MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 49–62

Iterative approximation of common fixed points for two quasi-ϕ-nonexpansive mappings in Banach spaces

Haiyun Zhou[1,*] and Xinghui Gao[2]

[1] *Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050 003, P. R. China*
[2] *College of Mathematics and Computer Science, Yanan University, Yanan 716 000, P. R. China*

Received April 29, 2009; accepted March 25, 2010

Abstract. In this paper, we introduce a new type of a projective algorithm for a pair of quasi-ϕ-nonexpansive mappings. We establish strong convergence theorems of common fixed points in uniformly smooth and strictly convex Banach spaces with the property(K). Our results improve and extend the corresponding results announced by many others.

AMS subject classifications: 47H09, 47H10

Key words: strong convergence, modified projection algorithm, quasi-ϕ-nonexpansive mapping, property(K)

1. Introduction

Throughout this paper, we assume that E is a Banach space, C is a nonempty closed convex subset of E and $T : C \to C$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping, see [3]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad x \in C, \tag{2}$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [3] proved the following well-known strong convergence theorem.

*Corresponding author. *Email addresses:* `witman66@yahoo.com.cn` (H. Y. Zhou), `yadxgaoxinghui@163.com` (X. Gao)

Theorem 1. *Let K be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on K. Fix $u \in K$ and define $z_t \in K$ as $z_t = tu+(1-t)Tz_t$ for $t \in (0,1)$, $\{z_t\}$ converges strongly to an element of $F(T)$ nearest to u.*

As motivated by Theorem 1, Halpern [6] considered the following explicit iteration:

$$x_0 \in K, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{3}$$

and proved the following theorem.

Theorem 2. *Let K be a bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on K. Define a real sequence $\{\alpha_n\}$ in $[0,1]$ by $\alpha_n = n^{-\theta}$, $0 < \theta < 1$. Define a sequence $\{x_n\}$ by (3). Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u.*

In 1977, Lions [10] improved the result of Halpern [8], still in Hilbert spaces, by proving the strong convergence of $\{x_n\}$ to a fixed point of T when the real sequence $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}^2} = 0$.

It was observed that both Halpern's and Lion's conditions on the real sequence $\{\alpha_n\}$ excluded the canonical choice $\alpha_n = \frac{1}{n+1}$. This was overcome in 1992 by Wittmann [16], who proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

In 2002, Xu [17] improved the result of Lion twofold. First, he weakened condition (C3) by removing the square in the denominator so that the canonical choice of $\{\alpha_n\} = \frac{1}{n}$ is possible. Secondly, he proved the strong convergence of the scheme (3) in the framework of real uniformly smooth Banach spaces. Xu also remarked ([18], Remark 3.2) that Halpern [8] observed that conditions (C1) and (C2) are necessary for the strong convergence of algorithm (3) for all nonexpansive mappings. It is well known that (3) is widely believed to have slow convergence because of the restriction of condition $(C2)$. Thus to improve the rate of convergence of the iterative process (3), one cannot rely only on the process itself. Martinez-Yanes and Xu [11] proposed the following modification of the Halpern iteration for a single nonexpansive mapping T in a Hilbert space. To be more precise, they proved the following theorem.

Theorem 3. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0,1)$ is such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases} \tag{4}$$

converges strongly to $P_{F(T)}x_0$.

Recently, Qin and Su [14] extended the main result of Martinez-Yanes and Xu [12] from Hilbert spaces to Banach spaces. To be more precise, they proved the following theorem.

Theorem 4. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in (0, 1) such that* $\lim_{n\to\infty} \alpha_n = 0$. *Define a sequence $\{x_n\}$ in C by the following algorithm*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{v \in C : \phi(v, y_n) \le \alpha_n \phi(v, x_0) + (1-\alpha_n)\phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{5}$$

where J is the single-valued duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges to $\Pi_{F(T)}x_0$.

Very recently, Plubtieng and Ungchittrakool [13], still in the framework of uniformly smooth and uniformly convex Banach spaces, introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x\|^2 + 2\langle z, Jx_n - Jx\rangle), \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \end{cases} \tag{6}$$

where T, S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved the sequence $\{x_n\}$ generated by (6) converges strongly to a common fixed point of T and S.

In this paper, motivated and inspired by the above works, we introduce a new type of a modified projective algorithm for a pair of quasi-ϕ-nonexpansive mappings which more general than relatively nonexpansive mappings to have a strong convergence theorem in uniformly smooth and strictly convex Banach spaces with the property(K). The main novelty of our results lies in the facts that we extend not only the spaces but also the mappings to a more general setting; in part, we extend Martinez-Yanes and Xu [11], Plubtieng and Ungchittrakool [13], Qin and Su [14] and some others.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let J be the normalized duality mapping from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\} \tag{7}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that E is uniformly smooth if and only if E^* is uniformly convex.

It is well known that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{8}$$

Observe that, in a Hilbert space H, (8) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x), \tag{9}$$

the existence and uniqueness of the operator Π_C follows from properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [1, 2, 7, 15]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{10}$$

Remark 1. *If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (10), we have $\|x\| = \|y\|$. This implies $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [7, 15] for more details.*

Let C be a closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [15] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive [4, 5, 7, 12] if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [4, 5].

T is said to be ϕ-nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$.

Remark 2. *The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings [4, 5, 12, 13, 14] which requires a strong restriction: $F(T) = \widetilde{F(T)}$.*

Next, we give some examples which are closed quasi-ϕ-nonexpansive due to Zhou et al. [18].

Example 1. *Let E be a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone mapping such that its zero set $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1}J$ is a closed quasi-ϕ-nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.*

Example 2. *Let Π_C be a generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E. Then, Π_C is a closed and quasi-ϕ- nonexpansive mapping from E onto C with $F(\Pi_C) = C$.*

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be a unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is said to be uniformly smooth if the limit is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Recall that a Banach space E has the Kadeč-Klee property (property(K) for brevity) if for any sequence $\{x_n\} \subset E$ and $x \in E$, if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. For more information concerning property(K) the reader is referred to [9] and references cited there in. It is well known that if E is a uniformly convex Banach space, then E has the property(K); Banach space E is uniformly smooth if and only if E^* is uniformly convex.

In order to prove our main results, we need the following lemmas.

Lemma 1 (See [2])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

Lemma 2 (See [2]). *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 3. *Let E be a reflexive, strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Proof. Let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$, we prove that $p \in F(T)$. From the definition of quasi$-\phi-$nonexpansive mappings, one has $\phi(p_n, Tp) \leq \phi(p_n, p)$, which implies that $\phi(p_n, Tp) \to 0$ as $n \to \infty$. Noticing that

$$\phi(p_n, Tp) = \|p_n\|^2 - 2\langle p_n, J(Tp)\rangle + \|Tp\|^2.$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \phi(p_n, Tp) = \|p\|^2 - 2\langle p, J(Tp)\rangle + \|Tp\|^2 = \phi(p, Tp).$$

Hence $\phi(p, Tp) = 0$. It implies that $p = Tp$. We next show that $F(T)$ is convex. To end this, for arbitrary $p_1, p_2 \in F(T), t \in (0, 1)$, putting $p_3 = tp_1 + (1-t)p_2$, we prove that $Tp_3 = p_3$. Indeed, by using the definition of $\phi(x, y)$, we have

$$\begin{aligned}
\phi(p_3, Tp_3) &= \|p_3\|^2 - 2\langle p_3, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2\langle tp_1 + (1-t)p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 - 2t\langle p_1, J(Tp_3)\rangle - 2(1-t)\langle p_2, J(Tp_3)\rangle + \|Tp_3\|^2 \\
&= \|p_3\|^2 + t\phi(p_1, Tp_3) + (1-t)\phi(p_2, Tp_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&\leq \|p_3\|^2 + t\phi(p_1, p_3) + (1-t)\phi(p_2, p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p_3\|^2 - 2\langle p_3, Jp_3\rangle + \|p_3\|^2 = 0.
\end{aligned}$$

This implies that $Tp_3 = p_3$. □

Lemma 4 (See [6]). *Let X be a uniformly convex Banach space, r a fixed positive number and $B_r(0)$ be a dosed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

3. Main results

Theorem 5. *Let E be a uniformly smooth and strictly convex Banach space with the property(K), C a nonempty closed convex subset of E. Let $T, S : C \to C$ be two closed quasi-ϕ-nonexpansive mappings such that the common fixed point set*

$F := F(T) \cap F(S)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J z_n), \\ z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \le \phi(u, x_n) + \alpha_n(\|x_1\|^2 + 2\langle u, Jx_n - Jx_1\rangle)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases} \tag{11}$$

where J is the duality mapping on E, $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$ $(i = 1, 2, 3)$ are sequences in (0,1) such that

(a) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$;

(b) $\lim_{n\to\infty} \alpha_n = 0$;

(c) $\liminf_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0$ *and* $\lim_{n\to\infty} \beta_n^{(1)} = 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. First, we show that C_n is closed and convex for every $n \ge 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the natural number set. For $u \in C_{k+1}$, one obtains that

$$\phi(u, y_k) \le \phi(u, x_k) + \alpha_k(\|x_1\|^2 + 2\langle u, Jx_k - Jx_1\rangle)$$

is equivalent to

$$2\langle u, Jx_k\rangle - 2\langle u, Jy_k\rangle - 2\alpha_k\langle u, Jx_k - Jx_1\rangle \le \|x_k\|^2 - \|y_k\|^2 + \alpha_k\|x_1\|^2.$$

It is easy to see that C_{k+1} is closed and convex. Then, for all $n \ge 1$, C_n are closed and convex. This shows that $\Pi_{C_{n+1}} x_1$ is well defined. Next, we prove $F \subset C_n$ for all $n \ge 1$. $F \subset C_1 = C$ is obvious. Suppose $F \subset C_k$ for some k. Then, for $\forall w \in F \subset C_k$, one has

$$\begin{aligned} \phi(w, z_k) &= \phi(w, J^{-1}(\beta_k^{(1)} Jx_k + \beta_k^{(2)} JTx_k + \beta_k^{(3)} JSx_k) \\ &= \|w\|^2 - 2\langle w, \beta_k^{(1)} Jx_k + \beta_k^{(2)} JTx_k + \beta_k^{(3)} JSx_k\rangle \\ &\quad + \|\beta_k^{(1)} Jx_k + \beta_k^{(2)} JTx_k + \beta_k^{(3)} JSx_k\|^2 \\ &\le \|w\|^2 - 2\beta_k^{(1)}\langle w, Jx_k\rangle - 2\beta_k^{(2)}\langle w, JTx_k\rangle - 2\beta_k^{(3)}\langle w, JSx_k\rangle \\ &\quad + \beta_k^{(1)}\|x_k\|^2 + \beta_k^{(2)}\|Tx_k\|^2 + \beta_k^{(3)}\|JSx_k\|^2 \\ &= \beta_k^{(1)}\phi(w, x_k) + \beta_k^{(2)}\phi(w, Tx_k) + \beta_k^{(3)}\phi(w, Sx_k) \\ &\le \beta_k^{(1)}\phi(w, x_k) + \beta_k^{(2)}\phi(w, x_k) + \beta_k^{(3)}\phi(w, x_k) \\ &= \phi(w, x_k) \end{aligned}$$

and then

$$\begin{aligned}\phi(w, y_k) &= \phi(w, J^{-1}[\alpha_k Jx_1 + (1-\alpha_k)Jz_k])\\ &= \|w\|^2 - 2\langle w, \alpha_k Jx_1 + (1-\alpha_k)Jz_k\rangle + \|\alpha_k Jx_1 + (1-\alpha_k)Jz_k\|^2\\ &\le \|w\|^2 - 2\alpha_k\langle w, Jx_1\rangle - 2(1-\alpha_k)\langle w, Jz_k\rangle + \alpha_k\|x_1\|^2 + (1-\alpha_k)\|z_k\|^2\\ &= \alpha_k\phi(w, x_1) + (1-\alpha_k)\phi(w, z_k)\\ &\le \alpha_k\phi(w, x_1) + (1-\alpha_k)\phi(w, x_k)\\ &= \phi(w, x_k) + \alpha_k[\phi(w, x_1) - \phi(w, x_k)]\\ &\le \phi(w, x_k) + \alpha_k(\|x_1\|^2 + 2\langle w, Jx_k - Jx_1\rangle).\end{aligned}$$

This shows $w \in C_{k+1}$. That is, $F \subset C_n$ for all $n \ge 1$. From $x_n = \Pi_{C_n}x_1$, one sees

$$\langle x_n - u, Jx_1 - Jx_n\rangle \ge 0, \quad \forall u \in C_n. \tag{12}$$

Since $F \subset C_n$ for all $n \ge 0$, we arrive at

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in F. \tag{13}$$

From Lemma 2, one has

$$\phi(x_n, x_1) = \phi(\Pi_{C_n}x_1, x_1) \le \phi(w, x_1) - \phi(w, x_n) \le \phi(w, x_1),$$

for each $w \in F \subset C_n$. Therefore, the sequence $\{\phi(x_n, x_1)\}$ is bounded, it follows from (10) that $\{x_n\}$ is bounded. On the other hand, noticing that $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_1) \le \phi(x_{n+1}, x_1)$$

for all $n \ge 1$. Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_1)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m}x_1 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_1)\\ &\le \phi(x_m, x_1) - \phi(\Pi_{C_n}x_1, x_1)\\ &= \phi(x_m, x_1) - \phi(x_n, x_1).\end{aligned} \tag{14}$$

Letting $m, n \to \infty$ in (14), one has $\phi(x_m, x_n) \to 0$. Hence $|\|x_m\| - \|x_n\|| \to 0$. This implies that $\{x_n\}$ is bounded. Since X is reflexive, without loss of generality, we can assume that $x_n \to p$ weakly as $n \to \infty$. Since $C_j \subset C_n$ for $j \ge n$, we have $x_j \in C_n$ for $j \ge n$. Since C_n is closed and convex, one has $p \in C_n$ for all $n \ge 1$. Hence $p \in \bigcap_{n=1}^{\infty} C_n = D$. Noticing that $\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \le \phi(p, x_0)$, $x_n \to p$ weakly as $n \to \infty$ and by using the fact that the norm is weakly lower semi-continuous, we have

$$\phi(p, x_0) \le \liminf_{n\to\infty} \phi(x_n, x_0) \le \limsup_{n\to\infty} \phi(x_n, x_0) \le \phi(p, x_0),$$

which implies that $\phi(x_n, x_0) \to \phi(p, x_0)$ as $n \to \infty$. Hence $\|x_n\| \to \|p\|$. By the property(K) of X, we have $x_n \to p \in C$ as $n \to \infty$.

Finally, we show that $p = \Pi_F x_1$. To show this, we first show $p \in F$. By taking $m = n+1$ in (14), one arrives at

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{15}$$

Since $x_n \to p$, one has

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{16}$$

Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \alpha_n(\|x_1\|^2 + 2\langle x_{n+1}, Jx_n - Jx_1\rangle).$$

It follows from (15) and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{17}$$

Noting that $0 \leq (\|x_{n+1}\| - \|y_n\|)^2 \leq \phi(x_{n+1}, y_n)$. Hence $\|y_n\| \to \|p\|$ and consequently $\|Jy_n\| \to \|Jp\|$. This implies that $\{J(y_n)\}$ is bounded. Since E is reflexive, E^* is also reflexive. So we can assume that

$$J(y_n) \to f_0 \in X^*$$

weakly. On the other hand, in view of the reflexivity of E, one has $J(E) = E^*$, which means that for $f_0 \in E^*$, there exists $x \in E$, such that $Jx = f_0$. In view of (17) and the weak lower semi-continuity of the norm, we have that

$$\begin{aligned} 0 = \liminf_{n\to\infty} \phi(x_{n+1}, y_n) &= \liminf_{n\to\infty} \{\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n||^2\} \\ &= \liminf_{n\to\infty} \{\|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n||^2\} \\ &\geq \|p\|^2 - 2\langle p, f_0\rangle + \|f_0||^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx||^2 \\ &= \phi(p, x) \geq 0. \end{aligned}$$

It follows that $\phi(p, x) = 0$ and then $p = x$, which implies that $f_0 = Jp$. Hence

$$Jy_n \to Jp \in E^*$$

weakly. Since $\|Jy_n\| \to \|Jp\|$ and E^* has the property(K), we have

$$\|Jy_n - Jp\| \to 0. \tag{18}$$

Noting that $J^{-1} : E^* \to E$ is demi-continuous, we have $y_n \to p$ weakly. Since $\|y_n\| \to \|p\|$ and E has the property(K), we obtain

$$y_n \to p \quad as \quad n \to \infty. \tag{19}$$

Noticing that

$$\begin{aligned}\phi(z_n,y_n) &= \|z_n\|^2 - 2\langle z_n, \alpha_n Jx_1 + (1-\alpha_n)Jz_n\rangle + \|\alpha_n Jx_1 + (1-\alpha_n)Jz_n\|^2\\ &\leq \|z_n\|^2 - 2\alpha_n\langle z_n, Jx_1\rangle - 2(1-\alpha_n)\langle z_n, Jz_n\rangle + \alpha_n\|x_1\|^2 + (1-\alpha_n)\|z_n\|^2\\ &= \alpha_n(\|z_n\|^2 - 2\langle z_n, Jx_1\rangle + \|x_1\|^2)\\ &= \alpha_n\phi(z_n,x_1).\end{aligned}$$

Since $\{x_n\}$ is bounded, from (10) and the definition of T and S, we can prove that $\{Tx_n\}$ and $\{Sx_n\}$ are all bounded, and hence $\{JTx_n\}$ and $\{JSx_n\}$ are all bounded. At this point, by using the definition of $\{z_n\}$, we know that $\{z_n\}$ is bounded. It follows from (10) that $\{\phi(z_n,x_1)\}$ is bounded, consequently, by condition (b) we have

$$\lim_{n\to\infty}\phi(z_n,y_n) = 0. \tag{20}$$

Noting that $0 \leq (\|z_n\| - \|y_n\|)^2 \leq \phi(z_n,y_n)$. It follows from (19) and (20) that $\|z_n\| \to \|p\|$. Since E is reflexive, $\{z_n\}$ is bounded, we can assume that

$$z_n \to g_0 \in X^*$$

weakly. By using (18), (19), (20) and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}0 = \liminf_{n\to\infty}\phi(z_n,y_n) &= \liminf_{n\to\infty}\{\|z_n\|^2 - 2\langle z_n, Jy_n\rangle + \|y_n\|^2\}\\ &\geq \|g_0\|^2 - 2\langle g_0, Jp\rangle + \|p\|^2\\ &= \phi(g_0,p) \geq 0,\end{aligned}$$

It follows that $\phi(g_0,p) = 0$. Hence $p = g_0$ and consequently

$$z_n \to p$$

weakly. Since $\|z_n\| \to \|p\|$ and E has the property (K), we have

$$z_n \to p \quad as \quad n \to \infty. \tag{21}$$

From $x_n \to p$, we have

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{22}$$

Since J is also uniformly norm-to-norm continuous on bounded sets, one sees

$$\lim_{n\to\infty}\|Jz_n - Jx_n\| = 0. \tag{23}$$

Let $r = \sup_{n\geq 1}\{\|x_n\|, \|Sx_n\|, \|Tx_n\|\}$. From Lemma 4, for any $w \in F$, one arrives at

$$\begin{aligned}\phi(w,z_n) &= \phi(w, J^{-1}[\beta_n^{(1)}Jx_n + \beta_n^{(2)}JTx_n + \beta_n^{(3)}JSx_n])\\ &= \|w\|^2 - 2\langle w, \beta_n^{(1)}Jx_n + \beta_n^{(2)}JTx_n + \beta_n^{(3)}JSx_n\rangle\\ &\quad + \|\beta_n^{(1)}Jx_n + \beta_n^{(2)}JTx_n + \beta_n^{(3)}JSx_n\|^2\\ &\le \|w\|^2 - 2\beta_n^{(1)}\langle w, Jx_n\rangle - 2\beta_n^{(2)}\langle w, JTx_n\rangle - 2\beta_n^{(3)}\langle w, JSx_n\rangle + \beta_n^{(1)}\|x_n\|^2\\ &\quad + \beta_n^{(2)}\|Tx_n\|^2 + \beta_n^{(3)}\|Sx_n\|^2 - \beta_n^{(2)}\beta_n^{(3)}g(\|JTx_n - JSx_n\|)\\ &= \beta_n^{(1)}\phi(w,x_n) + \beta_n^{(2)}\phi(w,Tx_n) + \beta_n^{(3)}\phi(w,Sx_n) - \beta_n^{(2)}\beta_n^{(3)}g(\|JTx_n - JSx_n\|)\\ &\le \beta_n^{(1)}\phi(w,x_n) + \beta_n^{(2)}\phi(w,x_n) + \beta_n^{(3)}\phi(w,x_n) - \beta_n^{(2)}\beta_n^{(3)}g(\|JTx_n - JSx_n\|)\\ &= \phi(w,x_n) - \beta_n^{(2)}\beta_n^{(3)}g(\|JSx_n - JTx_n\|).\end{aligned}$$

This implies that

$$\beta_n^{(2)}\beta_n^{(3)}g(\|JTx_n - JSx_n\|) \le \phi(w,x_n) - \phi(w,z_n). \tag{24}$$

On the other hand, one has

$$\begin{aligned}\phi(w,x_n) - \phi(w,z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle w, Jx_n - Jz_n\rangle\\ &\le \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|w\|\|Jx_n - Jz_n\|.\end{aligned}$$

It follows from (22) and (23) that

$$\phi(w,x_n) - \phi(w,z_n) \to 0, \quad as\ n \to \infty. \tag{25}$$

Observing that assumption $\liminf_{n\to\infty}\beta_n^{(2)}\beta_n^{(3)} > 0$, (24) and (25), one has

$$g(\|JTx_n - JSx_n\|) \to 0, \quad as\ n \to \infty.$$

It follows from the property of g that

$$\|JTx_n - JSx_n\| \to 0, \quad as\ n \to \infty. \tag{26}$$

Noticing that $|\|Tx_n\| - \|Sx_n\|| = |\|JTx_n\| - \|JSx_n\|| \le \|JTx_n - JSx_n\|$. It follows from (26) that

$$\|Tx_n\| - \|Sx_n\| \to 0, \quad as\ n \to \infty. \tag{27}$$

Observe that

$$\begin{aligned}\phi(Tx_n, Sx_n) &= \|Tx_n\|^2 - 2\langle Tx_n, JSx_n\rangle + \|Sx_n\|^2\\ &= \|Tx_n\|^2 - 2\langle Tx_n, JTx_n\rangle + 2\langle Tx_n, JTx_n - JSx_n\rangle + \|Sx_n\|^2\\ &\le \|Sx_n\|^2 - \|Tx_n\|^2 + 2\|Sx_n\|\|JTx_n - JSx_n\|\end{aligned}$$

It follows from (26) and (27) that

$$\lim_{n\to\infty}\phi(Tx_n, Sx_n) = 0. \tag{28}$$

On the other hand, one has

$$\begin{aligned}
\phi(Tx_n, z_n) &= \phi(Tx_n, J^{-1}[\beta_n^{(1)} Jx_n + \beta_n^{(1)} JTx_n + \beta_n^{(3)} JSx_n]) \\
&= \|Tx_n\|^2 - 2\langle Tx_n, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\rangle \\
&\quad + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\|^2 \\
&\le \|Tx_n\|^2 - 2\beta_n^{(1)}\langle Tx_n, Jx_n\rangle - 2\beta_n^{(2)}\langle Tx_n, JTx_n\rangle - 2\beta_n^{(3)}\langle Tx_n, JSx_n\rangle \\
&\quad + \beta_n^{(1)}\|x_n\|^2 + \beta_n^{(2)}\|Tx_n\|^2 + \beta_n^{(3)}\|Sx_n\|^2 \\
&\le \beta_n^{(1)}\phi(Tx_n, x_n) + \beta_n^{(3)}\phi(Tx_n, Sx_n). \qquad (29)
\end{aligned}$$

Noticing that $\beta_n^{(1)} \to 0$ as $n \to \infty$, (28) and (29), one arrives at

$$\lim_{n\to\infty} \phi(Tx_n, z_n) = 0. \qquad (30)$$

Since $J : X \to X^*$ is demi-continuous, from (21), we have

$$Jz_n \to Jp$$

weakly. Noticing that $|\|Jz_n\| - \|Jp\|| = |\|z_n\| - \|p\|| \le \|z_n - p\| \to 0$. Since X^* has the property(K), we have

$$Jz_n \to Jp \quad as \quad n \to \infty. \qquad (31)$$

On the other hand, noting that $0 \le (\|Tx_n\| - \|z_n\|)^2 \le \phi(Tx_n, z_n)$. It follows from (21) that $\|Tx_n\| \to \|p\|$. Since E is reflexive, $\{Tx_n\}$ is bounded, we can assume that

$$Tx_n \to g_0 \in X^*$$

weakly. By using (21), (30), (31) and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}
0 = \liminf_{n\to\infty} \phi(Tx_n, z_n) &= \liminf_{n\to\infty} \{\|Tx_n\|^2 - 2\langle Tx_n, Jz_n\rangle + \|z_n\|^2\} \\
&\ge \|g_0\|^2 - 2\langle g_0, Jp\rangle + \|p\|^2 \\
&= \phi(g_0, p) \ge 0.
\end{aligned}$$

It follows that $\phi(g_0, p) = 0$. Hence $p = g_0$ and consequently

$$Tx_n \to p$$

weakly. Since $\|Tx_n\| \to \|p\|$ and E has the property(K), we have

$$Tx_n \to p \quad as \quad n \to \infty. \qquad (32)$$

In view of $x_n \to p$ and the closedness of T, one gets $p \in F(T)$. In a similar way, we can obtain $p \in F(S)$. Hence $p \in F(T) \cap F(S) = F$.

Finally, we show that $p = \Pi_F x_1$. By taking the limit as $n \to \infty$ in (13), we obtain

$$\langle p - w, Jx_1 - Jp\rangle \ge 0, \quad \forall w \in F,$$

and hence $p = \Pi_F x_1$ by Lemma 1. This completes the proof. □

Remark 3. *Theorem 5 improves Theorem 3.5 of Plubtieng and Ungchittrakool [13] in the following senses:*

(1) *from uniformly convex and uniformly smooth Banach spaces to uniformly smooth and strictly convex Banach spaces with the property(K).*

(2) *from relatively nonexpansive mappings to more general quasi-ϕ-nonexpansive mappings, that is, we remove the strict restriction: $\widetilde{F(T)} = F(T)$;*

(3) *from computation point of view, algorithm (11) is also more simple and convenient to compute than the one given by [13]. To be more precise, we remove the set "W_n" in [13].*

As consequence of Theorem 5, we immediately obtain the following results.

If $\beta_n^{(1)} = 0$ for all $n \geq 0$ and $T = S$ in Theorem 5, then Theorem 5 reduces to the following.

Corollary 1. *Let E be a uniformly smooth and strictly convex Banach space, and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Assume the fixed point set $F(T)$ of T is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) JTx_n], \\ C_{n+1} = \{u \in C_n : \phi(u, y_n) \leq \phi(u, x_n) + \alpha_n(\|x_1\|^2 + 2\langle u, Jx_n - Jx_1\rangle)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where J is the duality mapping on E and $\{\alpha_n\}$ is a sequence in (0,1) such that $\limsup_{n\to\infty} \alpha_n = 0$. *Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.*

Remark 4. *Corollary 1 mainly improves the analogue of Qin and Su [14].*

In the framework of Hilbert spaces, Corollary 1 reduces to the following result.

Corollary 2. *Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a quasi-nonexpansive mapping. Assume the fixed point set $F(T)$ of T is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_1 + (1-\alpha_n) Tx_n, \\ C_{n+1} = \{u \in C_n : \|u - y_n\|^2 \leq \|u - x_n\|^2 + \alpha_n(\|x_1\|^2 + 2\langle u, x_n - x_1\rangle)\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in (0,1) such that $\limsup_{n\to\infty} \alpha_n = 0$. *Then $\{x_n\}$ converges strongly to $P_{F(T)} x_1$.*

Remark 5. *Corollary 2 extends the corresponding result announced by Martinez-Yanes and Xu [11] from nonexpansive mappings to quasi-nonexpansive mappings.*

References

[1] Ya. I. Alber, S. Reich, *An iterative method for solving a class of nonlinear operatorequations in Banach spaces*, Panamer. Math. J. **4**(1994), 39–54.

[2] Ya. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, in: *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, (A. G. Kartsatos, Ed.), Marcel Dekker, New York, 1996, 15–50.

[3] F. E. Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Natl. Acad. Sci. USA **53**(1965), 1272–1276.

[4] D. Butnariu, S. Reich, A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7**(2001), 151–174.

[5] D. Butnariu, S. Reich, A. J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **24**(2003), 489–508.

[6] Y. J. Cho, H. Y. Zhou, J.T. Guo, *Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl. **47**(2004), 707–717.

[7] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[8] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Am. Math. Soc. **73**(1967), 957–961.

[9] H. Hudzik, W. Kowalewski, G. Lewicki, *Approximative compactness and full rotundity in Musielak-Orlicz spaces and Lorentz-Orlicz spaces*, Z. Anal. Anwend. **25**(2006), 163–192.

[10] P. -L. Lions, *Approximation de points fixes de contractions*, C.R. Acad. Sci. Paris Ser. A-B **284**(1977), 1357–1359.

[11] C. Martinez-Yanes, H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64**(2006), 2400–2411.

[12] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134**(2005), 257–266.

[13] S. Plubtieng, K. Ungchittrakool, *Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **149**(2007), 103–115.

[14] X. Qin, Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Nonlinear Anal. **67**(2007), 1958–1965.

[15] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama-Publishers, Yokohama, 2000.

[16] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58**(1992), 486–491.

[17] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65**(2002), 109–113.

[18] H. Y. Zhou, G. L. Gao, B. Tan, *Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings*, J. Appl. Math. Computing. **32**(2010), 453–464.