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General rotational surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map

Uğur Dursun[1,*] and Nurretin Cenk Turgay[1]

[1] *Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 34 469 Maslak, Istanbul, Turkey*

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Abstract. In this paper, we study general rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map. We consider general rotational surfaces in $\mathbb{E}^4$ whose meridian curves lie in two-dimensional planes. We firstly obtain all general rotational surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind. Then we classify minimal rotational surfaces of $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the second kind.

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1. Introduction

In late 1970's, B. Y. Chen introduced the notion of finite type immersion into a Euclidean space. Since then many works have been writen in this field (see [5, 6], etc.). A submanifold M of a Euclidean space $\mathbb{E}^m$ is said to be of finite type if its position vector x can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is, $x = x_0 + x_1 + \cdots + x_k$, where x_0 is a constant map, $x_1, \ldots, x_k$ are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all different, then M is said to be of k-type. In [7], Chen and Piccinni similarly extended this definition to differentiable maps, in particular, to Gauss maps of submanifolds. They made a general study on compact submanifolds of Euclidean spaces with finite type Gauss map, and for hypersurfaces they proved that a compact hypersurface M of $\mathbb{E}^{n+1}$ has 1-type Gauss map if and only if M is a hypersphere in $\mathbb{E}^{n+1}$. Also, many geometers studied submanifolds with finite type Gauss map ([2, 3, 4, 7, 20] etc.).

If a submanifold M of a Euclidean space has 1-type Gauss map ν, then $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. However, the Laplacian of the Gauss map of several surfaces such as helicoid, catenoid and right cones in $\mathbb{E}^3$, and also some hypersurfaces take the form

$$\Delta\nu = f(\nu + C) \tag{1}$$

*Corresponding author. *Email addresses:* `udursun@itu.edu.tr` (U. Dursun), `turgayn@itu.edu.tr` (N. C. Turgay)

for some smooth function f on M and some constant vector C. A submanifold of a Euclidean space is said to have *pointwise 1-type Gauss map* if it satisfies (1). A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if C is the zero vector. Otherwise, it is said to be of *the second kind.* A pointwise 1-type Gauss map is called *proper* if the function f defined by (1) is non-constant. A non-proper pointwise 1-type Gauss map is just usual 1-type Gauss map.

Remark 1. *For an n-dimensional plane M in a Euclidean space, the Gauss map ν is constant and $\Delta\nu = 0$. For $f = 0$ if we write $\Delta\nu = 0 \cdot \nu$, then M has pointwise 1-type Gauss map of the first kind. If we choose $C = -\nu$ for any nonzero smooth function f, then* (1) *holds. In this case, M has pointwise 1-type Gauss map of the second kind. Therefore we say that an n-dimensional plane M in a Euclidean space is a trivial submanifold with pointwise 1-type Gauss map of the first kind or the second kind.*

Surfaces and some hypersurfaces in Euclidean spaces with pointwise 1-type Gauss map were recently studied in [1, 8, 9, 10, 12, 13, 14, 15, 17].

In [14], the characterizations of surfaces in the Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map were given. Also, in [1], simple rotational surfaces in $\mathbb{E}^4$ whose meridian curves lie in 3-spaces were considered, and the meridian curve of the flat rotational surfaces with pointwise 1-type Gauss map of the second kind was described.

In this paper, we study general rotational surfaces in $\mathbb{E}^4$ with meridian curves lying in two-dimensional planes and pointwise 1-type Gauss map. We firstly prove that there exists no non-planar minimal general rotational surface with pointwise 1-type Gauss map of the first kind. Then we obtain all general rotational surfaces with proper pointwise 1-type Gauss map of the first kind which includes the results given in [19]. We also classify minimal general rotational surfaces of $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the second kind.

2. Preliminaries

Let M be an oriented n-dimensional submanifold in an $(n+2)$-dimensional Euclidean space $\mathbb{E}^{n+2}$. We choose an oriented local orthonormal frame $\{e_1, \dots, e_{n+2}\}$ on M such that $e_1, \dots, e_n$ are tangent to M and e_{n+1}, e_{n+2} are normal to M. We use the following convention on the range of indices: $1 \leq i, j, k, \dots \leq n$, $n+1 \leq r, s, t, \dots \leq n+2$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^{n+2}$ and ∇ the induced connection on M. Denote by $\{w^1, \dots, w^{n+2}\}$ the dual frame and by $\{w_B^A\}, A, B = 1, \dots, n+2$, the connection forms associated to $\{e_1, \dots, e_{n+2}\}$. Then we have

$$\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{n} w_i^j(e_k) e_j + \sum_{r=n+1}^{n+2} h_{ik}^r e_r, \ \widetilde{\nabla}_{e_k} e_s = -A_s(e_k) + \sum_{r=n+1}^{n+2} w_s^r(e_k) e_r$$

and

$$D_{e_k} e_s = \sum_{r=n+1}^{n+2} w_s^r(e_k) e_r,$$

where D is the normal connection, h_{ij}^r the coefficients of the second fundamental form h, and A_r the Weingarten map in the direction e_r.

The mean curvature vector H and the squared length $\|h\|^2$ of the second fundamental form h are defined, respectively, by

$$H = \frac{1}{n}\sum_{r,i} h_{ii}^r e_r \quad \text{and} \quad \|h\|^2 = \sum_{r,i,j} h_{ij}^r h_{ji}^r.$$

A submanifold M is said to have parallel mean curvature vector if the mean curvature vector satisfies $DH = 0$ identically.

The Codazzi equation of M in $\mathbb{E}^{n+2}$ is given by

$$\begin{aligned} &h_{ij,k}^r = h_{jk,i}^r, \\ &h_{jk,i}^r = e_i(h_{jk}^r) + \sum_{s=n+1}^{n+2} h_{jk}^s w_s^r(e_i) - \sum_{\ell=1}^{n} \left(w_j^\ell(e_i) h_{\ell k}^r + w_k^\ell(e_i) h_{j\ell}^r \right). \end{aligned} \tag{2}$$

Also, from the Ricci equation of M in $\mathbb{E}^{n+2}$, we have

$$R^D(e_j, e_k; e_r, e_s) = \langle [A_{e_r}, A_{e_s}](e_j), e_k \rangle = \sum_{i=1}^{n} \left(h_{ik}^r h_{ij}^s - h_{ij}^r h_{ik}^s \right), \tag{3}$$

where R^D is the normal curvature tensor.

Let M be an oriented n-dimensional submanifold of a Euclidean space $\mathbb{E}^m$. The map $\nu : M \to G(m-n, m)$ defined by $\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \cdots \wedge e_m)(p)$ is called the *Gauss map* of M that is a smooth map which carries a point $p \in M$ into the oriented $(m-n)$-plane in $\mathbb{E}^m$ which is obtained from the parallel translation of the normal space of M at p in $\mathbb{E}^m$, where $G(m-n, m)$ denotes the Grassmannian manifold consisting of all oriented $(m-n)$-planes through the origin of $\mathbb{E}^m$. Since $G(m-n, m)$ is canonically embedded in $\bigwedge^{m-n} \mathbb{E}^m = \mathbb{E}^N$, $N = \binom{m}{m-n}$, then the notion of the type of the Gauss map is naturally defined.

2.1. General rotational surfaces

In [16], Moore introduced general rotational surfaces in the Euclidean space $\mathbb{E}^4$. A rotational surface in $\mathbb{E}^4$ is a surface left invariant by a rotation in $\mathbb{E}^4$ which is defined as a linear transformation of positive determinant preserving distance and leaving one point fixed.

In [11], Cole studied the general theory of rotations in $\mathbb{E}^4$. In $\mathbb{E}^4$, two planes which have no line in common are called completely (or absolutely) perpendicular to each other. A rotation in general leaves two completely perpendicular planes invariant not fixed point for point, but only as planes. A rotation which leaves one of the invariant planes fixed point for point and converts the other invariant plane to itself is called a simple rotation. In general, every general rotation (also called double rotation) of $\mathbb{E}^4$ can be reduced to a succession of two simple rotations whose fixed planes are completely perpendicular to each other (for details see [11]). By a suitable isometry of $\mathbb{E}^4$, two completely perpendicular planes at a point in $\mathbb{E}^4$ can be transformed to completely perpendicular xy- and zw-planes at the origin of $\mathbb{E}^4$.

Let $\beta(s) = (x(s), y(s), z(s), w(s))$ be a regular smooth curve in $\mathbb{E}^4$ on an open interval I in $\mathbb{R}$, and let a and b be some real numbers. Then, considering the equations of the general rotation given in [11], a general rotational surface M in $\mathbb{E}^4$ with the meridian curve β is given by

$$X(s,t) = \Big(x(s)\cos at - y(s)\sin at,\ x(s)\sin at + y(s)\cos at, \\ z(s)\cos bt - w(s)\sin bt,\ z(s)\sin bt + w(s)\cos bt\Big), \tag{4}$$

where a and b are the rates of rotation in fixed planes of the rotation, [16]. If a or b is zero, then a surface M defined by (4) is a simple rotational surface as the rotation subgroup which produces M is a simple rotation.

Let M be a general rotational surface in $\mathbb{E}^4$ whose meridians lie in 2-planes. Then these planes of meridians are perpendicular to the two fixed planes of the rotation that generates the surface M. If M with planar meridians is parametrized by (4), then the planes of meridians are perpendicular to the invariant xy- and zw-planes of the rotation which generates the surface M. Therefore, without loss of generality, we can choose a meridian curve β of M in the xz-plane as $\beta(s) = (x(s), 0, z(s), 0)$, and thus a general rotational surface M in $\mathbb{E}^4$ whose meridians lie in 2-planes is given by the parametrization

$$F(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cos bt, z(s)\sin bt) \tag{5}$$

with the rates of rotation a and b, where $s \in I \subset \mathbb{R},\ t \in (0, 2\pi)$. Throughout this work we suppose that $ab \neq 0$. Since β is a regular smooth curve, parametrization (5) is an immersion if and only if $a^2x^2(s) + b^2z^2(s) > 0$ on I.

Moreover, a rotational surface in $\mathbb{E}^4$ defined by (5) for $a = b = 1$, $x(s) = u(s)\cos s$ and $z(s) = u(s)\sin s$ is also known as a Vranceanu rotational surface [18], where u is a differentiable function.

From now on, since β is a plane curve, without loss of generality, we consider β of the form $\beta(s) = (x(s), z(s))$ on some open interval I.

Suppose that s is the arc length parameter of β. Then, ${x'}^2(s) + {z'}^2(s) = 1$, and the curvature function κ of β is given by $\kappa(s) = x'(s)z''(s) - x''(s)z'(s),\ s \in I$.

Let M be a general rotational surface in $\mathbb{E}^4$ defined by (5). We consider the following orthonormal moving frame field $\{e_1, e_2, e_3, e_4\}$ on M such that e_1, e_2 are tangent to M, and e_3, e_4 are normal to M:

$$e_1 = \frac{\partial}{\partial s}, \quad e_2 = \frac{1}{\sqrt{a^2x^2 + b^2z^2}}\frac{\partial}{\partial t}, \tag{6}$$

$$e_3 = (-z'\cos at, -z'\sin at, x'\cos bt, x'\sin bt), \tag{7}$$

$$e_4 = \frac{1}{\sqrt{a^2x^2 + b^2z^2}}(-bz\sin at, bz\cos at, ax\sin bt, -ax\cos bt). \tag{8}$$

By a direct computation we have components of the second fundamental form and the connection forms as

$$h^3_{11} = \kappa, \quad h^3_{22} = \frac{a^2xz' - b^2zx'}{a^2x^2 + b^2z^2}, \quad h^3_{12} = 0, \tag{9}$$

$$h^4_{12} = \frac{ab(zx' - xz')}{a^2x^2 + b^2z^2}, \quad h^4_{11} = h^4_{22} = 0, \tag{10}$$

$$w^2_1(e_1) = 0, \quad w^2_1(e_2) = \frac{a^2xx' + b^2zz'}{a^2x^2 + b^2z^2}, \tag{11}$$

$$w^3_4(e_1) = 0, \quad w^3_4(e_2) = \frac{ab(xx' + zz')}{a^2x^2 + b^2z^2}. \tag{12}$$

Thus, the shape operators of M are of the form

$$A_3 = \begin{pmatrix} h^3_{11} & 0 \\ 0 & h^3_{22} \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & h^4_{12} \\ h^4_{12} & 0 \end{pmatrix}, \tag{13}$$

from which we obtain the mean curvature vector and the normal curvature of M as follows:

$$H = \frac{1}{2}(h^3_{11} + h^3_{22})e_3, \tag{14}$$

$$R^D(e_1, e_2; e_3, e_4) = h^4_{12}(h^3_{22} - h^3_{11}). \tag{15}$$

On the other hand, from Codazzi equation (2) we have

$$e_1(h^3_{22}) = w^1_2(e_2)\left(h^3_{22} - \kappa\right) + h^4_{12}w^3_4(e_2), \tag{16}$$

$$e_1(h^4_{12}) = 2w^1_2(e_2)h^4_{12} - \kappa w^3_4(e_2). \tag{17}$$

Remark 2. *If the meridian curve β of M is a line $z = c_0x$ passing through the origin, and the rates of rotation a and b hold $a^2 = b^2$, then the rotational surface M is given by $F(x,t) = (x\cos t, x\sin t, c_0x\cos t, \varepsilon c_0x\sin t)$, $x > 0$, $\varepsilon = a/b = \pm 1$. It can be easily shown that M is an open part of a plane in $\mathbb{E}^4$.*

3. General rotational surfaces with pointwise 1-type Gauss map of the first kind

In this section, we obtain all general rotational surfaces defined by (5) with pointwise 1-type Gauss map of the first kind.

The Laplacian of the Gauss map ν for an n-dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+2}$ was given by

Lemma 1 (See [14])**.** *Let M be an n-dimensional submanifold of Euclidean space $\mathbb{E}^{n+2}$. Then, the Laplacian of the Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned}\Delta\nu =& \|h\|^2\nu + 2\sum_{j<k} R^D(e_j, e_k; e_{n+1}, e_{n+2})e_j \wedge e_k \\ &+ n\sum_{j=1}^{n} w^{n+1}_{n+2}(e_j)e_j \wedge H + \nabla(\mathrm{tr}A_{n+1}) \wedge e_{n+2} - \nabla(\mathrm{tr}A_{n+2}) \wedge e_{n+1},\end{aligned} \tag{18}$$

where $\|h\|^2$ *is the squared length of the second fundamental form,* R^D *the normal curvature tensor, and* $\nabla(\mathrm{tr}A_r)$ *the gradient of* $\mathrm{tr}A_r$.

In [14], the following results were given for the characterization of surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind.

Theorem 1 (See [14])**.** *An oriented minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has flat normal bundle.*

Theorem 2 (See [14])**.** *An oriented non-minimal surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector in $\mathbb{E}^4$.*

We will classify rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind by using the above theorems.

Theorem 3. *Let M be a general rotational surface in $\mathbb{E}^4$ defined by (5) for the rates of rotation a and b. Then, M is minimal, and its normal bundle is flat if and only if M is an open part of a plane.*

Proof. Let M be a general rotational surface given by (5). Then, we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M in $\mathbb{E}^4$ given by (6)-(8), and the shape operators A_3 and A_4 are given by (13). If M is minimal, and its normal bundle is flat, then (14) and (15) imply, respectively,

$$\kappa + h^3_{22} = 0, \tag{19}$$

$$h^4_{12}(h^3_{22} - \kappa) = 0, \tag{20}$$

as $h^3_{11} = \kappa$, where κ is the curvature of the meridian curve of M. By using these equations we get $h^4_{12}\kappa = 0$ which implies either $\kappa = 0$ or $h^4_{12} = 0$.

Case 1. $\kappa = 0$. Then the meridian curve of M is a line. We may put

$$x(s) = x_1 s + x_2, \quad z(s) = z_1 s + z_2 \tag{21}$$

for some constants $x_1,\ x_2,\ z_1,\ z_2$ with $x_1^2 + z_1^2 = 1$. From (19) we also have $h^3_{22} = 0$. By using the second equation in (9) and (21) we obtain

$$h^3_{22} = \frac{(a^2 - b^2)x_1 z_1 s + (a^2 x_2 z_1 - b^2 x_1 z_2)}{a^2(x_1 s + x_2)^2 + b^2(z_1 s + z_2)^2} = 0$$

which yields

$$(a^2 - b^2)x_1 z_1 = 0, \tag{22}$$

$$a^2 x_2 z_1 - b^2 x_1 z_2 = 0. \tag{23}$$

Equation (22) implies either $a^2 - b^2 = 0$ or $x_1 z_1 = 0$. If $x_1 = 0$, then $z_1 = \pm 1$. Also from (23) we get $x_2 = 0$. Thus, $x = 0$, and M is an open part of the x_3x_4-plane because of (5). By a similar argument, if $z_1 = 0$, then M is an open part of the x_1x_2-plane.

Now, assume that $x_1z_1 \neq 0$ and $a^2 - b^2 = 0$. Then, (23) implies $x_2z_1 = x_1z_2$ from which and (21) we get $x_1z = z_1x$, i.e., line (21) is passing through the origin. In view of Remark 2, M is an open part of a plane.

Case 2. $h^4_{12} = 0$. From the first equation in (10) we have $xz' - x'z = 0$, i.e., $z = c_0x$, where c_0 is a constant. Hence, β is an open part of a line passing through the origin. Therefore M is an open part of a plane because of Remark 2.

The converse of the proof is trivial. □

From Theorem 1 and Theorem 3 we state

Theorem 4. *There exists no non-planar minimal general rotational surface in* $\mathbb{E}^4$ *defined by* (5) *with pointwise* 1*-type Gauss map of the first kind.*

In [20], Yoon studied flat Vranceanu rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the first kind. He proved that a flat Vranceanu rotational surface M in $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M is a Clifford torus in $\mathbb{E}^4$, that is, the product of two plane circles with the same radius.

Now we investigate non-minimal general rotational surfaces in $\mathbb{E}^4$ with parallel mean curvature vector to obtain surfaces in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the first kind. For this reason we prove

Theorem 5. *A non-minimal general rotational surface* M *in* $\mathbb{E}^4$ *defined by* (5) *has parallel mean curvature vector if and only if it is an open part of the surface defined by*

$$F(s,t)=\left(r_0\cos(\frac{s}{r_0})\cos at,\ r_0\cos(\frac{s}{r_0})\sin at,\ r_0\sin(\frac{s}{r_0})\cos bt,\ r_0\sin(\frac{s}{r_0})\sin bt\right) \quad (24)$$

which is minimal in $S^3(r_0) \subset \mathbb{E}^4$.

Proof. Let M be a non-minimal general rotational surface in $\mathbb{E}^4$ defined by (5). Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal moving frame on M in $\mathbb{E}^4$ given by (6)-(8). From (13) we have $H = \frac{1}{2}(h^3_{11} + h^3_{22})e_3$. Suppose that the mean curvature vector H is parallel, i.e., $DH = 0$. By considering (12), we obtain that

$$D_{e_2}H = -\frac{ab(h^3_{11} + h^3_{22})(xx' + zz')}{2(a^2x^2 + b^2z^2)}e_4 = 0.$$

Since M is non-minimal, this equation yields $xx' + zz' = 0$, i.e., $x^2 + z^2 = r_0^2$, where r_0 is a positive real number. Hence, the meridian curve β is an open part of a circle which is parametrized by

$$x(s) = r_0\cos\frac{s}{r_0}, \quad z(s) = r_0\sin\frac{s}{r_0}.$$

Therefore, M is an open part of the surface given by (24).

The converse follows from a direct calculation. □

By Theorem 2 and Theorem 5 we have

Corollary 1. *A non-minimal general rotational surface M in $\mathbb{E}^4$ defined by* (5) *has pointwise 1-type Gauss map of the first kind if and only if it is an open part of the surface given by* (24).

By computation we have

$$\|h\|^2 = \operatorname{tr}(A_3)^2 + \operatorname{tr}(A_4)^2 = \frac{2}{r_0^2}\Big(1 + \frac{a^2 b^2}{(a^2\cos^2\frac{s}{r_0} + b^2\sin^2\frac{s}{r_0})^2}\Big)$$

for the rotational surface (24).

By combining the results obtained in this section we state a classification theorem:

Theorem 6. *Let M be a general rotational surface in $\mathbb{E}^4$ defined by* (5)*. Then M has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a plane or a surface given by* (24)*. Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the rotational surface* (24) *satisfies* (1) *for the function*

$$f = \frac{2}{r_0^2}\Big(1 + \frac{a^2 b^2}{(a^2\cos^2\frac{s}{r_0} + b^2\sin^2\frac{s}{r_0})^2}\Big).$$

Corollary 2. *The only general rotational surface M in $\mathbb{E}^4$ defined by* (5) *with proper pointwise 1-type Gauss map of the first kind is the surface given by* (24) *for $a^2 \neq b^2$.*

In particular, if the rates of rotation a and b in (5) meet $a^2 = b^2$, then the rotational surface (24) is a Clifford torus in $\mathbb{E}^4$ which has (global) 1-type Gauss map of the first kind studied in [19, 20].

4. Minimal general rotational surfaces with pointwise 1-type Gauss map of the second kind

In [16], Moore proved that a general rotational surface M defined by (5) for $a = b = 1$ is minimal if and only if its meridian curve is an open part of the hyperbola

$$c_1(z^2 - x^2) + 2xz + c_2 = 0, \tag{25}$$

where c_1 and c_2 are some real numbers. A direct calculation shows that this result still holds if $a^2 = b^2$.

In [14], a characterization of minimal surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map of the second kind was given as follows:

Theorem 7 (See [14])**.** *A non-planar minimal oriented surface M in the Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by*

$$A_3 = \begin{pmatrix} \rho & 0 \\ 0 & -\rho \end{pmatrix} \quad \text{and} \quad A_4 = \begin{pmatrix} 0 & \varepsilon\rho \\ \varepsilon\rho & 0 \end{pmatrix}, \tag{26}$$

where $\varepsilon = \pm 1$ and ρ is a smooth non-zero function on M.

By using Theorem 7 we classify non-planar minimal general rotational surfaces in $\mathbb{E}^4$ defined by (5) with pointwise 1-type Gauss map of the second kind.

Theorem 8. *Let M be a non-planar general rotational surface in $\mathbb{E}^4$ defined by* (5) *for the rates of rotation a and b. Then,*

1. *if $a^2 = b^2$, then the minimal surface M whose meridian curve is given by* (25) *has proper pointwise 1-type Gauss map of the second kind.*
2. *if $a^2 \neq b^2$, then M is minimal and its Gauss map is of pointwise 1-type of the second kind if and only if the meridian curve of M is given by*

$$z = cx^{\mp b/a}, \quad x > 0 \tag{27}$$

for some real number $c \neq 0$.

Proof. Let M be a non-planar general rotational surface in $\mathbb{E}^4$ defined by (5). Then we have an orthonormal moving frame $\{e_1, e_2, e_3, e_4\}$ on M in $\mathbb{E}^4$ given by (6)-(8), and the shape operators A_3 and A_4 are given by (13). For $a^2 = b^2$, assume that M is minimal. Thus we have $h^3_{11} + h^3_{22} = 0$ which gives the differential equation

$$x'z'' - z'x'' + \frac{xz' - zx'}{x^2 + z^2} = 0$$

that has a general solution given by (25). Also, from the second equation in (9) and the first equation in (10) we have $(h^3_{22})^2 = (h^4_{12})^2$. If we put $\rho = h^3_{11}$, then $h^3_{22} = -\rho$ and $h^4_{12} = \varepsilon\rho$, where $\varepsilon = \pm 1$. Thus, the shape operators A_3 and A_4 are of the form (26). A direct calculation (or see the proof of Theorem 7) shows that the function f satisfying (1) is given by $f = 8\rho^2 = 8\kappa^2$ as $\rho = h^3_{11} = \kappa$ from (9). Since κ is not constant for the hyperbola given by (25), f is not a constant function. As a result M has proper pointwise 1-type Gauss map of the second kind by Theorem 7. This gives case 1 of the theorem.

Now, for $a^2 \neq b^2$ assume that a non-planar general rotational surface M in $\mathbb{E}^4$ defined by (5) is minimal and its Gauss map $\nu = e_3 \wedge e_4$ is of pointwise 1-type of the second kind. Then, Theorem 7 implies that the shape operators A_3 and A_4 of M are of the form (26). Hence we have $h^3_{11} + h^3_{22} = 0$ and $h^4_{12} = \varepsilon h^3_{11} = -\varepsilon h^3_{22}$, where $\varepsilon = \pm 1$.

From the second equation in (9) and the first equation in (10) it is seen that $h^4_{12} = -\varepsilon h^3_{22}$ implies the differential equation $axz' = -\varepsilon bzx'$ as $a^2 \neq b^2$, and its solution gives (27).

Conversely, suppose that the meridian curve of the rotational surface M is given by (27). We will show that the shape operators A_3 and A_4 of M are of the form (26).

From (27) if we write $z = cx^{-\varepsilon b/a}$, then we have $axz' = -\varepsilon bzx'$ from which, the second equation in (9) and the first equation in (10) it is seen that $h^4_{12} = -\varepsilon h^3_{22}$. Now, let us show that the minimality condition holds, i.e., $h^3_{11} + h^3_{22} = 0$ or equivalently, $h^3_{11} - \varepsilon h^4_{12} = 0$. Using the second equation in (9) and the first equation in (10), the equation $h^3_{11} - \varepsilon h^4_{12} = 0$ produces the differential equation

$$x'z'' - z'x'' + \frac{\varepsilon ab(xz' - zx')}{a^2x^2 + b^2z^2} = 0$$

which is expressed as

$$\frac{d}{ds}\Big(\tan^{-1}\Big(\frac{z'}{x'}\Big)\Big)+\varepsilon\frac{d}{ds}\Big(\tan^{-1}\Big(\frac{bz}{ax}\Big)\Big)=0 \tag{28}$$

because of ${x'}^2+{z'}^2=1$. Since $\tan^{-1}$ is an odd function, it is easily seen that the equation $axz'=-\varepsilon bzx'$ which produces (27) satisfies (28). That is, the minimality condition holds.

If we put $\rho=h_{11}^3$, then $h_{22}^3=-\rho$ and $h_{12}^4=\varepsilon\rho$. Thus, the shape operators A_3 and A_4 are of the form (26). Therefore, M is minimal and its Gauss map is of pointwise 1-type of the second kind by Theorem 7. By a direct calculation it is easy to show that the Gauss map is of proper pointwise 1-type of the second kind. This completes the proof of case 2. □

Here, using (25) and (27) we give two examples of a general rotational surface in $\mathbb{E}^4$ which are minimal and have proper pointwise 1-type Gauss map of the second kind.

Example 1. *For $c_1=0$ and $c_2=-1$ in* (25) *we have the hyperbola $2xz=1$ or equivalently $x^2-z^2=1$. Let $x=\cosh u$, $z=\sinh u$ be the parametrization of the right-hand branch of the hyperbola $x^2-z^2=1$. Then, the general rotational surface M defined by*

$$F(u,t)=(\cosh u\cos t,\cosh u\sin t,\sinh u\cos t,\sinh u\sin t)$$

is minimal in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the second kind. Moreover, following the proof of Theorem 7, the Gauss map $\nu=e_3\wedge e_4$ satisfies (1) *for the function $f=8\operatorname{sech}^3(2u)$ and for the constant vector $C=-\frac{1}{2}e_1\wedge e_2-\frac{1}{2}e_3\wedge e_4$ for some suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$ on M.*

Example 2. *If we choose $a=1$, $b=2$ and $z=x^2$ from* (27)*, then the general rotational surface M defined by*

$$F(x,t)=(x\cos t,x\sin t,x^2\cos 2t,x^2\sin 2t),\ x>0$$

is minimal in $\mathbb{E}^4$ with proper pointwise 1-type Gauss map of the second kind. Also, the Gauss map $\nu=e_3\wedge e_4$ satisfies (1) *for the function $f=\frac{32}{(1+4x^2)^3}$ and for the constant vector $C=\frac{1}{2}e_1\wedge e_2-\frac{1}{2}e_3\wedge e_4$ for some suitable orthonormal frame $\{e_1,e_2,e_3,e_4\}$ on M.*

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