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Umbilical hypersurfaces of Minkowski spaces *

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Abstract. In this paper, by the Gauss equation of the induced Chern connection for Finsler submanifolds, we prove that if M is an umbilical hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$, then either M is a Riemannian space form or a locally Minkowski space.
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1. Introduction

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ the natural projection from the tangent bundle. Let (x, Y) be a point of TM with $x \in M, Y \in T_xM$ and let (x^i, Y^i) be the local coordinates on TM with $Y = Y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) Regularity: $F(x, Y)$ is smooth in $TM\backslash 0$;

(ii) Positive homogeneity: $F(x, \lambda Y) = \lambda F(x, Y)$ for $\lambda > 0$;

(iii) Strong convexity: The fundamental quadratic form $g = g_{ij}(x, Y)dx^i \otimes dx^j$ is positive definite, where $g_{ij} = \frac{\partial^2(F^2)}{2\partial Y^i \partial Y^j}$.

The simplest class of Finsler manifolds are Minkowski spaces. Let V^{n+1} be a real vector space. A Finsler metric $\overline{F} : TV^{n+1} \to [0, \infty)$ is called Minkowski if $\overline{F}$ is a function of $\overline{Y} \in V^{n+1}$ only and $(V^{n+1}, \overline{F})$ is called a Minkowski space.

Finsler manifolds are just Riemannian manifolds with metrics without the quadratic restriction. The geometry of Finsler submanifolds has been developed in the recent years. In 1990s, Z. Shen ([4]) studied the geometry of Finsler submanifolds by using the volume form. It is well known that the Gauss equation plays an important role in studying the Riemannian submanifolds. On the other hand, to the best of author's knowledge, no one has so far used the induced Chern connection in studies on Finsler submanifolds and isometric immersions. In this paper, by the Gauss and Codazzi equations of Finsler submanifolds on the induced Chern connection, we study the umbilical hypersurfaces of a Minkowski space $(V^{n+1}, \overline{F})$ and obtain the following classification theorem on the umbilical hypersurfaces of a Minkowski space

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Theorem 1 (Main theorem)**.** *If M is an umbilical hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$, then either M is a Riemannian space form or a locally Minkowski space.*

2. The Gauss formula

Let (M^n, F) be an n-dimensional Finsler manifold. F inherits the *Hilbert* form and the *Cartan* tensor as follows:

$$\omega = \frac{\partial F}{\partial Y^i} dx^i, \quad A = A_{ijk} dx^i \otimes dx^j \otimes dx^k, \quad A_{ijk} := \frac{F \partial g_{ij}}{2 \partial Y^k}.$$

It is well known that there exists uniquely the Chern connection ∇ on $\pi^* TM$ with $\nabla \frac{\partial}{\partial x^i} = \omega_i^j \frac{\partial}{\partial x^j}$ and $\omega_i^j = \Gamma_{ik}^j dx^k$ satisfying that

$$\begin{cases} \text{Torsion freeness: } d(dx^i) - dx^j \wedge \omega_j^i = -dx^j \wedge \omega_j^i = 0, \\ \text{Almost } g \text{ compatibility: } dg_{ij} - g_{ik}\omega_j^k - g_{jk}\omega_i^k = 2A_{ijk}\frac{\delta Y^k}{F}, \end{cases}$$

where $\delta Y^i = dY^i + N_j^i dx^j$, $N_j^i = \gamma_{jk}^i Y^k - \frac{1}{F} A_{jk}^i \gamma_{st}^k Y^s Y^t$ and γ_{jk}^i are the formal Christoffel symbols of the second kind for g_{ij}.

The curvature 2-forms of the Chern connection ∇ are

$$\omega_j^i - \omega_j^k \wedge \omega_k^i = \Omega_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l + \frac{1}{F} P_{jkl}^i dx^k \wedge \delta Y^l,$$

where R_{jkl}^i and P_{jkl}^i are the components of the hh-curvature tensor and hv-curvature tensor of the Chern connection, respectively.

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be a smooth map from a Finsler manifold to a Finsler manifold. φ is called an isometric immersion if $F(Y) = \overline{F}(\varphi_*(Y))$. We have that [3]

$$\begin{aligned} g_Y(U, V) &= \overline{g}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V)), \\ A_Y(U, V, W) &= \overline{A}_{\varphi_*(Y)}(\varphi_*(U), \varphi_*(V), \varphi_*(W)), \end{aligned} \tag{1}$$

where $Y, U, V, W \in TM, \overline{g}$ and $\overline{A}$ are the fundamental tensor and the *Cartan* tensor of $\overline{M}$, respectively.

It can be seen from (1) that $\varphi^*(\overline{\omega}) = \omega$, where $\overline{\omega}$ is the *Hilbert* form of $\overline{M}$.

In the following, any vector $U \in TM$ will be identified with the corresponding vector $\varphi_*(U) \in T\overline{M}$ and we will use the following convention:

$$1 \leq i, j, \cdots \leq n; \; n+1 \leq \alpha, \beta, \cdots \leq n+p; \; 1 \leq a, b, \cdots \leq n+p; \; 1 \leq \lambda, \mu, \cdots \leq n-1.$$

Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold. Take a $\overline{g}$-orthonormal frame form $\{e_a\}$ for each fibre of $\pi^* T\overline{M}$ and let $\{\omega^a\}$ be its local dual coframe, such that $\{e_i\}$ is a frame field for each fibre of $\pi^* TM$ and ω^n is the *Hilbert* form. Let θ_b^a and ω_j^i denote the Chern connection 1-form of $\overline{F}$ and F, respectively, i.e., $\overline{\nabla} e_a = \theta_a^b e_b$ and $\nabla e_i = \omega_i^j e_j$, where $\overline{\nabla}$ and ∇ are the Chern connection of $\overline{M}$ and M, respectively. We obtain

$$A(e_i, e_j, e_n) = 0, \quad \overline{A}(e_a, e_b, e_n) = 0, \quad \forall i, j, a, b. \tag{2}$$

The structure equations of $\overline{M}$ are given by

$$\begin{cases} d\omega^a = -\theta^a_b \wedge \omega^b, \\ d\theta^a_b = -\theta^a_c \wedge \theta^c_b + \frac{1}{2}\overline{R}^a_{bcd}\omega^c \wedge \omega^d + \overline{P}^a_{bcd}\omega^c \wedge \theta^d_n, \\ \theta^a_b + \theta^b_a = -2\overline{A}_{abc}\theta^c_n, \\ \theta^a_n + \theta^n_a = 0, \quad \theta^n_n = 0. \end{cases} \tag{3}$$

For any $\xi \in \Gamma(\pi^*TM)^\perp$ and $X \in \Gamma(\pi^*TM)$, we decompose $\overline{\nabla}_X\xi$ into a sum of the form

$$\overline{\nabla}_X\xi = -\mathcal{A}_\xi(X) + \nabla^\perp_X\xi,$$

where $\mathcal{A}_\xi(X) \in \Gamma(\pi^*TM)$ and $\nabla^\perp_X\xi \in \Gamma(\pi^*TM)^\perp$.

By $\omega^\alpha = 0$ and the structure equations of $\overline{M}$, we have that

$$\theta^\alpha_j \wedge \omega^j = 0. \tag{4}$$

It follows from (4) that

$$\theta^\alpha_j = h^\alpha_{ij}\omega^i, \qquad h^\alpha_{ij} = h^\alpha_{ji}. \tag{5}$$

We now establish the following

Theorem 2 (The Gauss formula)**.** *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold and let θ^a_b denote the Chern connection $\overline{\nabla}$ 1-form of $\overline{M}$. If*

$$\omega^j_i = \theta^j_i - \Psi_{jik}\omega^k, \tag{6}$$

where

$$\Psi_{jik} = h^\alpha_{jn}\overline{A}_{ki\alpha} - h^\alpha_{kn}\overline{A}_{ji\alpha} - h^\alpha_{in}\overline{A}_{kj\alpha} - h^\alpha_{nn}\overline{A}_{iks}\overline{A}_{sj\alpha} + h^\alpha_{nn}\overline{A}_{ijs}\overline{A}_{sk\alpha} + h^\alpha_{nn}\overline{A}_{jks}\overline{A}_{si\alpha}, \tag{7}$$

then ω^j_i are the 1-forms of the Chern connection ∇ of M.

Proof. (i) (Affine connection) Let $B(\bullet, V) = \overline{\nabla}^\perp V = v^i\theta^\alpha_i e_\alpha, \forall V = v^i e_i \in \Gamma(\pi^*TM)$. By (6) we have that $B(\bullet, V) = v^i h^\alpha_{ij}\omega^j \otimes e_\alpha$. Now, from (6) it follows that

$$\overline{\nabla}V = \nabla V + B(\bullet, V) + \sum_i v^j\Psi_{ijk}\omega^k \otimes e_i. \tag{8}$$

Because $\overline{\nabla}$ is an affine connection of $\overline{M}$, B and A are linear, by (8) we obtain that ∇ is an affine connection of M.

(ii) (Torsion freeness) Because of the torsion freeness of $\overline{M}$, using (6), we have that

$$d\omega^i = -\omega^i_j \wedge \omega^j - \Psi_{ijk}\omega^k \wedge \omega^j. \tag{9}$$

It can be seen from (7) that

$$\Psi_{ijk} = \Psi_{ikj}. \tag{10}$$

It follows from (9) and (10) that

$$d\omega^i = -\omega^i_j \wedge \omega^j. \tag{11}$$

(iii) (Almost g-compatibility) It is easy to see from (7) that

$$\Psi_{ijk} + \Psi_{jik} = -2h^\alpha_{kn}\overline{A}_{ij\alpha} + 2h^\alpha_{nn}\overline{A}_{ijs}\overline{A}_{sk\alpha}. \tag{12}$$

By (5), (12) and the third formula of (3), we obtain that

$$\begin{aligned} \omega^i_j + \omega^j_i &= (\theta^i_j - \Psi_{ijk}\omega^k) + (\theta^j_i - \Psi_{jik}\omega^k) \\ &= -2\overline{A}_{ij\lambda}\theta^\lambda_n - 2\overline{A}_{ij\alpha}\theta^\alpha_n + (2h^\alpha_{kn}\overline{A}_{ij\alpha} - 2h^\alpha_{nn}\overline{A}_{ijs}\overline{A}_{sk\alpha})\omega^k \\ &= -2A_{ij\lambda}\omega^\lambda_n. \end{aligned} \tag{13}$$

Combining (i), (ii) and (iii) completes the proof of Theorem 2. □

From (7, we get that

$$\theta^j_n = \omega^j_n - h^\alpha_{nn}\overline{A}_{j\lambda\alpha}\omega^\lambda, \qquad \theta^n_j = \omega^n_j + h^\alpha_{nn}\overline{A}_{j\lambda\alpha}\omega^\lambda. \tag{14}$$

By (5), (14) and the almost $\overline{g}$-compatibility, we have that

$$\begin{aligned} \theta^j_\alpha &= -\theta^\alpha_j - 2\overline{A}_{j\alpha a}\theta^a_n \\ &= (-h^\alpha_{ij} - 2h^\beta_{ni}\overline{A}_{j\alpha\beta} + 2h^\beta_{nn}\overline{A}_{j\lambda\alpha}\overline{A}_{i\lambda\beta})\omega^i - 2\overline{A}_{j\alpha\lambda}\omega^\lambda_n. \end{aligned} \tag{15}$$

In particular,

$$\theta^n_\alpha = -h^\alpha_{ni}\omega^i. \tag{16}$$

3. The Gauss and Codazzi equations

By the structure equations, we have that

$$\begin{aligned} d\theta^j_i &= \theta^k_i \wedge \theta^j_k + \theta^\alpha_i \wedge \theta^j_\alpha + \frac{1}{2}\overline{R}^j_{ikl}\omega^k \wedge \omega^l + \overline{P}^j_{ik\lambda}\omega^k \wedge \theta^\lambda_n + \overline{P}^j_{ik\alpha}\omega^k \wedge \theta^\alpha_n \\ &= \omega^k_i \wedge \omega^j_k + \Psi_{kil}\omega^l \wedge \omega^j_k + \Psi_{jkl}\omega^k_i \wedge \omega^l + \Big\{\Psi_{sik}\Psi_{jsl} - h^\alpha_{ik}h^\alpha_{jl} \\ &\quad -2h^\alpha_{ik}h^\beta_{nl}\overline{A}_{j\alpha\beta} + 2h^\alpha_{ik}h^\beta_{nn}\overline{A}_{js\alpha}\overline{A}_{ls\beta} + \frac{1}{2}\overline{R}^j_{ikl} \\ &\quad -h^\alpha_{nn}\overline{A}_{ls\alpha}\overline{P}^j_{iks} + h^\alpha_{nl}\overline{P}^j_{ik\alpha}\Big\}\omega^k \wedge \omega^l - \Big\{2h^\alpha_{ik}\overline{A}_{j\alpha\lambda} - \overline{P}^j_{ik\lambda}\Big\}\omega^k \wedge \omega^\lambda_n. \end{aligned} \tag{17}$$

On the other hand,

$$d\omega^j_i = \omega^k_i \wedge \omega^j_k + \frac{1}{2}R^j_{ikl}\omega^k \wedge \omega^l + P^j_{ik\lambda}\omega^k \wedge \omega^\lambda_n, \tag{18}$$

and

$$\begin{aligned} d(\Psi_{jik}\omega^k) &= d\Psi_{jik} \wedge \omega^k + \Psi_{jik}\omega^l \wedge \omega_l^k \\ &= \Psi_{jik|l}\omega^l \wedge \omega^k + \Psi_{jik;\lambda}\omega_n^\lambda \wedge \omega^k \\ &\quad + \Psi_{kil}\omega^l \wedge \omega_k^j + 2\Psi_{kil}A_{jks}\omega^l \wedge \omega_n^s + \Psi_{jkl}\omega_i^k \wedge \omega^l. \end{aligned} \tag{19}$$

where "|" and ";" denote the horizontal and the vertical covariant differentials with respect to the Chern connection, respectively.

Substituting (17), (18) and (19) into (6), we obtain that

$$\begin{aligned} &(-2h_{ik}^\alpha \overline{A}_{j\lambda\alpha} + \overline{P}_{ik\lambda}^j)\omega^k \wedge \omega_n^\lambda + (\Psi_{sik}\Psi_{jsl} - h_{ik}^\alpha h_{jl}^\alpha - 2h_{ik}^\alpha h_{nl}^\beta \overline{A}_{j\alpha\beta} \\ &+2h_{ik}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ls\beta} + \frac{1}{2}\overline{R}_{ikl}^j - h_{nn}^\alpha \overline{P}_{iks}^j \overline{A}_{sl\alpha} + h_{nl}^\alpha \overline{P}_{ik\alpha}^j)\omega^k \wedge \omega^l \\ &\quad = (P_{ik\lambda}^j - \Psi_{jik;\lambda} + 2\Psi_{sik}A_{js\lambda})\omega^k \wedge \omega_n^\lambda + (\frac{1}{2}R_{ikl}^j - \Psi_{jik|l})\omega^k \wedge \omega^l. \end{aligned} \tag{20}$$

Then we have the following result

Theorem 3 (The Gauss equations)**.** *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{cases} P_{ik\lambda}^j = \overline{P}_{ik\lambda}^j + \Psi_{jik;\lambda} - 2\Psi_{sik}A_{js\lambda} - 2h_{ik}^\alpha \overline{A}_{j\lambda\alpha}, \\ R_{ikl}^j = \overline{R}_{ikl}^j - h_{ik}^\alpha h_{jl}^\alpha + h_{il}^\alpha h_{jk}^\alpha + \Psi_{jik|l} - \Psi_{jil|k} \\ \qquad +\Psi_{sik}\Psi_{jsl} - \Psi_{sil}\Psi_{jsk} - 2h_{ik}^\alpha h_{nl}^\beta \overline{A}_{j\alpha\beta} + 2h_{il}^\alpha h_{nk}^\beta \overline{A}_{j\alpha\beta} \\ \qquad +2h_{ik}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ls\beta} - 2h_{il}^\alpha h_{nn}^\beta \overline{A}_{js\alpha}\overline{A}_{ks\beta} - h_{nn}^\alpha \overline{A}_{sl\alpha}\overline{P}_{iks}^j \\ \qquad +h_{nn}^\alpha \overline{A}_{sk\alpha}\overline{P}_{ils}^j + h_{nl}^\alpha \overline{P}_{ik\alpha}^j - h_{nk}^\alpha \overline{P}_{il\alpha}^j. \end{cases} \tag{21}$$

We now establish the Codazzi equations. The exterior derivative of the left-hand side term of (5) yields that

$$\begin{aligned} d\theta_i^\alpha &= \theta_i^k \wedge \theta_k^\alpha + \theta_i^\beta \wedge \theta_\beta^\alpha + \frac{1}{2}\overline{R}_{ikl}^\alpha \omega^k \wedge \omega^l + \overline{P}_{ik\lambda}^\alpha \omega^k \wedge \theta_n^\lambda + \overline{P}_{ik\beta}^\alpha \omega^k \wedge \theta_n^\beta \\ &= (\omega_i^k + \Psi_{kil}\omega^l) \wedge h_{kj}^\alpha \omega^j + h_{ik}^\beta \omega^k \wedge \theta_\beta^\alpha + \frac{1}{2}\overline{R}_{ikl}^\alpha \omega^k \wedge \omega^l \\ &\quad + \overline{P}_{ik\lambda}^\alpha \omega^k \wedge (\omega_n^\lambda - h_{nn}^\beta \overline{A}_{\lambda s\beta}\omega^s) + h_{nl}^\beta \overline{P}_{ik\beta}^\alpha \omega^k \wedge \omega^l \\ &= \Big\{ h_{sl}^\alpha \Psi_{sik} + \frac{1}{2}\overline{R}_{ikl}^\alpha - h_{nn}^\beta \overline{A}_{sl\beta}\overline{P}_{iks}^\alpha + h_{nl}^\beta \overline{P}_{ik\beta}^\alpha \Big\} \omega^k \wedge \omega^l \\ &\quad + \overline{P}_{ik\lambda}^\alpha \omega^k \wedge \omega_n^\lambda + h_{kj}^\alpha \omega_i^k \wedge \omega^j + h_{ik}^\beta \omega^k \wedge \theta_\beta^\alpha. \end{aligned} \tag{22}$$

Moreover, the right-hand side term of (5) gives that

$$d(h_{ij}^\alpha \omega^j) = h_{ij|k}^\alpha \omega^k \wedge \omega^j + h_{ij;\lambda}^\alpha \omega_n^\lambda \wedge \omega^j + h_{kj}^\alpha \omega_i^k \wedge \omega^j - h_{ij}^\beta \theta_\beta^\alpha \wedge \omega^j. \tag{23}$$

It follows from (22) and (23) that

$$\begin{aligned} &\Big\{ h_{sl}^\alpha \Psi_{sik} + \frac{1}{2}\overline{R}_{ikl}^\alpha - h_{nn}^\beta \overline{A}_{sl\beta}\overline{P}_{iks}^\alpha + h_{nl}^\beta \overline{P}_{ik\beta}^\alpha \Big\} \omega^k \wedge \omega^l + \overline{P}_{ik\lambda}^\alpha \omega^k \wedge \omega_n^\lambda \\ &\quad = h_{il|k}^\alpha \omega^k \wedge \omega^l + h_{ij;\lambda}^\alpha \omega_n^\lambda \wedge \omega^j. \end{aligned} \tag{24}$$

From (24), we may state the following theorem

Theorem 4 (The Codazzi equations). *Let $\varphi : (M^n, F) \to (\overline{M}^{n+p}, \overline{F})$ be an isometric immersion from a Finsler manifold to a Finsler manifold, then we have that*

$$\begin{cases} h^{\alpha}_{ij;\lambda} = -\overline{P}^{\alpha}_{ij\lambda}, \\ h^{\alpha}_{ij|k} - h^{\alpha}_{ik|j} = -\overline{R}^{\alpha}_{ijk} + h^{\beta}_{nj}\overline{P}^{\alpha}_{ik\beta} - h^{\beta}_{nk}\overline{P}^{\alpha}_{ij\beta} \\ \qquad -h^{\alpha}_{lk}\Psi_{lij} + h^{\alpha}_{lj}\Psi_{lik} - h^{\beta}_{nn}\overline{A}_{lj\beta}\overline{P}^{\alpha}_{ikl} + h^{\beta}_{nn}\overline{A}_{lk\beta}\overline{P}^{\alpha}_{ijl}. \end{cases} \tag{25}$$

4. Umbilical hypersurfaces of Minkowski space

Let (M^n, F) be a hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$. For $\xi = e_{n+1}$, the Weingarten transformation $\mathcal{A}_\xi : \Gamma(\pi^*TM) \to \Gamma(\pi^*TM)$ is defined by

$$\langle \mathcal{A}_\xi(X), Y\rangle = \langle B(X,Y), \xi\rangle, \tag{26}$$

where $X, Y \in \Gamma(\pi^*TM)$.

Definition 1. *A point $p \in M$ is called umbilical if all eigenvalues of the Weingarten transformation $\mathcal{A}_\xi$ are equal one to another. M is called an umbilical hypersurface if all points are umbilical points.*

Thus the hypothesis that M is an umbilical hypersurface means

$$h^{n+1}_{ij} = \langle B(e_i, e_j), \xi\rangle = \langle \mathcal{A}_\xi(e_i), e_j\rangle = \tau\delta_{ij}, \tag{27}$$

where τ is the eigenvalue of the Weingarten transformation $\mathcal{A}_\xi$.

Now we can prove the following

Theorem 5. *If M is an umbilical hypersurface of a Minkowski space $(V^{n+1}, \overline{F})$, then either M is a Riemannian space form or a locally Minkowski space.*

Proof. It can be seen from $h^{n+1}_{nn} = h^{n+1}_{ii}$ that

$$h^{n+1}_{nn|j}\omega^j + h^{n+1}_{nn;\lambda}\omega^{\lambda}_n + 2h^{n+1}_{n\lambda}\omega^{\lambda}_n - h^{n+1}_{nn}\theta^{n+1}_{n+1} \tag{28}$$

$$= h^{n+1}_{ii|j}\omega^j + h^{n+1}_{ii;\lambda}\omega^{\lambda}_n + 2h^{n+1}_{ij}\omega^j_i - h^{n+1}_{ii}\theta^{n+1}_{n+1}. \tag{29}$$

It follows from (27), (28) and $\omega^i_i = -A_{ii\lambda}\omega^{\lambda}_n$ that

$$h^{n+1}_{nn|j}\omega^j + h^{n+1}_{nn;\lambda}\omega^{\lambda}_n = h^{n+1}_{ii|j}\omega^j + (h^{n+1}_{ii;\lambda} - 2\tau A_{ii\lambda})\omega^{\lambda}_n, \tag{30}$$

which associated with the first formula of (25) implies that

$$\begin{cases} h^{n+1}_{nn|j} = h^{n+1}_{ii|j}, \\ \tau A_{ii\lambda} = 0. \end{cases} \qquad (\forall i, j) \tag{31}$$

On the other hand, it can be seen from $h_{ij}^{n+1} = 0$ $(i \neq j)$ that

$$\begin{aligned} 0 &= h_{ij|k}^{n+1}\omega^k + h_{ij;\lambda}^{n+1}\omega_n^\lambda + h_{kj}^{n+1}\omega_i^k + h_{ik}^{n+1}\omega_j^k \\ &= h_{ij|k}^{n+1}\omega^k + \tau(\omega_i^j + \omega_j^i) \\ &= h_{ij|k}^{n+1}\omega^k - 2\tau A_{ij\lambda}\omega_n^\lambda. \end{aligned} \tag{32}$$

Then we have that

$$\begin{cases} h_{ij|k}^{n+1} = 0, \\ \tau A_{ij\lambda} = 0. \end{cases} \quad (\forall i \neq j) \tag{33}$$

Substituting $\overline{R}_{ikl}^j = \overline{P}_{ik\lambda}^j = 0, \Psi_{lnn} = 0$ and the first formula of (33) into the second formula of (25) yields that

$$h_{nn|\lambda}^{n+1} = h_{n\lambda|n}^{n+1} - h_{l\lambda}^{n+1}\Psi_{lnn} + h_{ln}^{n+1}\Psi_{ln\lambda} = 0. \tag{34}$$

On the other hand, we also have that

$$\begin{aligned} h_{nn|n}^{n+1} &= h_{\lambda\lambda|n}^{n+1} \\ &= h_{\lambda n|\lambda}^{n+1} - h_{ln}^{n+1}\Psi_{l\lambda\lambda} + h_{l\lambda}^{n+1}\Psi_{l\lambda n} \\ &= -h_{nn}^{n+1}\Psi_{n\lambda\lambda} + h_{\lambda\lambda}^{n+1}\Psi_{\lambda\lambda n} \\ &= -2\tau^2\overline{A}_{\lambda\lambda n+1}. \end{aligned} \tag{35}$$

It follows from the second formulas of (31) and (33) that

$$0 = d(\tau A_{ijk}) = (\tau_{|l}\omega^l + \tau_{;\lambda}\omega_n^\lambda)A_{ijk} + \tau(A_{ijk|l}\omega^l + A_{ijk;\lambda}\omega_n^\lambda), \tag{36}$$

which together with $\tau A_{ijk} = 0$ implies that

$$\begin{cases} \tau A_{ijk|l} = 0, \qquad \text{i.e., } \tau P_{jk\lambda}^i = 0, \\ \tau A_{ijk;\lambda} = 0. \end{cases} \tag{37}$$

It can be seen from (7), (37) and the first formula of (25) that

$$\Psi_{\mu\nu\xi;\lambda} = 0. \tag{38}$$

Substituting (37) and (38) into the first formula of (21), we get that

$$\tau h_{\mu\nu}^{n+1}\overline{A}_{\xi\lambda n+1} = 0. \tag{39}$$

Substituting (39) into (35), we have that $h_{nn|n}^{n+1} = 0$, which, together with (34) and the first formula of (25), yields that h_{nn}^{n+1} is constant, i.e., τ is constant, which associated with the second formulas of (31) and (33) implies that $\tau = 0$ or $A_{ijk} = 0$.

(1) The first case. When $\tau = 0$, we have that $h_{ij}^{n+1} = 0$ and $\Psi_{ijk} = 0$. By (3.5) we obtain that $P_{ik\lambda}^j = 0$ and $R_{ikl}^j = 0$, i.e., M is a locally Minkowski space.

(2) The second case. When $A_{ijk} = 0$, i.e., M is Riemannian. Substituting (39) into (8), we get $\Psi_{ijk} = 0$. Then the second formula of (21) implies that

$$R_{ikl}^j = -h_{ik}^{n+1}h_{jl}^{n+1} + h_{il}^{n+1}h_{jk}^{n+1} = \tau^2(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}). \tag{40}$$

This completes the proof of Main theorem. □

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References

[1] D. Bao, S. S. Chern, Z. Shen, *An introduction to Riemann-Finsler geometry*, Springer-Verlag, New York, 2000.

[2] G. Berck, *Minimality of totally geodesic submanifolds in Finsler geometry*, Math. Ann. **343**(2009), 955–973.

[3] L. Maria Abatangelo, *On totally umbilical submanifolds of locally Minkowski manifold*, Collect. Math. **43**(1992), 151–175.

[4] Z. Shen, *On Finsler geometry of submanifolds*, Math. Ann. **311**(1998), 549–576.