}g(\|Jx_n - JT_ix_n\|) \\
&= \alpha_{n,0}\phi(w,x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(w,T_ix_n) - \alpha_{n,0}\alpha_{n,i}g(\|Jx_n - JT_ix_n\|) \\
&\leq \alpha_n\phi(w,x_n) + \beta_n\phi(w,x_n) + \gamma_n\phi(w,x_n) - \alpha_{n,0}\alpha_{n,i}g(\|Jx_n - JT_ix_n\|) \\
&= \phi(w,x_n) - \alpha_{n,0}\alpha_{n,i}g(\|Jx_n - JT_ix_n\|).
\end{aligned}$$

It follows that

$$\alpha_{n,0}\alpha_{n,i}g(\|Jx_n - JT_ix_n\|) \le \phi(w, x_n) - \phi(w, u_n). \tag{18}$$

On the other hand, one has

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle \\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|w\|\|Jx_n - Ju_n\|.\end{aligned}$$

It follows from (15) and (16) that

$$\phi(w, x_n) - \phi(w, u_n) \to 0 \quad as \quad n \to \infty. \tag{19}$$

In view of the assumption (b)$\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$, (3.13) and (3.14), we have

$$g(\|Jx_n - JT_ix_n\|) \to 0 \quad as \quad n \to \infty.$$

It follows from the property of g that

$$\|Jx_n - JT_ix_n\| \to 0 \quad as \quad n \to \infty. \tag{20}$$

Since $\|x_n - p\| \to 0 \quad as \quad n \to \infty$, noting that $J : E \to E^*$ is demi-continuous, we have

$$Jx_n \to Jp \in X^*$$

weakly. Noting that

$$|\|Jx_n\| - \|Jp\|| = |\|x_n\| - \|p\|| \le \|x_n - p\| \to 0,$$

which implies that $\|Jx_n\| \to \|Jp\|$. By using the property(K) of X^*, we have

$$\|Jx_n - Jp\| \to 0 \quad as \quad n \to \infty.$$

In view of (20), one has

$$\|JT_ix_n - Jp\| \to 0 \quad as \quad n \to \infty.$$

Noting that $J^{-1} : E^* \to E$ is demi-continuous, we have

$$T_ix_n \to p$$

weakly as $n \to \infty$ for all $i = 1, 2, \cdots, N$. Noting that

$$|\|T_ix_n\| - \|p\|| = |\|JT_ix_n\| - \|Jp\|| \le \|JT_ix_n - Jp\| \to 0,$$

which implies that $\|T_ix_n\| \to \|p\|$. By using the property(K) of E, we have

$$\|T_ix_n - p\| \to 0 \quad as \quad n \to \infty.$$

It follows from $x_n \to p$ and the closedness of T_i that $T_ip = p$, which means that $p \in \cap_{i=1}^N F(T_i)$.

Then we prove $p \in EF(f)$. From (6), we arrive at

$$\phi(w, y_n) \leq \phi(w, x_n). \tag{21}$$

From $u_n = T_{r_n} y_n$ and Lemma 7, one has

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n} y_n, y_n) \\ &\leq \phi(w, y_n) - \phi(w, T_{r_n} y_n) \\ &\leq \phi(w, x_n) - \phi(w, T_{r_n} y_n) \\ &= \phi(w, x_n) - \phi(w, u_n) \end{aligned} \tag{22}$$

It follows from (19) that

$$\phi(u_n, y_n) \to 0 \quad as \quad n \to \infty. \tag{23}$$

Noting that $0 \leq (\|u_n\| - \|y_n\|)^2 \leq \phi(u_n, y_n)$. It follows from (14) that $\|y_n\| \to \|p\|$ and consequently $\|Jy_n\| \to \|Jp\|$. This implies that $\{J(y_n)\}$ is bounded. Since E is reflexive, E^* is also reflexive. So we can assume that

$$J(y_n) \to h_0 \in X^*$$

weakly. On the other hand, in view of the reflexivity of E, one has $J(E) = E^*$, which means that for $h_0 \in E^*$, there exists $x \in E$, such that $Jx = h_0$. By using (23), $u_n \to p$ and the weak lower semi-continuity of the norm $\|\cdot\|$, we have that

$$\begin{aligned} 0 = \lim_{n\to\infty} \phi(u_n, y_n) &= \liminf_{n\to\infty} [\|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|y_n\|^2] \\ &= \liminf_{n\to\infty} [\|u_n\|^2 - 2\langle u_n, Jy_n\rangle + \|Jy_n\|^2] \\ &\geq \|p\|^2 - 2\langle p, h_0\rangle + \|h_0\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \phi(p, x) \geq 0, \end{aligned}$$

which gives that $\phi(p, x) = 0$ and hence $p = x$, which implies that $h_0 = Jp$. Consequently,

$$Jy_n \to Jp \in E^*$$

weakly. Since $\|Jy_n\| \to \|Jp\|$ and E^* has the property(K), we have

$$\|Jy_n - Jp\| \to 0.$$

Noting that $J^{-1} : E^* \to E$ is demi-continuous, we have $y_n \to p$ weakly. Since $\|y_n\| \to \|p\|$ and E has the property(K), we obtain that

$$y_n \to p \quad as \quad n \to \infty. \tag{24}$$

From (14), one can obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{25}$$

Since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{26}$$

In view of the assumption (c)$r_n \geq a$, one sees

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{27}$$

Noting that $u_n = T_{r_n} y_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall \quad y \in C.$$

From(A_2), one arrives at

$$\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in above inequality and from (A_2) and (3.9), one has

$$f(y, p) \leq 0, \quad \forall y \in C.$$

From $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noting that $y, p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A_1) and (A_4) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0.$$

Let $t \downarrow 0$, from (A_3), we obtain $f(p, y) \geq 0$, $\forall y \in C$. This implies that $p \in EP(f)$. This shows that $p \in F$.

Finally, we shows that $p = \Pi_F x_0$.

By the assumption on F, $F \neq \emptyset$. In view of Lemma 3 and Lemma 6, we know that F is a nonempty closed and convex subset of E, and hence $\Pi_F x_0$ is well defined for every $x_0 \in E$. By taking the limit in (8), one has

$$\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in F.$$

At this point, in view of Lemma 1, one sees that $p = \Pi_F x_0$. This completes the proof. □

If $T_i = S$ in Theorem 1, then we immediately obtain the following result.

Corollary 1. *Let E be a uniformly smooth and strictly convex Banach space with the property(K), C a nonempty closed convex subset of E. Let $S : C \to C$ be a closed quasi-ϕ-nonexpansive mapping and f a bifunction from $C \times C$ to R satisfying*

$(A_1) - (A_4)$ such that $F := F(S) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) JSx_n), \\ u_n \in C \quad \textit{such} \quad \textit{that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where J is the duality mapping on E, $\{\alpha_n\}$ is sequence in [0,1] such that $\liminf_{n\to\infty} \alpha_n (1-\alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Remark 4. *Corollary 1 improves and extends Theorem 3.1 of Takahashi and Zembayashi [19] in the following senses:*

(i) from uniformly convex and uniformly smooth Banach spaces extend to uniformly smooth and strictly convex Banach spaces with the property(K).

(ii) from relatively nonexpansive mappings extend to more general quasi$-\phi-$nonexpansive mappings, that is, the very strict restriction that $\widetilde{F(T)} = F(T)$ has been removed.

(iii) our algorithm is simpler than the one given by Takahashi and Zembayashi [19].

Remark 5. *We do not know whether the uniform smoothness of Banach space E can be weakened to smoothness in Theorem 1.*

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