Math. Commun. **17**(2012), 83–91

Voronovskaja type theorem for the Lupaş q-analogue of the Bernstein operators

Nazim Idrisoglu Mahmudov [1] and Pembe Sabancigil[1,*]

[1] *Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey*

Received October 18, 2010; accepted March 15, 2011

Abstract. In this paper, we estimate the third and the fourth order central moments for the difference of the Lupaş q-analogue of the Bernstein operator and the limit q-Lupaş operator. We also prove a quantitative variant of Voronovskaja's theorem for $R_{n,q}$.

AMS subject classifications: 65D10, 92C45

Key words: q-Bernstein polynomials, Lupaş q-analogue, limit q-Lupaş operator, Voronovskaja-type formulas

1. Introduction

Let $q > 0$. For any $n \in N \cup \{0\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + ... + q^{n-1}, \quad [0] := 0;$$

and the q-factorial $[n]! = [n]_q!$ by

$$[n]! := [1]\,[2] \ldots [n], \qquad [0]! := 1.$$

For integers $0 \le k \le n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\,[n-k]!} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix} = 0 \;\; \text{for } k > n.$$

In the last two decades interesting generalizations of the Bernstein polynomials based on the q-integers were proposed by Lupaş [5]

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)...(1-x+q^{n-1}x)}$$

and by Phillips [12]

$$B_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1 - q^s x).$$

*Corresponding author. *Email addresses:* `nazim.mahmudov@emu.edu.tr` (N. I. Mahmudov), `pembe.sabancigil@emu.edu.tr` (P. Sabancıgil)

The Phillips q-analogue of the Bernstein polynomials ($B_{n,q}$) attracted a lot of interest and was studied widely by a number of authors, see [2]-[4], [6]-[9], [12]-[17]. A survey of the obtained results and references on the subject can be found in [9]. The Lupaş operators ($R_{n,q}$) are less known, see [1, 10, 11, 18]. However, they have an advantage of generating positive linear operators for all $q > 0$, whereas Phillips polynomials generate positive linear operators only if $q \in (0, 1)$. Lupaş [5] investigated approximating properties of the operators $R_{n,q}(f, x)$ with respect to the uniform norm of $C[0, 1]$. In particular, he obtained some sufficient conditions for a sequence $\{R_{n,q}(f, x)\}$ to be approximating for any function $f \in C[0, 1]$ and estimated the rate of convergence in terms of the modulus of continuity. He also investigated behavior of the operators $R_{n,q}(f, x)$ for convex functions. In [10], several results on convergence properties of the sequence $\{R_{n,q}(f, x)\}$ are presented. In particular, it is proved that the sequence $\{R_{n,q_n}(f, x)\}$ converges uniformly to $f(x)$ on $[0, 1]$ if and only if $q_n \to 1$. On the other hand, for any $q > 0$ fixed, $q \neq 1$, the sequence $\{R_{n,q}(f, x)\}$ converges uniformly to $f(x)$ if and only if $f(x) = ax + b$ for some $a, b \in R$. In [18], the estimates for the rate of convergence of $R_{n,q}(f, x)$ by the modulus of continuity of f are obtained.

The paper is organized as follows. In Section 2, we estimate the third and the fourth order central moments for the difference of the Lupaş q-analogue of the Bernstein operator and the limit q-Lupaş operator. In Section 3, we discuss Voronovskaja-type theorems for the Lupaş q-analogue of the Bernstein operator for arbitrary fixed $q > 0$. Moreover, for the Voronovskaja's asymptotic formula we obtain the estimate of the remainder term.

2. Auxiliary results

It will be convenient to use the following transformations for $x \in [0, 1)$

$$v\left(q^j, x\right) := \frac{q^j x}{1 - x + q^j x}, \qquad v\left(q, v\left(q^j, x\right)\right) = v\left(q^{j+1}, x\right), \qquad j = 0, 1, 2, ...$$

Let $0 < q < 1$. We set

$$b_{nk}(q; x) := \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1 - x + qx)...(1 - x + q^{n-1}x)}, \quad x \in [0, 1],$$

$$b_{\infty k}(q; x) := \frac{q^{\frac{k(k-1)}{2}} \left(x/1-x\right)^k}{(1-q)^k\, [k]! \prod_{j=0}^{\infty}(1 + q^j\left(x/1-x\right))}, \quad x \in [0, 1).$$

It was proved in [5] and [10] that for $0 < q < 1$ and $x \in [0, 1)$,

$$\sum_{k=0}^{n} b_{nk}(q; x) = \sum_{k=0}^{\infty} b_{\infty k}(q; x) = 1.$$

Definition 1 (See [5]). *The linear operator $R_{n,q} : C[0,1] \to C[0,1]$ defined by*

$$R_{n,q}(f,x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) b_{nk}(q;x)$$

is called the q-analogue of the Bernstein operator.

Definition 2. *The linear operator defined on $C[0,1]$ given by*

$$R_{\infty,q}(f,x) = \begin{cases} \sum\limits_{k=0}^{\infty} f\left(1-q^k\right) b_{\infty k}(q;x) & \text{if} \quad x \in [0,1), \\ f(1) & \text{if} \quad x = 1. \end{cases}$$

is called the limit q-Lupaş operator.

It follows directly from the definition that operators $R_{n,q}(f,x)$ possess the end-point interpolation property, that is,

$$R_{n,q}(f,0) = f(0), \qquad R_{n,q}(f,1) = f(1)$$

for all $q > 0$ and all $n = 1, 2, \ldots$.

Lemma 1. *We have*

$$b_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{n-k-1} \left(1 - v\left(q^{k+j}, x\right)\right), \qquad x \in [0,1],$$

$$b_{\infty k}(q;x) = \frac{1}{(1-q)^k [k]!} \prod_{j=0}^{k-1} v\left(q^j, x\right) \prod_{j=0}^{\infty} \left(1 - v\left(q^{k+j}, x\right)\right), \qquad x \in [0,1].$$

It was proved in [5] and [10] that $R_{n,q}(f,x)$, $R_{\infty,q}(f,x)$ reproduce linear functions and $R_{n,q}\left(t^2,x\right)$ and $R_{\infty,q}\left(t^2,x\right)$ were explicitly evaluated. Using Lemma 1 we may write formulas for $R_{n,q}\left(t^2,x\right)$ and $R_{\infty,q}\left(t^2,x\right)$ in the compact form.

Lemma 2. *We have*

$$\begin{aligned} R_{n,q}(1,x) &= 1, R_{n,q}(t,x) = x, R_{\infty,q}(1,x) = 1, R_{\infty,q}(t,x) = x, \\ R_{n,q}\left(t^2,x\right) &= xv(q,x) + \frac{x(1-v(q,x))}{[n]}, \\ R_{\infty,q}\left(t^2,x\right) &= xv(q,x) + (1-q)x(1-v(q,x)) = x - qx(1-v(q,x)). \end{aligned}$$

Now define

$$L_{n,q}(f,x) := R_{n,q}(f,x) - R_{\infty,q}(f,x).$$

Theorem 1. *The following recurrence formulae hold*

$$R_{n,q}\left(t^{m+1},x\right) = R_{n,q}\left(t^m,x\right) - (1-x)\frac{[n-1]^m}{[n]^m} R_{n-1,q}\left(t^m, v(q,x)\right), \tag{1}$$

$$R_{\infty,q}\left(t^{m+1},x\right) = R_{\infty,q}\left(t^m,x\right) - (1-x) R_{\infty,q}\left(t^m, v(q,x)\right), \tag{2}$$

$$\begin{aligned} L_{n,q}(t^{m+1},x) &= L_{n,q}(t^m,x) + (1-x) \\ &\times \left(\left(1 - \frac{[n-1]^m}{[n]^m}\right) R_{\infty,q}(t^m, v(q,x)) - \frac{[n-1]^m}{[n]^m} L_{n-1,q}(t^m, v(q,x))\right). \end{aligned} \tag{3}$$

Proof. First we prove (1). We write explicitly

$$R_{n,q}(t^{m+1},x)=\sum_{k=0}^{n}\frac{[k]^{m+1}}{[n]^{m+1}}\begin{bmatrix} n\\ k\end{bmatrix}\prod_{j=0}^{k-1}v\left(q^{j},x\right)\prod_{j=0}^{n-k-1}\left(1-v\left(q^{k+j},x\right)\right) \tag{4}$$

and rewrite the first two factors in the following form:

$$\begin{aligned}\frac{[k]^{m+1}}{[n]^{m+1}}\begin{bmatrix} n\\ k\end{bmatrix} &= \frac{[k]^{m}}{[n]^{m}}\left(1-q^{k}\frac{[n-k]}{[n]}\right)\begin{bmatrix} n\\ k\end{bmatrix}\\ &= \frac{[k]^{m}}{[n]^{m}}\begin{bmatrix} n\\ k\end{bmatrix}-\frac{[n-1]^{m}}{[n]^{m}}\frac{[k]^{m}}{[n-1]^{m}}\begin{bmatrix} n-1\\ k\end{bmatrix}q^{k}. \end{aligned}\tag{5}$$

Finally, if we substitute (5) in (4) we get (1):

$$\begin{aligned}R_{n,q}(t^{m+1},x) &= \sum_{k=0}^{n}\frac{[k]^{m}}{[n]^{m}}\begin{bmatrix} n\\ k\end{bmatrix}\prod_{j=0}^{k-1}v\left(q^{j},x\right)\prod_{j=0}^{n-k-1}\left(1-v\left(q^{k+j},x\right)\right)\\ &\quad -\frac{[n-1]^{m}}{[n]^{m}}\left(1-x\right)\sum_{k=0}^{n-1}\frac{[k]^{m}}{[n-1]^{m}}\begin{bmatrix} n-1\\ k\end{bmatrix}\\ &\quad \times\prod_{j=0}^{k-1}v\left(q^{j},v\left(q,x\right)\right)\prod_{j=0}^{n-k-2}\left(1-v\left(q^{k+j},v\left(q,x\right)\right)\right)\\ &= R_{n,q}\left(t^{m},x\right)-\frac{[n-1]^{m}}{[n]^{m}}\left(1-x\right)R_{n-1,q}\left(t^{m},v\left(q,x\right)\right).\end{aligned}$$

Next we prove (3)

$$\begin{aligned}L_{n,q}(t^{m+1},x) &= R_{n,q}\left(t^{m+1},x\right)-R_{\infty,q}\left(t^{m+1},x\right)\\ &= R_{n,q}\left(t^{m},x\right)-\left(1-x\right)\frac{[n-1]^{m}}{[n]^{m}}R_{n-1,q}\left(t^{m},v\left(q,x\right)\right)\\ &\quad -R_{\infty,q}\left(t^{m},x\right)+\left(1-x\right)R_{\infty,q}\left(t^{m},v\left(q,x\right)\right)\\ &= L_{n,q}(t^{m},x)+\left(1-x\right)\\ &\quad \times\left(\left(1-\frac{[n-1]^{m}}{[n]^{m}}\right)R_{\infty,q}\left(t^{m},v\left(q,x\right)\right)-\frac{[n-1]^{m}}{[n]^{m}}L_{n-1,q}\left(t^{m},v\left(q,x\right)\right)\right).\end{aligned}$$

Formula (2) can be obtained from (1), by taking the limit as $n\to\infty$. $\square$

Moments $R_{n,q}\left(t^{m},x\right)$, $R_{\infty,q}\left(t^{m},x\right)$ are of particular importance in the theory of approximation by positive operators. In what follows we need explicit formulas for moments $R_{n,q}\left(t^{3},x\right)$, $R_{\infty,q}\left(t^{3},x\right)$.

Lemma 3. *We have*

$$\begin{aligned}R_{n,q}\left(t^{3},x\right) &= xv\left(q,x\right)+\frac{x\left(1-v\left(q,x\right)\right)}{[n]^{2}}-\frac{[n-1]\left[n-2\right]q^{2}}{[n]^{2}}x\left(1-v\left(q,x\right)\right)v\left(q^{2},x\right),\\ R_{\infty,q}\left(t^{3},x\right) &= xv\left(q,x\right)+\left(1-q\right)^{2}x\left(1-v\left(q,x\right)\right)-q^{2}x\left(1-v\left(q,x\right)\right)v\left(q^{2},x\right).\end{aligned}$$

Proof. Note that explicit formulas for $R_{n,q}(t^m,x)$, $R_{\infty,q}(t^m,x)$, $m=0,1,2$ were proved in [5, 10]. Now we prove an explicit formula for $R_{n,q}\left(t^3,x\right)$, since the formula for $R_{\infty,q}\left(t^3,x\right)$ can be obtained by taking limit as $n\to\infty$. The proof is based on the recurrence formula (1). Indeed,

$$\begin{aligned}
R_{n,q}\left(t^3,x\right) &= R_{n,q}\left(t^2,x\right)-(1-x)\frac{[n-1]^2}{[n]^2}R_{n-1,q}\left(t^2,v(q,x)\right)\\
&= xv(q,x)+\frac{x(1-v(q,x))}{[n]}-(1-x)\frac{[n-1]^2}{[n]^2}v(q,x)\,v\left(q^2,x\right)\\
&\quad -(1-x)\frac{[n-1]}{[n]^2}v(q,x)+(1-x)\frac{[n-1]}{[n]^2}v(q,x)\,v\left(q^2,x\right).
\end{aligned}$$

Using the identity $(1-x)\,v(q,x)=qx(1-v(q,x))$ we obtain

$$\begin{aligned}
R_{n,q}\left(t^3,x\right) &= xv(q,x)+\frac{x(1-v(q,x))}{[n]}\left(1-\frac{q[n-1]}{[n]}\right)\\
&\quad -\frac{[n-1]}{[n]^2}\left([n-1]-1\right)qx(1-v(q,x))\,v\left(q^2,x\right)\\
&= xv(q,x)+\frac{x(1-v(q,x))}{[n]^2}-\frac{[n-1]\,[n-2]\,q^2}{[n]^2}x(1-v(q,x))\,v\left(q^2,x\right).
\end{aligned}$$

□

In order to prove the Voronovskaja type theorem for $R_{n,q}(f,x)$ we also need explicit formulas and inequalities for $L_{n,q}(t^m,x)$, $m=2,3,4$.

Lemma 4. *Let $0<q<1$. Then*

$$L_{n,q}(t^2,x) = \frac{q^n}{[n]}x(1-v(q,x)), \tag{6}$$

$$\begin{aligned}
L_{n,q}(t^3,x) &= \frac{q^n}{[n]^2}x(1-v(q,x))\\
&\quad\times\left[2-q^n+[n-1](1+q)\,v\left(q^2,x\right)+[n]\,qv\left(q^2,x\right)\right],
\end{aligned}\tag{7}$$

$$L_{n,q}(t^4,x) = \frac{q^n}{[n]^2}x(1-v(q,x))\,M\left(q,v\left(q^2,x\right),v\left(q^3,x\right)\right), \tag{8}$$

where M is a function of $\left(q,v\left(q^2,x\right),v\left(q^3,x\right)\right)$.

Proof. First we find a formula for $L_{n,q}\left(t^3,x\right)$. To do this we use the recurrence formula (3):

$$\begin{aligned}
L_{n,q}\left(t^3,x\right) &= L_{n,q}\left(t^2,x\right)+(1-x)\\
&\quad\times\left[\left(1-\frac{[n-1]^2}{[n]^2}\right)R_{\infty,q}\left(t^2,v(q,x)\right)-\frac{[n-1]^2}{[n]^2}L_{n-1,q}\left(t^2,v(q,x)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^n}{[n]} x\left(1-v\left(q,x\right)\right) + (1-x)\left(1-\frac{[n-1]^2}{[n]^2}\right)\left[(1-q)\,v\left(q,x\right)+qv\left(q,x\right)v\left(q^2,x\right)\right] \\
&\quad -(1-x)\frac{[n-1]^2}{[n]^2}\frac{q^{n-1}}{[n-1]}v\left(q,x\right)\left(1-v\left(q^2,x\right)\right) \\
&= \frac{q^n}{[n]^2} x\left(1-v\left(q,x\right)\right) \\
&\quad \times\left[[n]+\left(\frac{[n]^2-[n-1]^2}{q^{n-1}}\right)\left(1-q+qv\left(q^2,x\right)\right)-[n-1]\left(1-v\left(q^2,x\right)\right)\right] \\
&= \frac{q^n}{[n]^2} x\left(1-v\left(q,x\right)\right) \\
&\quad \times\left[[n]+([n-1]+[n])\left(1-q+qv\left(q^2,x\right)\right)-[n-1]\left(1-v\left(q^2,x\right)\right)\right] \\
&= \frac{q^n}{[n]^2} x\left(1-v\left(q,x\right)\right) \\
&\quad \times\left[[n]+1-q^{n-1}+1-q^n+[n-1](1+q)\,v\left(q^2,x\right)+[n]\,qv\left(q^2,x\right)-[n-1]\right] \\
&= \frac{q^n}{[n]^2} x\left(1-v\left(q,x\right)\right)\left[2-q^n+[n-1](1+q)\,v\left(q^2,x\right)+[n]\,qv\left(q^2,x\right)\right].
\end{aligned}
$$

The proof of equation (8) is also elementary, but tedious and complicated. Just notice that we use the recurrence formula for $L_{n,q}\left(t^4,x\right)$ and clearly each term of the formula contains $\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right)$. □

Lemma 5. *We have*

$$
\begin{aligned}
L_{n,q}\left((t-x)^2,x\right) &= \frac{q^n}{[n]}x\left(1-v\left(q,x\right)\right), && (9)\\
L_{n,q}\left((t-x)^3,x\right) &= \frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right) && (10)\\
&\quad \times\left[2-q^n+[n-1](1+q)\,v\left(q^2,x\right)+[n]\,qv\left(q^2,x\right)-3\,[n]\,x\right], \\
L_{n,q}\left((t-x)^4,x\right) &\leq K_1\left(q\right)\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right), && (11)
\end{aligned}
$$

where $K_1\left(q\right)$ is a positive constant which depends on q.

Proof. Proofs of (10) and (11) are based on (7), (8) and on the following identities:

$$
\begin{aligned}
L_{n,q}\left((t-x)^3,x\right) &= L_{n,q}\left(t^3,x\right)-3xL_{n,q}\left((t-x)^2,x\right), \\
L_{n,q}\left((t-x)^4,x\right) &= L_{n,q}\left(t^4,x\right)-4xL_{n,q}\left((t-x)^3,x\right)-6x^2L_{n,q}\left((t-x)^2,x\right).
\end{aligned}
$$

□

3. Voronovskaja type results

Theorem 2. *Let* $0 < q < 1$, $f \in C^2[0,1]$. *Then there exists a positive constant* $K(q)$ *such that*

$$\left|\tfrac{[n]}{q^n}\left(R_{n,q}(f,x) - R_{\infty,q}(f,x)\right) - \tfrac{f''(x)}{2}x\left(1 - v\left(q,x\right)\right)\right| \\ \leq K\left(q\right)x\left(1 - v\left(q,x\right)\right)\omega(f'',[n]^{-\frac{1}{2}}). \tag{12}$$

Proof. Let $x \in (0,1)$ be fixed. We set

$$g(t) = f(t) - \left(f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2\right).$$

It is known that (see [5]) if the function h is convex on $[0,1]$, then

$$R_{n,q}(h,x) \geq R_{n+1,q}(h,x) \geq ... \geq R_{\infty,q}(h,x),$$

and therefore,

$$L_{n,q}(h,x) := R_{n,q}(h,x) - R_{\infty,q}(h,x) \geq 0.$$

Thus $L_{n,q}$ is positive on the set of convex functions on $[0,1]$. But in general $L_{n,q}$ is not positive on $C\left[0,1\right]$.

Simple calculation gives

$$L_{n,q}\left(g,x\right) = \left(R_{n,q}(f,x) - R_{\infty,q}(f,x)\right) - \frac{q^n}{[n]}\frac{f''(x)}{2}x\left(1 - v\left(q,x\right)\right).$$

In order to prove the theorem, we need to estimate $L_{n,q}\left(g,x\right)$. To do this, it is enough to choose a function $S\left(t\right)$ such that functions $S\left(t\right) \pm g\left(t\right)$ are convex on $[0,1]$. Then $L_{n,q}\left(S \pm g, x\right) \geq 0$, and therefore,

$$\left|L_{n,q}\left(g(t),x\right)\right| \leq L_{n,q}\left(S(t),x\right).$$

So the first thing to do is to find such function $S\left(t\right)$. Using the well-known inequality $\omega(f,\lambda\delta) \leq (1+\lambda^2)\omega(f,\delta)$ $(\lambda,\delta > 0)$, we get

$$\begin{aligned}|g''(t)| &= |f''(t) - f''(x)| \leq \omega(f'',|t-x|) \\ &= \omega\left(f'',\frac{1}{[n]^{\frac{1}{2}}}[n]^{\frac{1}{2}}|t-x|\right) \leq \omega\left(f'',\frac{1}{[n]^{\frac{1}{2}}}\right)\left(\left(1+[n]\left(t-x\right)^2\right)\right).\end{aligned}$$

Define $S\left(t\right) = \omega\left(f'',[n]^{-\frac{1}{2}}\right)\left[\frac{1}{2}(t-x)^2 + \frac{1}{12}[n]\left(t-x\right)^4\right]$. Then

$$|g''(t)| \leq \frac{1}{6}\omega\left(f'',[n]^{-\frac{1}{2}}\right)\left(3(t-x)^2 + \frac{1}{2}[n]\left(t-x\right)^4\right)''_t = S''(t).$$

Hence functions $S(t) \pm g(t)$ are convex on $[0,1]$, and therefore,

$$\left|L_{n,q}\left(g(t),x\right)\right| \leq L_{n,q}\left(S(t),x\right),$$

and

$$L_{n,q}\left(S(t),x\right)=\frac{1}{6}\omega\left(f'',[n]^{-\frac{1}{2}}\right)\left(\frac{3q^n}{[n]}x\left(1-v\left(q,x\right)\right)+\frac{1}{2}\left[n\right]L_{n,q}\left((t-x)^4,x\right)\right).$$

Since by formula (11)

$$L_{n,q}\left((t-x)^4,x\right)\leq K_1\left(q\right)\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right), \tag{13}$$

we have

$$L_{n,q}\left(S(t),x\right)\leq\frac{1}{6}\omega\left(f'',[n]^{-\frac{1}{2}}\right)$$
$$\times\left(3\frac{q^n}{[n]}x\left(1-v\left(q,x\right)\right)+\frac{1}{2}\left[n\right]K_1\left(q\right)\frac{q^n}{[n]^2}x\left(1-v\left(q,x\right)\right)\right). \tag{14}$$

By (13) and (14), we obtain (12). The theorem is proved. □

When $q>1$, the following relations (see [10], Theorem 3) allow us to reduce to the case $q\in(0,1)$.

$$R_{n,q}\left(f,x\right)=R_{n,\frac{1}{q}}\left(g,1-x\right),$$

where $g\left(x\right)=f\left(1-x\right)$. For $q>1$, the limit q-Lupaş operator is defined by

$$R_{\infty,q}\left(f,x\right)=\begin{cases}\sum_{k=0}^{\infty}f\left(1/q^k\right)b_{\infty k}(1/q;1-x) & \text{if} \quad x\in(0,1],\\ f\left(0\right) & \text{if} \quad x=0.\end{cases}$$

Corollary 1. *Let $q>1$, $f\in C^2[0,1]$. Then there exists a positive constant $K\left(q\right)$ such that*

$$\left|q^n\left[n\right]_{\frac{1}{q}}\left(R_{n,q}(f,x)-R_{\infty,q}(f,x)\right)-\frac{f''(1-x)}{2}v\left(q,x\right)\left(1-x\right)\right|$$
$$\leq K\left(q\right)v\left(q,x\right)\left(1-x\right)\omega(g'',[n]_{\frac{1}{q}}^{-\frac{1}{2}}).$$

Remark 1. *For the function $f(t)=t^2$, the exact equality*

$$\frac{[n]_q}{q^n}\left(R_{n,q}(t^2,x)-R_{\infty,q}(t^2,x)\right)=x\left(1-v\left(q,x\right)\right), \qquad 0<q<1,$$
$$q^n\left[n\right]_{\frac{1}{q}}\left(R_{n,q}(t^2,x)-R_{\infty,q}(t^2,x)\right)=v\left(q,x\right)\left(1-x\right), \qquad q>1,$$

takes place without passing to the limit, but in contrast to the Phillips q-analogue of the Bernstein polynomials, the right-hand side depends on q. In contrast to classical Bernstein polynomials and the Phillips q-analogue of the Bernstein polynomials, the exact equality

$$[n]\left(B_{n,q}(t^2,x)-x^2\right)=\left(x^2\right)''x\left(1-x\right)/2$$

does not hold for the Lupaş q-analogue of the Bernstein polynomials.

Acknowledgments

The authors would like to thank the referees for their helpful suggestions.

References

[1] O. Agratini, *On certain q-analogues of the Bernstein operators*, Carpathian J. Math. **24**(2008), 281-286

[2] T. N. T. Goodman, H. Oruc, G. M. Phillips, *Convexity and generalized Bernstein polynomials*, Proc. Edinb. Math. Soc. (2) **42**(1999), 179–190.

[3] A. Ilinskii, S. Ostrovska, *Convergence of generalized Bernstein polynomials*, J. Approx. Theory **116**(2002), 100–112.

[4] S. Lewanowicz, P. Wozny, *Generalized Bernstein polynomials*, BIT **44**(2004), 63–78.

[5] A. Lupa, *A q-analogue of the Bernstein operator*, University of Cluj-Napoca, Seminar on numerical and statistical calculus, 1987.

[6] N. I. Mahmudov, *Korovkin-type Theorems and Applications*, Cent. Eur. J. Math. **7**(2009), 348–356.

[7] F. Meng, H. Wang, *The rate of convergence of q-Bernstein polynomials for* $0 < q < 1$, J. Approx. Theory **136**(2005), 151–158.

[8] H. Oruc, G. M. Phillips, *A generalization of the Bernstein polynomials*, Proc. Edinb. Math. Soc. (2) **42**(1999), 403–413.

[9] S. Ostrovska, *The first decade of the q-Bernstein polynomials: results and perspectives*, J. Math. Anal. Approx. Theory **2**(2007), 35–51.

[10] S. Ostrovska, *On the Lupaş q-analogue of the Bernstein operator*, Rocky Mountain J. Math. **36**(2006), 1615–1629.

[11] S. Ostrovska, *On the Lupaş q-trnasform*, Comput. Math. Appl. **61**(2011), 527-532.

[12] G. M. Phillips, *Bernstein polynomials based on the q-integers*, Ann. Numer. Math. **4**(1997), 511–518.

[13] G. M. Phillips, *On generalized Bernstein polynomials*, in: *Numerical analysis: A. R. Mitchell 75th Birthday Volume*, (World Sci. Publ., River Edge, NJ, 1996) 263–269.

[14] V. S. Videnskii, *On some classes of q-parametric positive linear operators*, Oper. Theory Adv. Appl. **158**(2005), 213–222.

[15] H. Wang, *Voronovskaya-type formulas and saturation of convergence for q-Bernstein polynomials for* $0 < q < 1$, J. Approx. Theory **145**(2007), 182–195.

[16] H. Wang, *Properties of convergence for the q-Meyer-König and Zeller operators*, J. Math. Anal. Appl. **335** (2007), 1360–1373.

[17] H. Wang, *Korovkin-type theorem and application*, J. Approx. Theory **132** (2005), 258-264.

[18] H. Wang, Y. Zhang, *The rate of convergence for the Lupaş q-analogue of the Bernstein operator*, Preprint 08/312, Middle East Technical University, Ankara, Turkey, 2008.