$k-$type partially null and pseudo null slant helices in Minkowski 4-space

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Abstract. We introduce the notion of a k-type slant helix in Minkowski space $\mathbf{E}_1^4$. For partially null and pseudo null curves in $\mathbf{E}_1^4$, we express some characterizations in terms of their curvature and torsion functions.
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1. Introduction

The notion of a slant helix is due to Izumiya and Takeuchi [12]. A curve α with non-vanishing curvature is called a slant helix in Euclidean space $\mathbf{E}^3$ if the principal normal lines of α make a constant angle with a fixed direction of the ambient space. Later, spherical images, the tangent and the binormal indicatrix and some characterizations of such curves were presented in [14]. Recently, further characterizations and position vectors of such curves are given in [1, 15, 17].

In recent years, by the coming of theory of relativity, researchers extended some topics of classical differential geometry to Lorentzian manifolds and there exists extensive literature on this subject. For instance, for slant helices on Lorentzian geometry, we refer to [2, 3, 4, 5, 8, 9, 10, 11, 13, 18].

Now we focus on Minkowski 4-dimensional space $\mathbf{E}_1^4$, that is, the real vector 4-dimensional space $\mathbf{R}^4$ equipped with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in $\mathbf{R}^4$. A spacelike curve $\alpha : I \subset \mathbf{R} \to \mathbf{E}_1^4$ is called spacelike if the induced metric is Riemannian. For spacelike curves parameterized by the length-arc, one has defined a Frenet frame $\{V_1, \dots, V_4\}$, where $V_1(s) = \alpha'(s)$. In Minkowski 4-dimensional space $\mathbf{E}_1^4$, we extend the concept of a slant helix as follows:

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Definition 1. *Let $\alpha : I \to \mathbf{E}_1^4$ be a spacelike curve, with Frenet frame $\{V_1, V_2, V_3, V_4\}$. We say that α is a k-type slant helix if there exists a (non-zero) constant vector field $U \in \mathbf{E}_1^4$ such that $g(V_{k+1}, U)$ is constant, for $0 \le k \le 3$. The vector U is called an* axis *of the curve.*

In particular, 0-type slant helices are general helices and 1-type slant helices are slant helices. In this work we consider k-type slant helices for partially null curves and pseudo null curves. Recall that a partially null curve is a (spacelike) curve where V_2 is spacelike and V_3 is a lightlike vector. On the other hand, a pseudo null curve is a (spacelike) curve if V_2 is a lightlike vector. In all cases, we characterize k-type slant helices in terms of the curvatures of the curve and we determine the axis of the curve. Next we focus on k-type pseudo null slant helices in hyperbolic space. Finally, we point out that type-3 slant helices in $\mathbf{E}_1^4$ were studied in [18] for those curves where the Frenet frame are non-lightlike vectors.

2. Preliminaries

The Lorentzian metric g in Minkowski space $\mathbf{E}_1^4$ is indefinite. Therefore, a vector $v \in \mathbf{E}_1^4$ can have one of the three causal characters. We say that v is spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and lightlike (or null) if $g(v, v) = 0$ and $v \ne 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbf{E}_1^4$ is called spacelike, timelike or lightlike, if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or lightlike, respectively. The norm of a vector $v \in \mathbf{E}_1^4$ is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, v is a unit vector if $g(v, v) = \pm 1$. A (spacelike or timelike) curve is parametrized by the arclength if $\alpha'(s)$ is a unit vector for any s. Also, we say that the vectors v, w in $\mathbf{E}_1^4$ are orthogonal if $g(v, w) = 0$.

Consider $\alpha = \alpha(s)$ a spacelike curve where s is the length-arc parameter. Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then T, N, B_1, B_2 are called the tangent, the principal normal, the first binormal and the second binormal vector fields of α, respectively. The fact that the metric g is indefinite causes that the vector N, B_1 and B_2 have different causal characters (here T is a spacelike vector, since α is a spacelike curve).

In this paper, we are interested in partially null curves and pseudo null curves (see [19]). A *partially null curve* is a spacelike curve where N is spacelike and B_1 is lightlike. In such case, the vector B_2 is the unique lightlike vector orthogonal to T and N such that $g(B_1, B_2) = 1$. The Frenet equations are given by ([7, 19]):

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & -\tau & 0 & -\sigma \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{1}$$

Here, κ, τ and σ are first, second and third curvature of the curve α, respectively. In fact, one can prove that after a null rotation of the ambient space, the curvature σ can be chosen to be zero (and τ is determined up to a constant). This means that any partially null curve lies in a three-dimensional lightlike subspace (orthogonal to B_1).

A spacelike curve $\alpha(s)$ is called a *pseudo null curve* if $\alpha''(s)$ is a lightlike vector for any s. Then the normal vector is $N = T'$. If N' is lightlike, then α is included in a lightlike plane: we discard this trivial case. In the rest of cases, B_1 is a unit spacelike vector orthogonal to $\{T, N\}$ and B_2 is the unique lightlike vector orthogonal to T and B_1 such that $g(N, B_2) = 1$. The Frenet equations are ([7, 19]):

$$
\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\tau \\ -\kappa & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}
$$

Here the first curvature κ can take only two values: $\kappa \equiv 0$ when the curve is a straight-line, or $\kappa \equiv 1$ in all other cases. We will assume non-trivial cases, that is, that $\kappa \equiv 1$, as well as, $\sigma, \tau \neq 0$. The classification of k-type slant helices that we present in this paper corresponds to partially null curves (Section 3) and pseudo null curves (Section 4). In the last setting, we also consider pseudo null curves in a pseudo-hyperbolic space.

We end this section about some of notations. If U is an axis of a k-type slant helix, we decompose U with respect to the Frenet frame $\{T, N, B_1, B_2\}$ as $U = u_1 T + u_2 N + u_3 B_1 + u_4 B_2$, where $u_i = u_i(s)$ are differentiable functions on s.

3. k-type partially null slant helices

In this section, we study k-type partially null slant helices. Recall that in the Frenet equations (1), the value of σ is zero. We will assume that $\kappa, \tau \neq 0$. In particular, B_1 is a constant vector. If we take $U = B_1$, then $g(T, U) = g(N, U) = (B_1, U) = 0$ and $g(B_2, U) = 1$. This means that any partially null curve is a k-type slant helix for any $k \in \{0, 1, 2, 3\}$. In this section, we discard the trivial case that $U = B_1$.

Theorem 1. *Let α be a partially null curve in $\mathbf{E}_1^4$. Then α is a 0-type slant helix (or general helix) if and only if*

$$
\frac{\tau}{\kappa} = constant. \tag{3}
$$

Moreover, α is also a k-type slant helix, for $k \in \{1, 2, 3\}$.

Proof. Assume that α is a 0-type slant helix. Then for a constant vector field U, the function $g(T, U) = c$ is constant. Differentiating this equation and from Frenet equations, we obtain that $\kappa g(N, U) = 0$. So U is orthogonal to N and we decompose U as

$$
U = c\,T + u_3\,B_1 + u_4\,B_2. \tag{4}
$$

Differentiating (4) and using the Frenet equations (1), one arrives to

$$
\begin{cases} c\kappa - \tau\,u_4 = 0, \\ u_3' = 0, \\ u_4' = 0. \end{cases}
$$

Thus u_3 and u_4 are constant. From the first equation, we obtain (3).

Conversely, suppose that relation (3) holds. We define the vector field

$$U = \frac{\tau}{\kappa} T + c B_1 + B_2, \ \ c \in \mathbf{R}. \tag{5}$$

It is shown that there are infinitive axes for all $c \in \mathbf{R}$. Differentiating (5) and considering (3) and (1), we have $U' = 0$, that is, U is a constant vector field. Because $g(T, U) = \tau/\kappa$ is constant, we have that α is a 0-type slant helix.

Finally, and because the coefficients of U are constant, we have $g(V_{k+1}, U) = 0$, for any $0 \leq k \leq 3$, that is, α is a k-type slant helix. $\qquad\square$

From the above theorem, we have

Corollary 1. *If α is a 0-type partially null slant helix in $\mathbf{E}_1^4$, an axis of α is*

$$D = \frac{\tau}{\kappa} T + c B_1 + B_2.$$

A particular case of 0-type partially null slant helices are those with κ and τ constant. In such case, α parametrizes as (see [6])

$$\alpha(s) = (cs, \frac{1}{\kappa} \cos(\kappa s), \frac{1}{\kappa} \sin(\kappa s), cs).$$

Theorem 2. *Let α be a partially null curve in $\mathbf{E}_1^4$. Then α is a 1-type slant helix if and only if there exists a constant C such that*

$$\frac{\tau(s)}{\kappa(s)} - C \int_0^s \kappa(t) \, dt = 0. \tag{6}$$

Moreover, α is a 2-type slant helix.

Proof. Assume that α is a 1-type slant helix. There exists a constant vector field U such that $g(N, U) = c$ is constant. We decompose U as $U = u_1 T + c N + u_3 B_1 + u_4 B_2$. Differentiating this equation and using (1), we have the following system of ordinary differential equations

$$\begin{cases} u_1' - c\kappa = 0, \\ \kappa \, u_1 - \tau \, u_4 = 0, \\ u_3' + c\tau = 0, \\ u_4' = 0. \end{cases} \tag{7}$$

In particular, u_4 is constant. Then we obtain

$$\begin{cases} u_1 = c \int_0^s \kappa(t) dt, \\ u_3 = -c \int_0^s \tau(t) dt, \end{cases} \tag{8}$$

Using the second equation in (7) together (8) and letting $C = \dfrac{c}{u_4}$ we conclude (6). Conversely, assume that relation (6) holds. Then, let us consider the following vector

$$U = (\int_0^s \kappa(t) dt) \, T + N - \frac{1}{C} (\int_0^s \tau(t) dt) B_1 + B_2,$$

for the real number C given in (6). Differentiating U and using the Frenet equations (1), we have $U' = 0$, that is, U is a constant vector field. Moreover, $g(N, U) = 1$ which means that α is a 1-type slant helix.

Because the coefficient u_4 is constant, then $g(B_1, U)$ is constant and α is a 2-type slant helix.   □

As a consequence of the above theorem, we have

Corollary 2. *Let α be a partially null curve in $\mathbf{E}_1^4$. If α is a 1-type slant helix, then*

$$D = (\int_0^s \kappa(t)dt) \, T + N - \frac{1}{C}(\int_0^s \tau(t)dt)B_1 + B_2, \ \ C \in \mathbf{R} - \{0\},$$

is a constant vector.

We now consider 2-type slant helices.

Theorem 3. *Let α be a 2-type partially null slant helix. Then the axes of α are*

$$U = v_1 \, T + \frac{v_1'}{\kappa} \, N + \left(c_3 - \int \frac{\tau}{\kappa} v_1' \, ds \right) B_1 + B_2, \tag{9}$$

where

$$v_1(s) = \cos \left[\int \kappa \, ds \right] \left(c_1 - \int \tau \sin \left[\int \kappa \, ds \right] ds \right)$$
$$+ \sin \left[\int \kappa \, ds \right] \left(c_2 + \int \tau \cos \left[\int \kappa \, ds \right] ds \right). \tag{10}$$

Proof. We know that there exists a constant vector field U such that $g(B_1, U) = c$, c is a constant. We decompose U as

$$U = u_1 \, T + u_2 \, N + u_3 \, B_1 + c \, B_2.$$

By differentiating U we have the following system of ordinary differential equations

$$\begin{cases} u_1' - \kappa \, u_2 = 0, \\ u_2' + \kappa \, u_1 - c \, \tau = 0, \\ u_3' + \tau \, u_2 = 0. \end{cases} \tag{11}$$

From the first equation, we have

$$u_2 = \frac{u_1'}{\kappa} = u'(\theta), \tag{12}$$

where $\theta = \int \kappa(s)ds$ is a new variable. Substituting (12) in the second equation of the system (11), we have the following ordinary differential equation

$$u_1''(\theta) + u_1(\theta) = c \, f(\theta),$$

where $f(\theta) = \frac{\tau}{\kappa}(\theta)$. By solving the above equation we obtain

$$
\begin{cases}
u_1 = c\left[\cos\theta\left(c_1 - \int f(\theta)\sin\theta\,d\theta\right) + \sin\theta\left(c_2 + \int f(\theta)\cos\theta\,d\theta\right)\right], \\
u_2 = c\left[\cos\theta\left(c_2 + \int f(\theta)\cos\theta\,d\theta\right) - \sin\theta\left(c_1 - \int f(\theta)\sin\theta\,d\theta\right)\right], \\
u_3 = c\left[c_3 - \int\left[\cos\theta\left(c_2 + \int f(\theta)\cos\theta\,d\theta\right) - \sin\theta\left(c_1 - \int f(\theta)\sin\theta\,d\theta\right)\right]d\theta\right].
\end{cases}
$$

This gives (9) and (10). □

Now we study 3-type partially null slant helices. Recall that for partially null curves, the vector B_1 is constant.

Theorem 4. *Let α be a partially null curve in $\mathbf{E}_1^4$. If α is a 3-type slant helix, then α is a k-type slant helix for $k \in \{0, 1, 2\}$.*

Proof. Let U be the constant vector field such that $g(B_2, U) = c$ is constant. A differentiation of $g(B_2, U) = c$ leads to $\tau\, g(N, U) = 0$, that is, $g(N, U) = 0$. We write U as $U = u_1 T + cB_1 + u_4 B_2$. We differentiate U obtaining

$$
\begin{cases}
u_1' = 0, \\
u_1\kappa - \tau u_4 = 0, \\
u_4' = 0.
\end{cases}
$$

This implies that u_1 and u_4 are constant, and from the second equation, the quotient τ/κ is constant too. By using Theorem 1, α is a 0-type slant helix.

Moreover, the coefficients of U with respect to the Frenet frame are all constant, which means that α is a k-type slant helix for $0 \leq k \leq 3$. □

As a consequence of the proof of Theorems 1, 2 and 4, we obtain the following

Corollary 3. *Let α be a partially null curve in $\mathbf{E}_1^4$. Then α is a 0-type slant helix if and only if α is a 3-type slant helix.*

Corollary 4. *Let α be a partially null curve in $\mathbf{E}_1^4$. Assume that there exists a constant vector field U such that $g(V_{k+1}, U) = 0$ for some $k \in \{0, 1, 3\}$. Then α is a k-type slant helix for any $k \in \{0, 1, 3\}$.*

Corollary 5. *Let α be a partially null curve in $\mathbf{E}_1^4$. Assume that the curvatures κ and τ are constant. Then α is a k-type slant helix for $k \in \{0, 1, 3\}$.*

As a consequence of our study, Figure 1 shows a set of relations between k-type partially null slant curves.

4. k-type pseudo null slant helices

In this section we study pseudo null curves that are k-type slant helices. Recall that we assume $\kappa \equiv 1$ in (2). We begin with general helices.
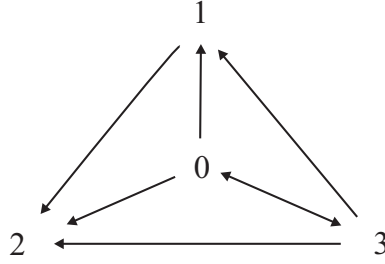
Figure 1: *The arrows mean the implication between different k-type partially null slant helices*

Theorem 5. *There do not exist* 0-*type pseudo null slant helices in* $\mathbf{E}_1^4$.

Proof. If such curve does exist, there exists a constant vector field U such that $g(T, U) = c$ is constant. Differentiating this equation and using the Frenet equations (2), we obtain $g(N, U) = 0$. Then the decomposition of U in terms of the Frenet frame is $U = c\,T + u_2\,N + u_3\,B_1$. As $U' = 0$, by using Frenet equations (2), we have

$$\begin{cases} u_2' + \sigma\,u_3 + c = 0, \\ u_3' + \tau\,u_2 = 0, \\ \tau\,u_3 = 0. \end{cases}$$

From the third equation, $u_3 = 0$, and thus, $u_2 = c = 0$, that is, $U = 0$: contradiction. $\qquad\square$

We characterize 1-type pseudo null slant helices as follows:

Theorem 6. *Let α be a pseudo null curve in $\mathbf{E}_1^4$. Then α is a 1-type slant helix if and only if*

$$\frac{\sigma(s)}{\tau(s)} = -\frac{s^2}{2} + as + b. \tag{13}$$

for some constants $a, b \in \mathbf{R}$. Moreover, α is also a 2-type slant helix.

Proof. Assume that α is a 1-type slant helix and let U be a constant vector field such that $g(N, U) = c$ is constant. Differentiation of this equation and by Frenet equations, $\tau g(B_1, U) = 0$, and so $g(B_1, U) = 0$. If we write U as a linear combination of the Frenet frame, we have $U = u_1 T + u_2 N + cB_2$. Differentiating U we obtain the following system of ordinary differential equations

$$\begin{cases} u_1' - c = 0, \\ u_2' + u_1 = 0, \\ \tau\,u_2 - c\sigma = 0, \end{cases}$$

where u_i are the coefficients of U in the decomposition with respect to the Frenet frame. If $c = 0$, $u_1 = u_2 = 0$, that is, $U = 0$: contradiction. Thus, $c \neq 0$. From the last equation $u_2 = c\left(\frac{\sigma}{\tau}\right)$ and from the first two equations, u_2 satisfies $u_2'' + c = 0$. The solution of this equation is

$$u_2 = -c\frac{s^2}{2} + \lambda s + \mu, \quad \lambda, \mu \in \mathbf{R}.$$

Thus σ/τ satisfies condition (13), with $a = \lambda/c$ and $b = \mu/c$.

Conversely, let us assume that relation (13) holds. Define the following vector field

$$U = -\left(\frac{\sigma}{\tau}\right)' T + \left(\frac{\sigma}{\tau}\right) N + B_2. \tag{14}$$

Differentiating (14) and using (2), we easily have $U' = 0$, that is, U is a constant vector. Moreover, $g(N, U) = 1$, showing that α is a 1-type slant helix. Finally, as $g(B_1, U) = 0$, then α is a 2-type slant helix. $\qquad\square$

Corollary 6. *Let α be a 1-type pseudo null slant helix. Then an axis of α is*

$$D = -\left(\frac{\sigma}{\tau}\right)' T + \left(\frac{\sigma}{\tau}\right) N + B_2.$$

Theorem 7. *Let α be a pseudo null curve in $\mathbf{E}_1^4$. Then α is a 2-type slant helix if and only if*

$$\int \tau\, ds + \frac{d}{ds}\left[\sigma + \frac{d}{ds}\left(\frac{\sigma}{\tau}\int \tau\, ds\right)\right] = 0. \tag{15}$$

Proof. If α is a 2-type slant helix, there exists a constant vector U such that $g(B_1, U) = c$ is constant. We write U as $U = u_1 T + u_2 N + cB_1 + u_4 B_2$. Differentiating U we obtain

$$\begin{cases} u_1' - u_4 = 0, \\ u_2' + u_1 + c\sigma = 0, \\ \tau\, u_2 - \sigma u_4 = 0, \\ u_4' - c\tau = 0. \end{cases} \tag{16}$$

Remark that $c \neq 0$. By solving this system, we conclude from the last equation that

$$\begin{cases} u_1 = -c\left[\sigma + \dfrac{d}{ds}\left(\dfrac{\sigma}{\tau}\int \tau\, ds\right)\right], \\ u_2 = c\dfrac{\sigma}{\tau}\int \tau\, ds, \\ u_4 = c\int \tau\, ds. \end{cases}$$

Hence the first two equations of (16) together, we conclude (15).

Conversely, assume that (15) holds. Define

$$U = -\left[\sigma + \frac{d}{ds}\left(\frac{\sigma \int \tau\, ds}{\tau}\right)\right] T + \left(\frac{\sigma \int \tau\, ds}{\tau}\right) N + B_1 + \left[\int \tau\, ds\right] B_2. \tag{17}$$

Considering (15) and differentiating (17), by using (2), we have that $U' = 0$, that is, U is constant. Moreover, $g(B_1, U) = 1$, which shows that α is a 2-type slant helix. $\qquad\square$

Corollary 7. *An axis of a 2-type pseudo null slant helix is the vector given by*

$$D = -\left[\sigma + \frac{d}{ds}\left(\frac{\sigma \int \tau\, ds}{\tau}\right)\right] T + \left(\frac{\sigma \int \tau\, ds}{\tau}\right) N + B_1 + \left[\int \tau\, ds\right] B_2.$$

Theorem 8. *Let α be a pseudo null curve in $\mathbf{E}_1^4$. Then α is a 3-type slant helix if and only if*

$$\frac{\tau}{\sqrt{1+\sigma^2}} + \frac{d}{ds}\left[\frac{\sqrt{1+\sigma^2}\left(\sigma\tau'(1+\sigma^2)+\tau\sigma'(2-\sigma^2)\right)}{\tau(1+\sigma^2)^2-3\tau\tau'^2+\sigma''(1+\sigma^2)}\right] = 0. \tag{18}$$

Proof. Suppose that α is a 3-type slant helix and let U be the constant vector field such that $g(B_2,U) = c$ is constant. If we write $U = u_1\,T + cN + u_3\,B_1 + u_4\,B_2$, a differentiation of this expression leads to

$$\begin{cases} u_1' - u_4 = 0, \\ u_1 + \sigma\,u_3 = 0, \\ u_3' + c\tau - \sigma\,u_4 = 0, \\ u_4' - \tau\,u_3 = 0. \end{cases} \tag{19}$$

The first and second equation of (19) give

$$\begin{cases} u_4 = u_1', \\ u_3 = -\frac{u_1}{\sigma}, \end{cases}$$

Substituting u_3 and u_4 into the two last equations of (19), we have

$$\begin{cases} u_1' = \frac{\sigma'}{\sigma(1+\sigma^2)}u_1 + \frac{\tau\sigma}{1+\sigma^2}c, \\ u_1'' + \frac{\tau}{\sigma}u_1 = 0. \end{cases}$$

By the last two expressions of u_1, we obtain (18). From the above system of equations, we have the following two equations

$$u_1 = \frac{c\sigma\left(\sigma\tau'(1+\sigma^2)+\tau\sigma'(2-\sigma^2)\right)}{\tau(1+\sigma^2)^2-3\sigma\sigma'+\sigma''(1+\sigma^2)}.$$

and

$$u_1 = \frac{c\sigma}{\sqrt{1+\sigma^2}}\int\frac{\tau}{\sqrt{1+\sigma^2}}ds.$$

Conversely, assume that (18) holds. Then we define the following vector

$$U = \frac{\sigma}{\sqrt{1+\sigma^2}}\epsilon\,T - \frac{1}{\sqrt{1+\sigma^2}}\epsilon\,B_1 + \left(\tau\sigma - \frac{\sigma'}{\sqrt{1+\sigma^2}}\epsilon\right)u_2\,B_2, \tag{20}$$

where $\epsilon = \int\frac{\tau}{\sqrt{1+\sigma^2}}ds$. Differentiating (20), we easily have $U' = 0$. Moreover, one obtains $g(B_2,U) = 0$, which means that α is a 3-type slant helix. $\square$

From the above theorem, one concludes

Corollary 8. *If α is a 3-type pseudo null slant helix in $\mathbf{E}_1^4$, an axis of α is the vector*

$$D = \frac{\sigma}{\sqrt{1+\sigma^2}}\epsilon\, T - \frac{1}{\sqrt{1+\sigma^2}}\epsilon\, B_1 + \left(\tau\sigma - \frac{\sigma'}{\sqrt{1+\sigma^2}}\epsilon\right)u_2\, B_2,$$

where $\epsilon = \int \frac{\tau}{\sqrt{1+\sigma^2}}ds$.

We end this section by focusing on k-type pseudo null slant helices in the pseudohyperbolic space. Recall that the pseudohyperbolic space of $\mathbf{E}_1^4$ of radius r and centered at $x_0 \in \mathbf{E}_1^4$ is defined by $\mathbf{H}_0^3 = \{x \in \mathbf{E}_1^4; g(x - x_0, x - x_0) = -r^2\}$. The metric g induced into $\mathbf{H}_0^3$ is Riemannian with a constant negative intrinsic curvature. In [7], it is proved that a pseudo null curve α lies in $\mathbf{H}_0^3$ if and only if the quotient σ/τ is a negative constant. We particularize our results for pseudo null curves of $\mathbf{H}_0^3$. As a consequence of Theorem 6, we have

Corollary 9. *There does not exist a 1-type pseudo null slant helix in pseudohyperbolic spaces of $\mathbf{E}_1^4$.*

As a consequence of Theorem 7 we have

Corollary 10. *If α is a 2-type pseudo null slant helix in $\mathbf{H}_0^3$, then*

$$\tau = \lambda e^{\frac{s}{\sqrt{-2c}}} + \mu e^{-\frac{s}{\sqrt{-2c}}}, \quad \lambda, \mu \in \mathbf{R}, \tag{21}$$

where c is a negative constant given by $\sigma = c\tau$.

Proof. From [7] we know that $\sigma = c\tau$ for some real number $c < 0$. If we put this into (15), we have that τ satisfies $2c\tau'' + \tau = 0$, whose solutions are given by (21). □

If we put $\sigma = c\tau$ for pseudo null curves in $\mathbf{H}_0^3$, Theorem 8 implies

Corollary 11. *If α is a 3-type pseudo null slant helix in $\mathbf{H}_0^3$, the second curvature function $\tau(s)$ satisfies*

$$2c^2\tau\tau'\tau''' = c\tau''\left[5\tau^2(1+c^2\tau^2) + c(3\tau'^2 + 4\tau\tau'')\right] + c^2\tau^5(2+c^2\tau^2) + \tau^3(1 - 15c^3\tau'^2).$$

Acknowledgments

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