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On the stability of linear mappings between essential Banach modules

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Abstract. We establish the generalized Hyers-Ulam-Rassias stability of the linear mappings between essential Banach modules over Banach algebras with bounded approximate identity and over C^*-algebras.

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1. Introduction

The stability problem of functional equations appeared by Ulam [16] in 1940. In the next year, Hyers [10] solved a version of this problem for additive mappings between Banach spaces. Since then the topic of stability of functional equations was studied in several ways by some mathematicians like: Aoki, Bourgin, Rassias, Gavruta, Forti and Park (see [6, 11, 14]). The stability of $\mathbb{R}$-linear mapping between Banach spaces was given by Rassias [15] in 1978.

Let E_1 and E_2 be Banach spaces. Suppose that $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$, the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$ and let there exist $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E_1$. Then there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2 - 2^p|}\|x\|^p$$

for all $x \in E_1$.
In 1994, Gavruta [9] generalized the Rassias' result by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ in (1) by a general control function $\phi : E_1 \times E_1 \to [0, \infty)$ satisfying

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty,$$

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for all $x, y \in E_1$. This result is significant in the development of what we now call generalized Hyers-Ulam-Rassias stability of functional equations.
Park [13] provided a generalization of Gavruta's result to show the generalized Hyers-Ulam-Rassias stability of the linear mappings between unit linked Banach modules over a unital Banach algebra. The results of Park imply the stability of $\mathbb{C}$-linear mappings in complex Banach spaces.

Recently, the stability of functional equations in Banach algebras with approximate identity has been investigated in several papers (for example, see [1, 2, 8]). In this paper, we establish the generalized Hyers-Ulam-Rassias stability of linear mappings between essential Banach modules over Banach algebras with bounded approximate identity and over C^*-algebras.

Let B be a Banach algebra with a bounded left approximate identity $(e_\lambda)_{\lambda\in\Lambda}$ and X a left Banach B-module. Then X is said to be essential if $e_\lambda x \to x$ for all $x \in X$. It is known that if X is an essential left Banach B-module, then for every bounded left approximate identity (f_γ) in B, $e_\gamma x \to x$ for all $x \in X$ (see [7]).

Also let Y be a left Banach B-module. The mapping $f : X \to Y$ is said to be B-linear if for all $a, b \in B$ and $x, y \in X$, $f(ax + by) = af(x) + bf(y)$.

2. Main results

Let B be a Banach algebra with a bounded left approximate identity, M_1 an essential left Banach B-module and M_2 a left Banach B-module. Our first result can be regarded as a generalization of [13, Theorem 2.1].

Theorem 1. *Let $\alpha \in \mathbb{C} \setminus \{0\}$. Suppose that $f : M_1 \to M_2$ is a mapping for which there exist two functions $\phi : M_1 \to [0,\infty)$ and $\psi : M_1 \times M_1 \to [0,\infty)$ such that*

$$\widetilde{\phi}(x) := \sum_{n=0}^{\infty} |\alpha|^{-n}\phi(\alpha^n x) < \infty, \tag{1}$$

$$\lim_{n\to\infty} |\alpha|^{-n}\psi(\alpha^n x, \alpha^n y) = 0, \tag{2}$$

$$\|\alpha^{-1}f(\alpha x) - f(x)\| \le \phi(x), \tag{3}$$

$$\|f(ax + ay) - af(x) - af(y)\| \le \psi(x, y) \tag{4}$$

for all $x, y \in M_1$ and all $a \in B_1 := \{x \in B : \|x\| = 1\}$. Then there exists a unique B-linear mapping $F : M_1 \to M_2$ such that

$$\|F(x) - f(x)\| \le \widetilde{\phi}(x) \tag{5}$$

for all $x \in M_1$.

Proof. By Proposition 1 of [4], for every $x \in M_1$, $F(x) := \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ exists and $F : M_1 \to M_2$ is a unique function with $\|F(x) - f(x)\| \le \widetilde{\phi}(x)$ and $F(\alpha x) = \alpha F(x)$ for all $x \in M_1$.

Putting $x = y$ in (4) and replacing x by $\alpha^n x$, we get

$$\|f(2\alpha^n ax) - 2af(\alpha^n x)\| \le \psi(\alpha^n x, \alpha^n x),$$

and so,

$$\left\| \frac{f(2\alpha^n ax)}{\alpha^n} - 2a\frac{f(\alpha^n x)}{\alpha^n} \right\| \leq \frac{1}{|\alpha|^n}\psi(\alpha^n x, \alpha^n x)$$

for all $n \in \mathbb{N}$ and $x \in M_1$ and $a \in B_1$.
By taking the limit as $n \to \infty$, it follows from (2) that

$$F(2ax) = 2aF(x) \tag{6}$$

for all $x \in M_1$ and all $a \in B_1$.
Let $\beta \in \mathbb{T} := \{\mu \in \mathbb{C} : |\mu| = 1\}$ and $x \in M_1$. Since B is a Banach algebra with a bounded left approximate identity and M_1 is essential, it follows from Cohen factorization theorem [7, Theorem 16.1] that there exist $c \in B$ and $x_1 \in M_1$ such that $x = cx_1$. Since $\beta\frac{c}{\|c\|} \in B_1$, by (6) we have

$$F\left(2\beta\frac{c}{\|c\|}\|c\|x_1\right) = 2\beta\frac{c}{\|c\|}F(\|c\|x_1).$$

Thus

$$\begin{aligned} F(2\beta x) &= F(2\beta cx_1) = F\left(2\beta\frac{c}{\|c\|}\|c\|x_1\right) \\ &= 2\beta\frac{c}{\|c\|}F(\|c\|x_1) = \beta F\left(2\frac{c}{\|c\|}\|c\|x_1\right) \\ &= \beta F(2x). \end{aligned} \tag{7}$$

Replacing x by $\frac{x}{2}$ in (7), we get

$$F(\beta x) = \beta F(x) \quad (\beta \in \mathbb{T}, x \in M_1). \tag{8}$$

Replacing x, y by $\alpha^n x, \alpha^n y$ respectively, in (4), we get

$$\|f(\alpha^n ax + \alpha^n ay) - af(\alpha^n x) - af(\alpha^n y)\| \leq \psi(\alpha^n x, \alpha^n y),$$

and so,

$$\left\| \frac{f(\alpha^n ax + \alpha^n ay)}{\alpha^n} - a\frac{f(\alpha^n x)}{\alpha^n} - a\frac{f(\alpha^n y)}{\alpha^n} \right\| \leq \frac{1}{|\alpha|^n}\psi(\alpha^n x, \alpha^n y)$$

for all $n \in \mathbb{N}$ and $x, y \in M_1$ and $a \in B_1$. Taking the limit as $n \to \infty$, we obtain

$$F(ax + ay) = aF(x) + aF(y) \tag{9}$$

for all $x, y \in M_1$ and all $a \in B_1$.

Putting $\beta = -1$ and $x = 0$ in (8), we get $F(0) = 0$. Now putting $y = 0$ in (9), we have

$$F(ax) = aF(x) \quad (a \in B_1, x \in M_1). \tag{10}$$

Let $x, y \in M_1$. By [7, Theorem 17.1], there exist $b \in B$ and $x_1, y_1 \in M_1$ such that $x = bx_1$ and $y = by_1$. Hence it follows from (9) and (10) that

$$\begin{aligned} F(x+y) &= F(bx_1 + by_1) = F\left(\|b\|\frac{b}{\|b\|}x_1 + \|b\|\frac{b}{\|b\|}y_1\right) \\ &= \frac{b}{\|b\|}F(\|b\|x_1) + \frac{b}{\|b\|}F(\|b\|y_1) = F(bx_1) + F(by_1) \\ &= F(x) + F(y). \end{aligned}$$

Therefore, F is an additive mapping. Since the additive mapping F satisfies (8), by [5, Lemma 2.4] we get

$$F(\lambda x) = \lambda F(x) \quad (\lambda \in \mathbb{C}, x \in M_1). \tag{11}$$

Finally, let $c \in B$. Then by (10) and (11), we obtain

$$F(cx) = F\left(\frac{c}{\|c\|}\|c\|x\right) = \frac{c}{\|c\|}F(\|c\|x) = cF(x) \quad (x \in M_1, c \in B). \tag{12}$$

Therefore, F is a unique B-linear mapping satisfying (5). □

Remark 1. *We note that in Theorem 1, the result has been obtained without the assumption that the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$; (compare [13, Theorem 2.1]).*

Corollary 1. *Let $f : M_1 \to M_2$ be a mapping satisfying*

$$f(ax + ay) = af(x) + af(y)$$

for all $a \in B_1$ and all $x, y \in M_1$. Then, f is a B-linear mapping.

Proof. Let $\alpha = 1$, $\phi = 0$, $\psi = 0$ and apply Theorem 1. □

Now, we investigate the generalized Hyers-Ulam-Rassias stability of linear mappings between essential Banach modules over a C^*-algebra. In the following theorem, A is a C^*-algebra, M_1 is an essential left Banach A-module and M_2 is a left Banach A-module. Also, by A^+ we denote the set of all positive elements in C^*-algebra A and suppose $A_1 = \{x \in A : \|x\| = 1\}$.

Theorem 2. *Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Suppose $f : M_1 \to M_2$ is a mapping for which there exist functions $\phi : M_1 \to [0, \infty)$ and $\psi : M_1 \times M_1 \to [0, \infty)$ satisfying (1), (2) and (3) such that*

$$\|f(ax + ay) - af(x) - af(y)\| \le \psi(x, y) \tag{13}$$

for all $x, y \in M_1$ and all $a \in A_1 \bigcap A^+$. If for each fixed $x \in M_1$, the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique A-linear mapping $F : M_1 \to M_2$ such that

$$\|F(x) - f(x)\| \le \widetilde{\phi}(x) \tag{14}$$

for all $x \in M_1$.

Proof. By the same reasoning as Theorem 1, the function $F : M_1 \to M_2$ defined by $F(x) = \lim_{n\to\infty} \frac{f(\alpha^n x)}{\alpha^n}$ $(x \in M_1)$ is unique function that $F(\alpha x) = \alpha F(x)$ for all $x \in M_1$ and satisfies (14).
By the same method as in the proof of Theorem 1, it is obtained that

$$F(ax + ay) = aF(x) + aF(y) \quad (x, y \in M_1, a \in A_1 \bigcap A^+). \tag{15}$$

Since $\alpha \neq 1$ and $F(\alpha x) = \alpha F(x)$ for all $x \in M_1$, we have $F(0) = 0$. Putting $y = 0$ in (15), we obtain

$$F(ax) = aF(x) \quad (x \in M_1, a \in A_1 \bigcap A^+). \tag{16}$$

Now we prove that the mapping F is additive.

Let $x, y \in M_1$. Since M_1 is an essential left Banach A-module, by [7, Theorem 17.1] there exist $c \in A$ and $x_1, y_1 \in M_1$ such that $x = cx_1$ and $y = cy_1$.
We claim that there exist $a \in A^+$ and $b \in A$ such that $c = ab$. To see this, it is known that C^*-algebra A has a bounded approximate identity $(e_\lambda)_{\lambda\in\Lambda}$, where $e_\lambda \in A^+$ for all $\lambda \in \Lambda$ [12, Theorem 3.1.1]. By Cohen factorization theorem, there exist $a, b \in A$ such that $c = ab$. By the proof of Cohen factorization theorem (see the proof of [3, Theorem 10.11]), there exist a real number $0 < \gamma < 1$ and a sequence (λ_n) in Λ such that

$$a = \sum_{n=1}^{\infty} \gamma(1-\gamma)^{n-1} e_{\lambda_n}.$$

Hence $a \in A^+$ since the sum of two positive elements in C^*-algebra A is a positive element and A^+ is closed in A (see [12]).
Therefore, there exist $a \in A^+$, $b \in A$ and $x_1, y_1 \in M_1$ such that $x = abx_1$ and $y = aby_1$. Now, by (15) and (16) we get

$$\begin{aligned} F(x + y) &= F(abx_1 + aby_1) = F\left(\frac{a}{\|a\|}\|a\|bx_1 + \frac{a}{\|a\|}\|a\|by_1\right) \\ &= \frac{a}{\|a\|}F(\|a\|bx_1) + \frac{a}{\|a\|}F(\|a\|by_1) = F(abx_1) + F(aby_1) \\ &= F(x) + F(y). \end{aligned}$$

Thus, F is an additive mapping. Hence by the same reasoning as in the proof of Rassias Theorem [15], the mapping F is $\mathbb{R}$-linear and so F is $\mathbb{C}$-linear since $F(\alpha x) = \alpha F(x)$.
Now it is clear that

$$F(ax) = aF(x) \quad (x \in M_1, a \in A^+),$$

and by the same method as the proof of [13, Theorem 2.2], one can obtain that

$$F(ax + by) = aF(x) + bF(y) \quad (a, b \in A, x, y \in M_1).$$

Thus F is an A-linear mapping, as desired. □

Remark 2. *In Theorem 2, one can replace the condition that f satisfies (13) for all $a \in A_1 \bigcap A^+$ by the condition that f satisfies (13) for all $a \in A_1 \bigcap (iA^+)$. The proof is similar to the proof of Theorem 2.*

Finally, let E_1 and E_2 be complex Banach spaces. In [13], the stability of $\mathbb{C}$-linear mappings has been obtained by the assumption that the mapping $f : E_1 \to E_2$ satisfies

$$\|f(\lambda x + \lambda y) - \lambda f(x) - \lambda f(y)\| \leq \varphi(x, y) \tag{17}$$

for $\lambda = 1, i$ and all $x, y \in E_1$. In the following result, we apply Remark 2 and prove the stability of $\mathbb{C}$-linear mappings in the case that $f : E_1 \to E_2$ satisfies (17), only for $\lambda = i$.

Corollary 2. *Let $f : E_1 \to E_2$ be a mapping for which there exists a function $\varphi : E_1 \times E_1 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty, \tag{18}$$

$$\|f(ix + iy) - if(x) - if(y)\| \leq \varphi(x, y)$$

for all $x, y \in E_1$. If the mapping $t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-linear mapping $F : E_1 \to E_2$ satisfying

$$\|F(x) - f(x)\| \leq \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi((2i)^n x, (2i)^n x)$$

for all $x \in E_1$.

Proof. It is easy to see that the mapping φ satisfying (18) satisfies

$$\sum_{n=0}^{\infty} 2^{-n} \varphi((2i)^n x, (2i)^n y) < \infty$$

for all $x, y \in E_1$. Let $\alpha = 2i$, $\phi(x) = \frac{1}{2}\varphi(x, x)$ and $\psi(x, y) = \varphi(x, y)$. Now apply Remark 2. ☐

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