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Spherical f-tilings by two non congruent classes of isosceles triangles - I

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Abstract. The theory of f-tilings is related to the theory of isometric foldings, initiated by S. Robertson [8] in 1977. The study of dihedral f-tilings of the Euclidean sphere S^2 by triangles and r-sided regular polygons was initiated in 2004, where the case $r = 4$ was considered [4]. In a subsequent paper [1], the study of all spherical f-tilings by triangles and r-sided regular polygons, for any $r \geq 5$, was described. Recently, in [2] and [3] a classification of all triangular dihedral spherical f-tilings for which one of the prototiles is an equilateral triangle is given. In this paper, we extend these results considering the dihedral case of two non congruent isosceles triangles in a particular way of adjacency ending up to a class of f-tilings composed by three parametrised families, denoted by $\mathcal{F}_{k,\alpha}$, $\mathcal{E}_\alpha$ and $\mathcal{L}_k$, $k \geq 3$, $\alpha > \frac{\pi}{2}$, respectively, and one isolated tiling, denoted by $\mathcal{G}$. The combinatorial structure including the symmetry group of each tiling is also given.
Dawson and Doyle in [6], [7] have also been working on spherical tilings, relaxing the edge to edge condition.

AMS subject classifications: 52C20, 05B45

Key words: Dihedral f-tilings, isometric foldings, symmetry groups

1. Introduction

Spherical folding tilings or *f-tilings* for short, are edge-to-edge decompositions of the sphere by geodesic polygons, such that all vertices satisfy the angle folding relation, i.e. all vertices are of even valency and the sum of their alternate angles is π. An f-tiling τ is said to be *monohedral* if it is composed by congruent cells and *dihedral* if every tile of τ is congruent to one of two fixed sets X and Y (the prototiles of τ). We shall denote by $\Omega(X, Y)$ the set, up to an isomorphism, of all dihedral f-tilings of S^2 whose prototiles are X and Y.

The classification of all spherical folding tilings by rhombi and triangles was obtained in 2005, [5]. However, the corresponding study considering two triangular (non-isomorphic) prototiles is not yet completed. This is not surprising, since it is much harder. The cases studied, until now are the ones corresponding to prototiles of the following type:

- an equilateral triangle and an isosceles triangle and

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- an equilateral triangle and a scalene triangle.

When the prototiles are two non congruent isosceles triangles, there are 3 distinct types of adjacency, see Figure 1. Here, our interest is focused on dihedral f-tilings by isosceles triangles with adjacency of type I.

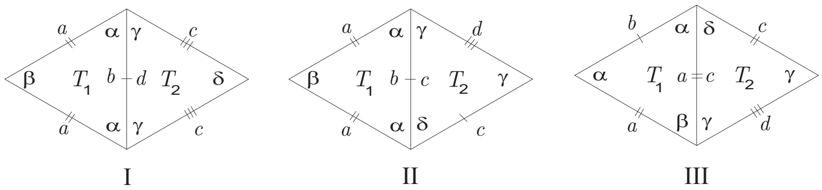

Figure 1: *Different types of adjacency*

The type I edge-adjacency condition can be analytically described by the equation

$$\frac{\cos\beta + \cos^2\alpha}{\sin^2\alpha} = \frac{\cos\delta + \cos^2\gamma}{\sin^2\gamma}. \tag{1}$$

Observe that if $\alpha = \gamma$, then $\beta = \delta$, which is impossible. Therefore, we shall consider, without loss of generality, $\alpha > \gamma$.

From now on, T_1 denotes an isosceles spherical triangle of angles α, α, β with sides a (opposite to α) and b (opposite to β) and T_2 another isosceles spherical triangle, not congruent to T_1, of angles γ, γ, δ with sides c (opposite to γ) and d (opposite to δ).

In order to get a dihedral f-tiling $\tau \in \Omega(T_1, T_2)$, we find it useful to start by considering a *local representation*, beginning with a common vertex to each one of the isosceles triangles in adjacent positions. In the diagrams that follows, it is convenient to label the tiles according to the following procedures:

(i) The tiles by which we begin the local representation of a tiling $\tau \in \Omega(T_1, T_2)$ are labelled by 1 and 2, respectively;

(ii) For $j \geq 2$, the location of tile j can be deduced from the configuration of tiles $(1, 2, \ldots, j-1)$ and from the hypothesis that the configuration is part of a complete local representation of an f-tiling (except in the cases explicitly indicated).

Let us begin by showing that it is impossible to have side b equal to side d and side a equal to side c.

In fact, assuming that $b = d$ and $a = c$, what is equivalent to have,

$$\frac{\cos\beta + \cos^2\alpha}{\sin^2\alpha} = \frac{\cos\delta + \cos^2\gamma}{\sin^2\gamma},$$

and

$$\frac{\cos\alpha(1 + \cos\beta)}{\sin\alpha\sin\beta} = \frac{\cos\gamma(1 + \cos\delta)}{\sin\gamma\sin\delta}, \tag{2}$$

we may conclude, by the relation

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\gamma}{\sin\delta} \tag{3}$$

that α and γ are in the same quadrant; $\alpha+\beta \neq \pi$, $\alpha+\gamma \neq \pi$, $\beta+\delta \neq \pi$ and $\gamma+\delta \neq \pi$.

Starting a configuration as shown in Figure 2, a decision about the angle $\lambda_1 \in \{\alpha, \gamma\}$ must be taken.

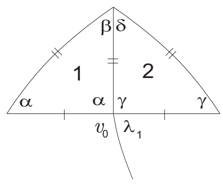

Figure 2: *Local configuration*

1.1. Suppose that $\lambda_1 = \alpha$. Then, $2\alpha < \pi$, since $\alpha = \frac{\pi}{2}$ implies, by equation (2), that $\gamma = \alpha$, which contradicts our assumption that $\alpha > \gamma$. As, $\alpha < \frac{\pi}{2}$ then $\gamma < \frac{\pi}{2}$.

Looking at the possible order relations between α, β and γ, δ, one and only one of the following conditions is satisfied,

- $\alpha < \beta$ and $\delta < \gamma$,
- $\beta < \alpha$ and $\gamma < \delta$,
- $\alpha < \beta$ and $\gamma < \delta$.

Note that the condition $\beta < \alpha \ \wedge \ \delta < \gamma$ is not considered, since it prevents the existence of vertices of valency four and these must exit, see [4].

1.1.1. Assume that $\alpha < \beta$ and $\delta < \gamma$. Then, $\delta < \gamma < \alpha < \frac{\pi}{2}$ and so $\beta \geq \frac{\pi}{2}$.
If $\beta \geq \frac{\pi}{2}$, then the angle λ_2 is necessarily δ (any other combination violates the angle folding relation). But, with $\lambda_2 = \delta$ at vertex $\widetilde{v}_1$ (see Figure 3) the sum $\alpha+\gamma+\mu$ also violates the angle folding relation, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$.

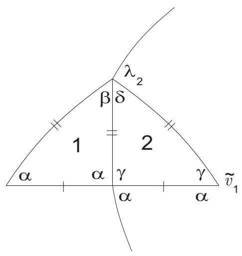

Figure 3: *Local configuration*

1.1.2. Suppose now that $\beta < \alpha$ and $\gamma < \delta$. As $\beta < \alpha < \frac{\pi}{2}$ and $\gamma < \frac{\pi}{2}$, then $\delta \geq \frac{\pi}{2}$. Taking into account that $\lambda_1 = \alpha$ (see Figure 2), $2\alpha+\beta > \pi$ and $2\gamma+\delta > \pi$, around vertex v_0 the sum of the alternate angles containing λ_1 is $2\alpha + t\gamma = \pi$, for some $t \geq 1$. Accordingly, $\gamma < \beta < \alpha < \frac{\pi}{2} \leq \delta$.

Assuming that $\delta = \frac{\pi}{2}$, then $\gamma > \frac{\pi}{4}$ and consequently $t = 1$. Using this information $(2\alpha + \gamma = \pi,\ \delta = \frac{\pi}{2})$ in equations (1) and (2) we get $\alpha \approx 337.5^o$ and $\beta \approx 225^o$ contradicting $\alpha < \frac{\pi}{2}$.

If $\delta > \frac{\pi}{2}$, the angle λ_3 in Figure 4, must be β and the vertex must be of valency four, taking in consideration the order and angle folding relations. However, by relation (3), it is impossible to have $\beta + \delta = \pi$.

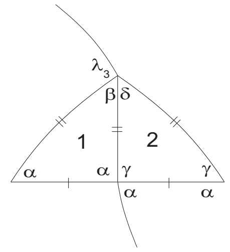

Figure 4: *Local configuration*

1.1.3. If $\alpha < \beta$ and $\gamma < \delta$, a decision about the angle $\widetilde{\lambda}_2 \in \{\alpha, \beta, \gamma, \delta\}$ must be taken, as shown in the next figure.

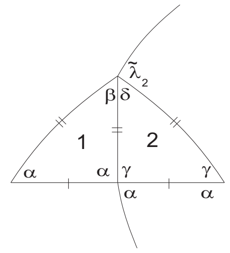

Figure 5: *Local configuration*

If $\widetilde{\lambda}_2 = \alpha$, then $\beta + \widetilde{\lambda}_2 < \pi$, by (3). In order to satisfy the angle folding relation, the sum must be $\alpha + \beta + t\gamma = \pi,\ t \geq 1$, since $\alpha < \beta,\ 2\alpha + \beta > \pi$ and $\gamma < \alpha < \beta,\ \beta > \frac{\pi}{3},\ \delta > \frac{\pi}{3}$. However, the angle arrangement leads us to the sums $\delta + (t+1)\alpha$ or $\delta + \gamma + t\alpha$ and both violate the angle folding relation (note that $\gamma < \alpha$).

If $\widetilde{\lambda}_2 = \beta$, then $2\beta < \pi$, since $\beta = \frac{\pi}{2}$ lead us to $\beta + \delta = \pi$ or $\beta = \delta$, both impossible by (3). As $\gamma < \alpha < \beta < \frac{\pi}{2} \leq \delta$, the sum of the alternate angles containing $\widetilde{\lambda}_2$ is $2\beta + m\gamma = \pi,\ m \geq 1$. By the angle arrangement (Figure 6), we conclude that the other sum of alternate angles is $\delta + \beta + ... + \alpha = \pi$ or $\delta + \beta + ... + \gamma = \pi$ or $\delta + \alpha + ... + \alpha = \pi$ or $\delta + \alpha + ... + \gamma = \pi$, all impossible by the order relation between the angles and $2\gamma + \delta > \pi$.

Supposing that $\widetilde{\lambda}_2 = \gamma$, then $\beta + \widetilde{\lambda}_2 < \pi$, since $\beta + \widetilde{\lambda}_2 = \pi$ implies that $\delta + \alpha = \pi$. By relation (3), one has $\sin\delta = \pm\sin\gamma$, an absurdity. As $\beta + \gamma < \pi$, the angle λ_4

is either γ or α. In the first case, the sum $\delta + \lambda_4$, see Figure 7, does not satisfy the angle folding relation, by the order relation $\gamma < \alpha < \beta$ and $\gamma < \delta$. In the second case, the same impossibility arises.

Finally, if $\widetilde{\lambda}_2 = \delta$, in order to fulfilled the angle folding relation, the sum $\beta + \widetilde{\lambda}_2 + \mu$ must be $\beta + \delta + \alpha = \pi$ or $\beta + \delta + \gamma = \pi$, both impossible since $2\alpha + \beta > \pi$, $2\gamma + \delta > \pi$ and $\beta, \delta > \frac{\pi}{3}$.

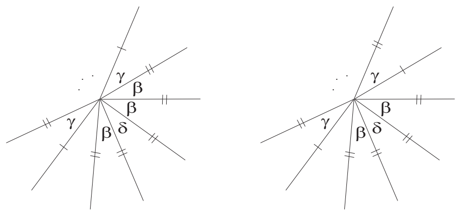

Figure 6: *Angle arrangement*

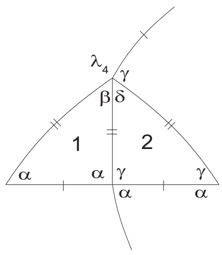

Figure 7: *Local configuration*

1.2. If $\lambda_1 = \gamma$ in Figure 2, then $\alpha + \lambda_1 < \pi$ by (3) and again $\gamma < \alpha < \frac{\pi}{2}$. The order relation $\beta < \alpha < \frac{\pi}{2}$ and $\delta < \gamma < \frac{\pi}{2}$ prevents the existence of vertices of valency four, so we have to analyse the other three possibilities, namely,

- $\alpha < \beta$ and $\delta < \gamma$,
- $\beta < \alpha$ and $\gamma < \delta$,
- $\alpha < \beta$ and $\gamma < \delta$.

1.2.1. In the first case, as $\frac{\pi}{3} < \gamma < \alpha < \beta$ and $2\gamma + \delta > \pi$, the sum $\alpha + \gamma + \mu > \pi$, for all $\mu \in \{\alpha, \beta, \gamma, \delta\}$ and so the local configuration cannot be extended.

1.2.2. In the second case, as $\beta < \alpha < \frac{\pi}{2}$ and $\gamma < \frac{\pi}{2}$, then $\delta \geq \frac{\pi}{2}$ and angle $\widetilde{\lambda}_3 \in \{\alpha, \beta, \gamma\}$, in Figure 8.

If $\widetilde{\lambda}_3 = \alpha$, then by the configuration the sum of alternate angles containing δ and $\widetilde{\lambda}_3$ is $\delta + \alpha + m\beta = \pi$, $m \geq 1$, contradicting $2\alpha + \beta > \pi$.

If $\widetilde{\lambda}_3 = \beta$, then the referred sum is $\delta + k\beta = \pi$, $k \geq 2$ or $\delta + \gamma + n\beta = \pi$, $n \geq 1$. In both cases, we conclude that $\beta < \gamma < \alpha < \frac{\pi}{2} \leq \delta$.
Assuming that $\delta = \frac{\pi}{2}$, the sum containing the alternate angles α and γ at vertex $\widetilde{v}_2$ is $\alpha + \gamma + q\beta = \pi$, $q \geq 1$ or $\alpha + 2\gamma + r\beta = \pi$, $r \geq 0$, since $\gamma > \frac{\pi}{4}$ and $\alpha > \frac{\pi}{3}$.

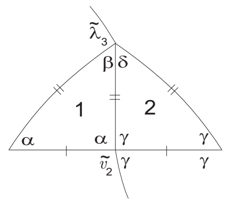

Figure 8: *Local configuration*

The first possibility leads to a sum of alternate angles of the form $2\gamma + q\beta = \pi$, see Figure 9 (observe that at vertex $\widetilde{v}_3$, $2\alpha + \mu > \pi$ for all $\mu \in \{\alpha, \beta, \gamma, \delta\}$) which implies the absurdity of $\alpha = \gamma$.

The second possibility leads us to $\gamma = \frac{5\pi}{3}, \beta = \frac{\pi}{2}$ for $r = 0$ and $2\alpha + \beta < \pi$, for $r \geq 1$ contradicting $\beta < \gamma < \frac{\pi}{2}$ and $2\alpha + \beta > \pi$.

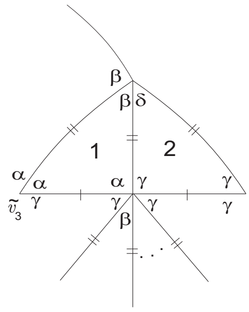

Figure 9: *Local configuration*

If $\delta > \frac{\pi}{2}$, the vertices of valency four are surrounded by alternate angles δ and α leading us to $\gamma > \frac{\pi}{6}$. The sum of alternate angles containing α and γ at vertex $\widetilde{v}_2$ is of the form $\alpha+\gamma+q\beta = \pi,\ q \geq 1$ or $\alpha+2\gamma+r\beta = \pi,\ r \geq 0$ or $\alpha+3\gamma+s\beta = \pi,\ s \geq 0$. In the first case, the angle arrangement is exactly the same as in the previous figure and so an absurdity is achieved.
In the second case, for $r = 0$ we get $\gamma = \frac{5\pi}{3}, \beta = \frac{4\pi}{3}$ and for $r \geq 1$ one has $2\alpha+\beta < \pi$. In both situations we find an impossibility.
In the third case, for $s = 0$, we conclude that $\gamma = \frac{7\pi}{4}, \beta = \frac{5\pi}{4}$ contradicting $\gamma < \frac{\pi}{2}$ and for $s \geq 1$ one gets the absurdity $2\alpha + \beta < \pi$.

If $\widetilde{\lambda}_3 = \gamma$, the sum of alternate angles containing δ and $\widetilde{\lambda}_3$ must be $\delta + \gamma + k\beta = \pi,\ k \geq 1$. Accordingly, $\beta < \gamma < \alpha < \frac{\pi}{2} \leq \delta$.

Suppose firstly, that $\delta = \frac{\pi}{2}$. Then, $\gamma > \frac{\pi}{4}$ and in Figure 8 the sum of alternate angles containing α and γ, at vertex $\widetilde{v}_2$, is of the form $2\alpha + \gamma = \pi$ or $\alpha + 2\gamma + r\beta = \pi,\ r \geq 0$ or $\alpha + \gamma + q\beta = \pi,\ q \geq 1$.
The first possibility leads us to conclude that $\gamma < \beta$, contradicting the order relation between the angles.
The second possibility, for $r = 0$ we get $\gamma = \frac{5\pi}{3}, \beta = \frac{\pi}{2}$ and for $r \geq 1$ one has $2\alpha + \beta < \pi$. In both situations an absurdity is achieved.

What remains is the third possibility and extending the configuration in Figure 8, we end up at a vertex ($\widetilde{v}_4$), surrounded by alternate angles δ and α, as illustrated in Figure 10. Taking into account the order relation between the angles, this vertex must be of valency four, contradicting the fact that $\alpha < \frac{\pi}{2} = \delta$.

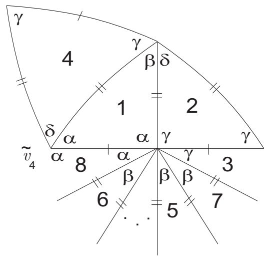

Figure 10: *Local configuration*

Note that in the previous configuration the tile labeled 8 is unique, otherwise, around vertex $\widetilde{v}_2$, we get sums $\alpha+\gamma+q\beta=\pi$ and $2\gamma+q\beta=\pi$ implying that $\alpha=\gamma$, which is an impossibility.

If $\delta > \frac{\pi}{2}$ and taking into account that the sum of alternate angles containing δ and $\widetilde{\lambda}_3$ satisfies $\delta+\gamma+k\beta=\pi,\ k\geq 1$, vertices of valency four must be surrounded by alternate angles δ and α. Consequently, they satisfy the condition $\delta+\alpha=\pi$. As $2\gamma+\delta>\pi$, one gets $\gamma>\frac{\pi}{6}$ and so at vertex $\widetilde{v}_2$ the sum containing the alternate angles α and γ must be of the form $2\alpha+\gamma=\pi$ or $\alpha+\gamma+q\beta=\pi,\ q\geq 1$ or $\alpha+2\gamma+r\beta=\pi,\ r\geq 0$ or $\alpha+3\gamma+s\beta=\pi,\ s\geq 0$.

In the first case we get $\gamma<\beta$, contradicting once again the order relation between the angles.

In the second case, extending the configuration in Figure 8, tile 7 has two possibilities, as shown in Figure 11:

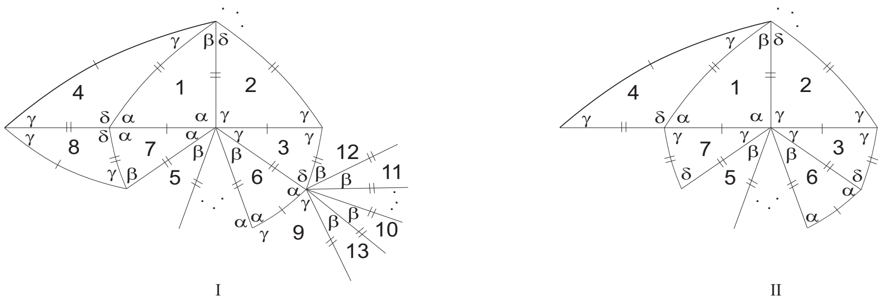

Figure 11: *Local configuration*

Observe that in Figure 11-I, tile 9 is unique, in order to avoid two alternate angles α, whose sum $2\alpha+\mu$ does not satisfy the angle folding relation for any $\mu\in\{\alpha,\beta,\gamma,\delta\}$.

The extended configuration ends up at a vertex surrounded by angles $\delta,\alpha,\gamma,\beta....,\beta$.

At this vertex the sums of alternate angles satisfy $\delta+\gamma+k\beta=\pi=\alpha+(k+1)\beta$. Therefore, we get $\delta+\gamma=\alpha+\beta$ and since vertices of valency four satisfy the condition $\delta+\alpha=\pi$, one has $\gamma+2\delta=\pi+\beta$. As $\beta<\gamma$, then $2\delta<\pi$, an absurdity.

In Figure 11-II, one has both sums $\alpha+\gamma+q\beta=\pi$ and $\gamma+\gamma+q\beta=\pi$ implying that $\alpha=\gamma$, also an absurdity.

In the third case, $\alpha+2\gamma+r\beta=\pi,\ r\geq 0$, we get $\gamma=\frac{5\pi}{3}$ and $\beta=\frac{4\pi}{3}$ for $r=0$ and for $r\geq 1$ we get $2\alpha+\beta<\pi$, both situations impossible.

In the last case, $\alpha+3\gamma+s\beta=\pi,\ s\geq 0$, one has $\gamma=\frac{7\pi}{4},\ \beta=\frac{5\pi}{4}$ for $s=0$ and $2\alpha+\beta<\pi$ for $s\geq 1$, contradicting $\gamma<\frac{\pi}{2}$ and the fact that $2\alpha+\beta>\pi$.

1.2.3. Finally, assume that $\alpha<\beta$ and $\gamma<\delta$ and consider again the angle $\widetilde{\lambda}_3$ (as shown in Figure 8).

If $\widetilde{\lambda}_3=\alpha$, $\alpha+\delta<\pi$ (otherwise, we get $\beta=\delta$ or $\alpha=\beta$) and consequently, the sum of alternate angles containing α and δ is one of the following ones: $m\alpha+\delta=\pi,\ m\geq 2$ or $n\alpha+2\delta=\pi,\ n\geq 1$ or $\alpha+\delta+\beta=\pi$. None of them can occur, since $\beta,\ \delta>\frac{\pi}{3},\ 2\gamma+\delta>\pi$ and $\gamma<\alpha$.

If $\widetilde{\lambda}_3=\beta$, by relation (3), $\delta+\beta<\pi$ and consequently $\delta+\beta+\alpha=\pi$ or $\delta+\beta+\gamma=\pi$, both conditions are impossible ($2\alpha+\beta>\pi,\ 2\gamma+\delta>\pi$).

If $\widetilde{\lambda}_3=\gamma$, the sum $\delta+\gamma+\mu=\pi,\ \mu\in\{\alpha,\beta,\gamma,\delta\}$ is not satisfied, since $\gamma<\alpha<\beta,\ \gamma<\delta$ and $2\gamma+\delta>\pi$.

If $\widetilde{\lambda}_3=\delta$, then $\delta<\frac{\pi}{2}$ (otherwise, $\beta=\delta$, an impossibility). However, $2\delta+\mu>\pi$, for all $\mu\in\{\alpha,\beta,\gamma,\delta\}$.

As a direct consequence of the type of adjacency I, we get the following result:

Lemma 1 (Elimination Lemma)**.** *In any local configuration of a dihedral f-tiling whose prototiles are isosceles triangles with adjacency of type I, there are no vertices surrounded by sequences of alternate angles containing:*

(a) p angles β and q angles δ with $p,q\geq 1$, or

(b) one angle α (resp. γ) and q angles δ (resp. β), $q\geq 1$, or

(c) one angle α (resp. γ), p angles β and q angles δ with $p,q\geq 1$.

2. Triangular dihedral f-tilings with adjacency of type I

Starting a local configuration of $\tau\in\Omega(T_1,T_2)$ with two adjacent cells congruent to T_1 and T_2, respectively, see Figure 12, a choice for angle $\theta_1\in\{\gamma,\delta\}$ must be made.

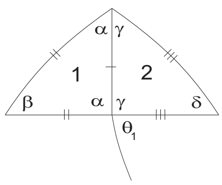

Figure 12: *Local configuration*

In order to facilitate the construction of dihedral f-tilings, we distinguish the different order relations between the angles. Therefore, in the first subsection, we shall consider the order relation $\alpha > \beta$ and $\gamma > \delta$, in the second subsection, $\alpha > \beta$ and $\delta > \gamma$, in the third subsection $\beta > \alpha$ and $\gamma > \delta$ and in the fourth subsection $\beta > \alpha$ and $\delta > \gamma$.

2.1. $\alpha > \beta$ and $\gamma > \delta$

With the above terminology one has:

Proposition 1. *If $\theta_1 = \gamma$, then $\Omega(T_1, T_2) \neq \varnothing$ if and only if $\beta = \delta = \frac{\pi}{k}$, $k \geq 3$ and $\alpha + \gamma = \pi$, $\alpha \in]\frac{\pi}{2}, \frac{2\pi}{3}[$. In this case, we get a 2-parameter family of dihedral f-tilings denoted by $\mathcal{F}_{k,\alpha}$ with D_{2k}, the dihedral group of order $4k$, as a group of symmetry. The action of D_{2k}, respectively, on the faces and vertices of $\mathcal{F}_{k,\alpha}$ gives rise to 2 classes of isohedrality and 3 classes of isogonality. A 3D representation of a member of this family $\mathcal{F}_{3,\alpha}$ is given in Figure 15.*

Proof. In order to have $\Omega(T_1, T_2) \neq \emptyset$, necessarily $\alpha + \theta_1 \leq \pi$.
1. Let us assume that $\alpha + \theta_1 = \pi$, with $\theta_1 = \gamma$. By adjacency condition (1) and taking into account that $\alpha > \gamma$, one has $\alpha > \frac{\pi}{2} > \gamma > \delta$ and $\beta = \delta$.
Expanding the configuration illustrated in Figure 12, we obtain the following one with $\theta_2 \in \{\alpha, \beta\}$.

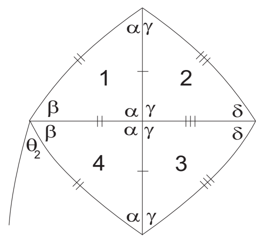

Figure 13: *Local configuration*

1.1. Suppose firstly, that $\theta_2 = \alpha$. Then, $\theta_2 + \beta < \pi$, otherwise, from $\alpha + \gamma = \pi$, we get $\beta = \delta = \gamma$, which is impossible. Taking into account the Elimination Lemma, the sum containing the alternate angles θ_2 and β must be of the form $\alpha + k\beta = \pi$, for $k \geq 2$.

If $\alpha + k\beta = \pi$, $k \geq 2$, the configuration extends a bit more, to the one shown in Figure 14. But due to incompatibility of the sides length, it is impossible to pursue the construction of the dihedral f-tiling.
1.2. Assume that $\theta_2 = \beta$.

If β is a submultiple of π, say $\beta = \frac{\pi}{k}$, then $k \geq 3$ and we may expand (in a unique way) the configuration given in Figure 13 by obtaining a global representation of a tiling $\tau \in \Omega(T_1, T_2)$ denoted by $\mathcal{F}_{k,\alpha}$. For each $k \geq 3$ and $\alpha \in]\frac{\pi}{2}, \frac{2\pi}{3}[$, $\mathcal{F}_{k,\alpha}$ has $4k$ isosceles triangles ($2k$ of each type). In Figure 15, we show a 2D and a $3D$ representation of $\mathcal{F}_{3,\alpha}$.

The symmetry group of $\mathcal{F}_{k,\alpha}$ is D_{2k}, the dihedral group of order $4k$ generated by the rotation, $R^z_{\frac{\pi}{k}}$, around the zz axis, through $\frac{\pi}{k}$ and the reflection ρ^{yz}, on the plane yOz. D_{2k} acts on the set of vertices of $\mathcal{F}_{k,\alpha}$ partitioning it into three classes. Each vertex of valency $2k$ forms a class of isogonality and the vertices of valency four are all in the same vertex transitive class. In relation to the faces of $\mathcal{F}_{k,\alpha}$, the action of the dihedral group D_{2k}, gives rise to two classes of isohedrality, each class containing $2k$ congruent isosceles triangles.

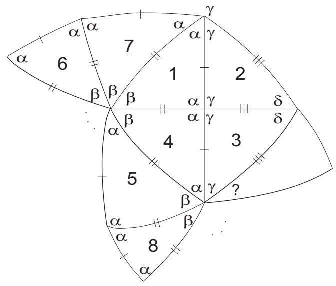

Figure 14: *Local configuration*

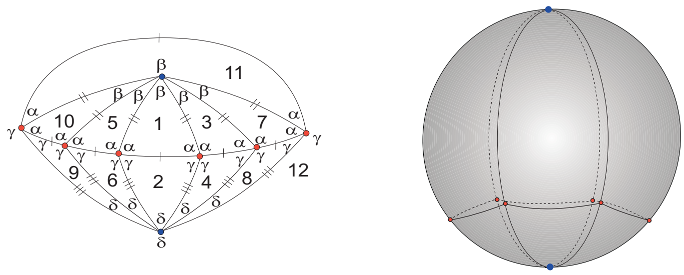

Figure 15: *2D and 3D representation of* $\mathcal{F}_{3,\alpha}$

Assuming now $k\beta \neq \pi,\ k \geq 2$, then, in order to satisfy the angle folding relation, one should have $k\beta + \alpha = \pi$ for $k \geq 2$, by the Elimination Lemma. In this case, the angle α has several possible positions to be placed and we shall study every one of these possibilities.

1.2.1. Assume firstly, that the angle α is placed in the tile numbered 6, as in Figure 16-I (by a symmetry argument it is not necessary to consider the case in which angle α is placed next to tile 5). In Figure 16-I the angle labelled θ_3 is either β or α.

By the Elimination Lemma, the sum of alternate angles β and γ does not satisfy the angle folding relation. Therefore, $\theta_3 = \beta$.

Expanding a bit more the configuration in Figure 16-I, we get a vertex surrounded by alternate angles β and γ, whose sum $\beta + \gamma + \mu,\ \mu \in \{\alpha, \beta, \gamma, \delta\}$ leads to the angle folding relation infringement or it is wiped out by the Elimination Lemma (Figure 16-II).

1.2.2. If the angle α is placed in the tile labelled 10 (Figure 17), the vertices located on the border of the configuration in Figure 17 are all of valency four giving rise to

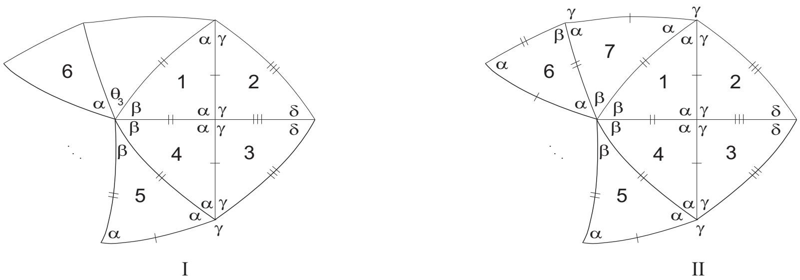

Figure 16: *Local configuration*

a vertex surrounded by alternate angles β and γ, which is an impossibility, as we have seen before.

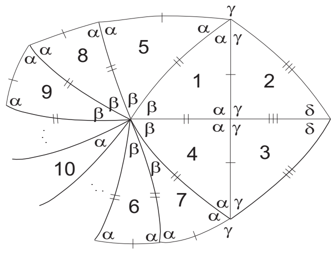

Figure 17: *Local configuration*

2. Assume that $\alpha + \theta_1 < \pi$, with $\theta_1 = \gamma$ (Figure 12). Taking into account that $\alpha > \gamma,\ \alpha > \beta$ and $\gamma > \delta$, then $\delta < \gamma < \frac{\pi}{2}, \alpha + \delta < \pi,\ \beta + \gamma < \pi$ and $\beta + \delta < \pi$. Since vertices of valency four must exist, then $\alpha \geq \frac{\pi}{2}$.

2.1. Assume that $\alpha = \frac{\pi}{2}$. Then, $\beta < \frac{\pi}{2}$ and vertices of valency four are surrounded exclusively by angles α. Taking into account that $\frac{\pi}{2} = \alpha > \gamma >,\ 2\alpha + \beta > \pi$ and $2\gamma + \delta > \pi$, the sum of the alternate angles α and γ satisfies $\alpha + \gamma + k\beta = \pi,\ k \geq 1$. Extending the configuration in Figure 12, a decision must be taken about the angle $\theta_4 \in \{\alpha, \gamma\}$, Figure 18.

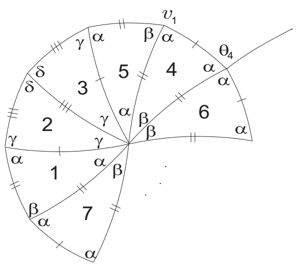

Figure 18: *Local configuration*

If $\theta_4 = \alpha$, we are led to a vertex surrounded by alternate angles α and β, whose sum is of the form $\alpha+\beta+t\beta=\pi,\ t\geq 1$ or $\alpha+\gamma+k\beta=\pi,\ k\geq 1$, since other sums are impossible by using the Elimination Lemma and the fact that $\alpha>\gamma$. Assuming that $\alpha+\gamma+k\beta=\pi, k\geq 1$, the angle arrangement is illustrated in the next figure, where we reach an incompatibility:

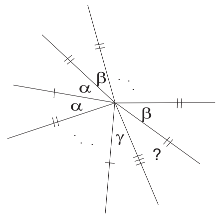

Figure 19: *Angle arrangement*

Accordingly, the vertex v_1 surrounded by the angles β,α,α satisfies $\alpha+t\beta=\pi,\ t\geq 2$ and by expanding the local representation in Figure 18, we are led to another incompatibility of the sides length, as shown in Figure 20-I.

If $\theta_4=\gamma$, then the configuration extends a bit more, but we are led to a similar impossibility, Figure 20-II.

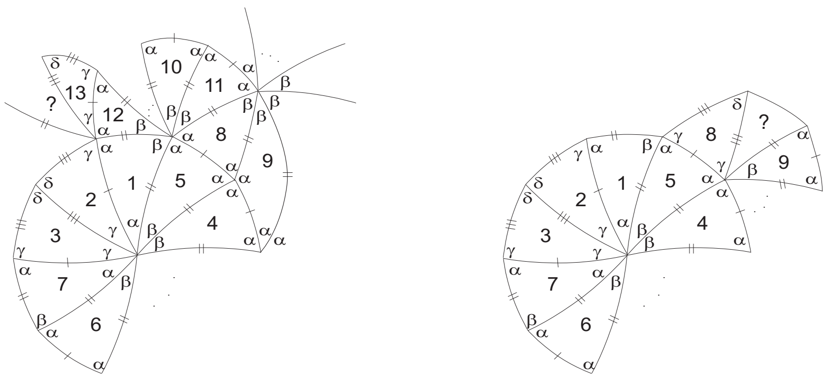

Figure 20: *Local configuration*

2.2. If $\alpha>\frac{\pi}{2}$, as $\delta<\gamma<\frac{\pi}{2}$, vertices of valency four are surrounded by alternate angles α and β since $\alpha+\delta<\alpha+\gamma<\pi$. Additionally, the sum containing the alternate angles α and γ is $\alpha+\gamma+m\delta=\pi, m\geq 1$, which is impossible by the sides length. ☐

Proposition 2. *If $\theta_1=\delta$ (Figure 12), then $\Omega(T_1,T_2)=\emptyset$.*

Proof. If $\theta_1=\delta$, then $\alpha+\theta_1<\pi$ (observe the incompatibility of the sides length if $\alpha+\delta=\pi$). The sum containing these two angles $\alpha+\delta+\mu$ violates the angle folding relation for any $\mu\in\{\alpha,\beta,\gamma,\delta\}$, taking into account the Elimination Lemma, $\alpha>\gamma$ and $2\gamma+\delta>\pi$. ☐

2.2. $\alpha > \beta$ and $\delta > \gamma$

Using the same terminology as in Figure 12, we get:

Proposition 3. *If $\theta_1 = \gamma$, then $\Omega(T_1, T_2)$ is composed by one parameter continuous family of an f-tiling $\mathcal{E}_\alpha$, $\frac{\pi}{2} < \alpha < \pi$, one discrete family of isolated dihedral f-tilings $\mathcal{L}_k$, $k \geq 3$, and one isolated dihedral f-tiling $\mathcal{G}$ such that the sum of alternate angles around vertices are respectively of the form:*

$$\alpha + \gamma = \pi,\ \beta = \delta = \frac{\pi}{2}\ \text{ for }\ \mathcal{E}_\alpha;$$
$$\alpha + 2\gamma = \pi,\ \delta = \frac{\pi}{2}\ \text{ and }\ \beta = \frac{\pi}{k},\ k \geq 3\ \text{ for }\ \mathcal{L}_k;$$

$$2\alpha + \gamma = \pi,\ \alpha = \arccos\frac{\sqrt{6}}{6},\ \delta = \frac{\pi}{2}\ \text{ and }\ \beta = \frac{\pi}{3}\ \text{ for }\ \mathcal{G}.$$

3D representations of a member of $\mathcal{E}_\alpha$, $\mathcal{L}_3$, and $\mathcal{G}$ are given in Figures 22, 27 and 28, respectively.

Proof. Assume that $\theta_1 = \gamma$, in Figure 12. Then, $\alpha + \theta_1 \leq \pi$.
1. If $\alpha + \theta_1 = \pi$, with $\theta_1 = \gamma$, then by adjacency condition (1), one has $\beta = \delta$ and since $\delta > \gamma$, then $\beta = \delta > \frac{\pi}{3}$. Adding a new cell to the configuration in Figure 12, a decision must be made about the angle $\theta_5 \in \{\alpha, \beta\}$, as in Figure 21.

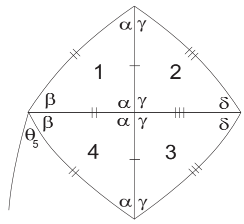

Figure 21: *Local configuration*

1.1. Suppose firstly that $\theta_5 = \alpha$. Then, $\theta_5 + \beta < \pi$, otherwise $\beta = \gamma = \delta$, which is an absurdity. As $\alpha + \beta < \pi$ and $\alpha > \beta$, then $\beta < \frac{\pi}{2}$ and consequently $\gamma < \delta = \beta < \frac{\pi}{2} < \alpha$. Therefore, the sum $\alpha + \beta + \mu$ violates the angle folding relation for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$.
1.2. If $\theta_5 = \beta$, then $\beta \leq \frac{\pi}{2}$.
1.2.1. Suppose that $\beta = \frac{\pi}{2} = \delta$. Then, $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ and $\alpha > \frac{\pi}{2}$. The configuration expands and we get a global representation of an f-tiling denoted by $\mathcal{E}_\alpha$ and illustrated in Figure 22. It is composed by four isosceles triangles of each.
1.2.2. If $\beta = \delta < \frac{\pi}{2}$, then $\gamma < \frac{\pi}{2} < \alpha$.
Taking into account the order relation between the angles and the Elimination Lemma, the sum $2\beta + \mu$ violates the angle folding relation, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$.
2. If $\alpha + \theta_1 < \pi$, with $\theta_1 = \gamma$, a decision about the angle $\theta_6 \in \{\gamma, \delta\}$ must be made.

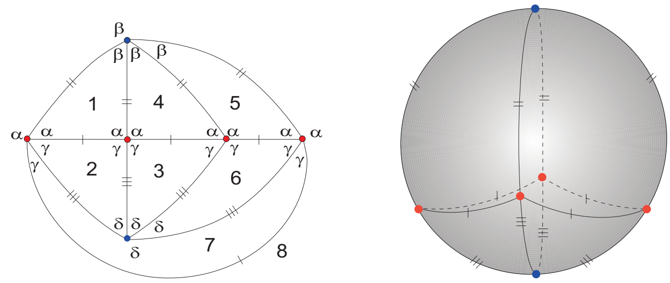

Figure 22: *2D and 3D representation of a member of $\mathcal{E}_\alpha$*

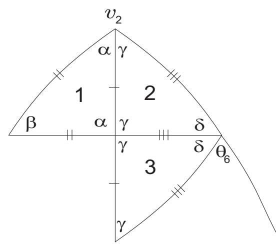

Figure 23: *Local configuration*

2.1. If $\theta_6 = \gamma$ and $\delta + \theta_6 = \pi$, the configuration ends up at a vertex (v_2 in Figure 24) surrounded by angles α, γ, δ, whose sum $\alpha + \delta$ must be of the form $\alpha + \delta + k\beta = \pi,\ k \geq 1$, since $\delta + \gamma = \pi,\ \delta > \frac{\pi}{3}$ and $\alpha > \frac{\pi}{3}$. However, it is impossible to arrange the angles in order to satisfy this sum, due to the Elimination Lemma.

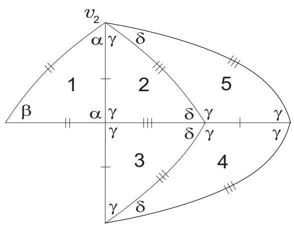

Figure 24: *Local configuration*

2.2. Suppose that $\theta_6 = \gamma$ and $\delta + \theta_6 < \pi$. Taking into account that $\delta > \gamma,\ 2\gamma + \delta > \pi$ and $\alpha > \frac{\pi}{3}$, the sum $\delta + \gamma$ satisfies $\delta + \gamma + k\beta = \pi,\ k \geq 1$. However, it is impossible to arrange the angles in order to satisfy the sum $\delta + \gamma + k\beta = \pi,\ k \geq 1$.
2.3. Assume now that $\theta_6 = \delta$. We shall study the cases $\delta = \frac{\pi}{2}$ and $\delta < \frac{\pi}{2}$ separately.
2.3.1. If $\delta = \frac{\pi}{2}$, then $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$. Since $\alpha > \gamma$, one of the sums of the alternate angles at vertex v_3 in Figure 25 is $\alpha + 2\gamma = \pi$ or $2\alpha + \gamma = \pi$ or $\alpha + \gamma + k\beta = \pi,\ k \geq 1$ or $\alpha + 2\gamma + p\beta = \pi,\ p \geq 1$.
2.3.1.1. If $\alpha + 2\gamma = \pi$ at vertex v_3, then $\frac{\pi}{4} < \gamma < \frac{\pi}{3}$. We may expand the configuration in Figure 25 and get the one shown in Figure 26-I. Note that in the construction

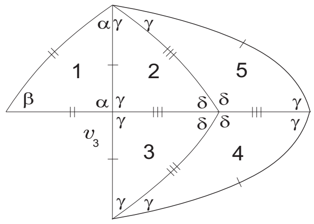

Figure 25: *Local configuration*

of the configuration, vertices surrounded by a sequence of angles containing the subsequence (γ, γ, γ) must be of valency six, with the sum $2\gamma + \alpha = \pi$. Additionally, one of the sums of alternate angles at vertices surrounded by a sequence of angles containing the subsequence (α, γ, γ) is $\alpha + \gamma + k\beta = \pi, k \geq 1$ or $\alpha + 2\gamma = \pi$.

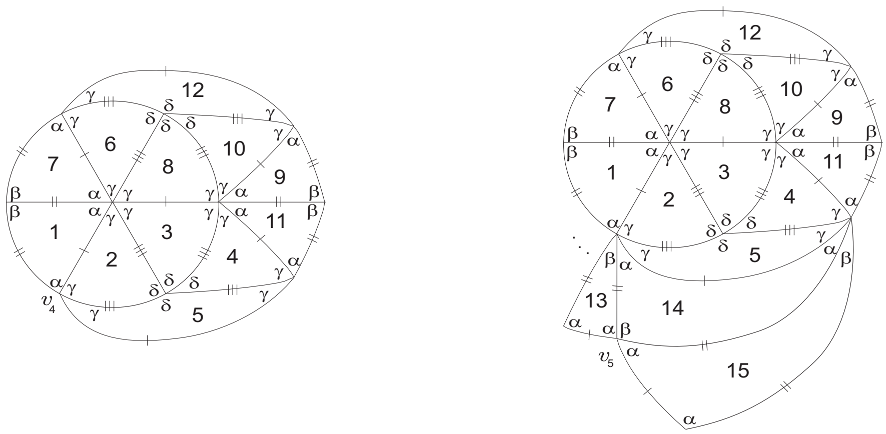

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Figure 26: *Local configuration*

If $\alpha + \gamma + k\beta = \pi,\ k \geq 1$, at vertex v_4, for instance, we extend in a unique way the configuration and we are led to another vertex (v_5) surrounded by angles α, β, α, whose sum 2α satisfies $2\alpha + t\gamma = \pi, t \geq 1$, Figure 26-II. Therefore, from $\alpha + 2\gamma = \pi$, we conclude that $t = 1$, which is impossible. Accordingly, every vertices surrounded by angles α, γ, γ are of valency six, with the sum $\alpha + 2\gamma = \pi$ and the configuration expands in a unique and symmetric way.

Figure 27 shows a global representation of an f-tiling $\tau \in \Omega(T_1, T_2)$, with three types of vertices: vertices of valency four surrounded exclusively by angles δ, vertices of valency six surrounded by alternate angles α, γ, γ and vertices of valency six surrounded exclusively by angles β. It is denoted by $\mathcal{L}^3$ and composed of twelve isosceles triangles of angles α, α, β and twenty four isosceles triangles of angles γ, γ, δ.

For $k \geq 3$, we get a family of discrete dihedral f-tilings denoted by $\mathcal{L}^k$ with $4k$ isosceles triangles of angles α, α, β and $8k$ isosceles triangles of angles γ, γ, δ.

2.3.1.2. If $2\alpha + \gamma = \pi$ at vertex v_3, then $\frac{\pi}{4} < \gamma < \beta < \alpha < \frac{\pi}{2} = \delta$ and all the

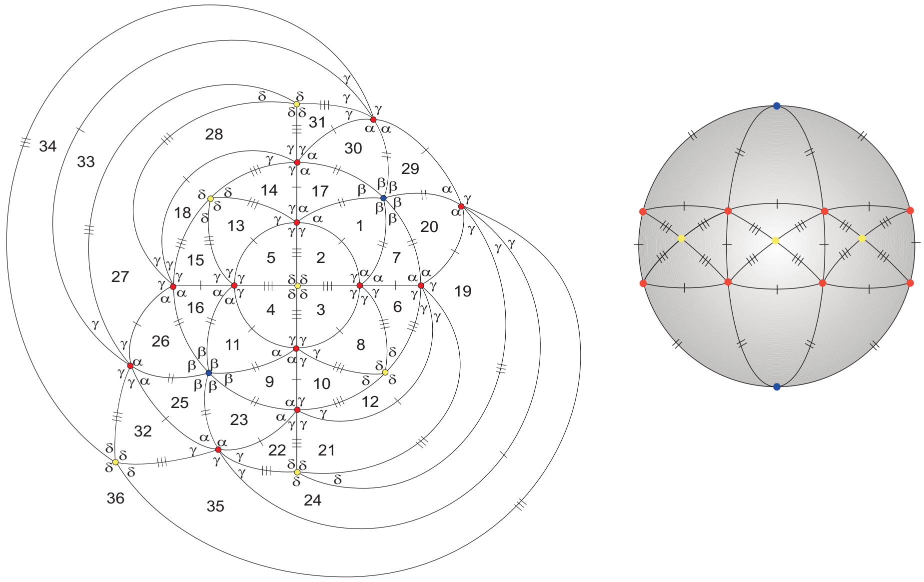

Figure 27: *2D and 3D representation of $\mathcal{L}_3$*

sums containing angles α and γ must satisfy $2\alpha + \gamma = \pi$, since other possibilities ($\alpha + 2\gamma = \pi$ and $\alpha + \gamma + \beta = \pi$) imply that $\alpha = \gamma$ and $\alpha = \beta$, which are both impossible. Expanding the configuration we get a global representation of a tiling denoted by $\mathcal{G} \in \Omega(T_1, T_2)$, see Figure 28.

$\mathcal{G}$ is composed of forty eight isosceles triangles of angles α, α, β and twenty four isosceles triangles of angles γ, γ, δ, with $\beta = \frac{\pi}{3}$, $\delta = \frac{\pi}{2}$, $\alpha = \arccos \frac{\sqrt{6}}{6} \approx 65.905°$ and $\gamma \approx 48.19°$.

2.3.1.3. Assuming that $\alpha + \gamma + k\beta = \pi$ at vertex v_3, the angle θ_7 in Figure 29-I, is β, since $\theta_7 = \alpha$ implies that $2\alpha + \gamma = \pi$ and so $\frac{\pi}{4} < \gamma < \beta$. Consequently, $k = 1$ and $\alpha = \beta$, which is impossible. Extending the configuration in Figure 29-I with $\theta_7 = \beta$, we end up at a vertex surrounded by alternate angles α, γ, γ, whose sum is $\alpha + 2\gamma \leq \pi$ (Figure 29-II).

In case $\alpha + 2\gamma = \pi$, we may add some new cells to the configuration above and the sum of the alternate angles at vertex v_6 (see Figure 30) is of the form $3\gamma + q\beta = \pi$, $q \geq 1$, which is impossible by the Elimination Lemma. Therefore $\alpha + 2\gamma < \pi$ and it remains the possibility $\alpha + 2\gamma + p\beta = \pi$, $p \geq 1$, which is also impossible due to the incompatibility of the edge sides.

2.3.1.4. If $\alpha + 2\gamma + p\beta = \pi$, $p \geq 1$, at vertex v_3, then $\alpha < \alpha + p\beta < \delta = \frac{\pi}{2}$ so $2\gamma \leq 2\gamma + (p-1)\beta < \alpha < \frac{\pi}{2}$ taking into account that $2\gamma + \delta > \pi$ and $2\alpha + \beta > \pi$. Therefore, $\gamma < \frac{\pi}{4}$ (since $2\gamma < \alpha < \frac{\pi}{2}$), contradicting $\delta = \frac{\pi}{2}$ and $2\gamma + \delta > \pi$.

2.3.2. In case $\delta < \frac{\pi}{2}$, then $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ and so $\alpha \geq \frac{\pi}{2}$ or $\beta \geq \frac{\pi}{2}$. Either way, $\alpha \geq \frac{\pi}{2}$.

The sum of alternate angles α and γ in Figure 23 is $\alpha + \gamma + k\beta = \pi, k \geq 1$, since other possibilities ($2\alpha + \gamma = \pi$ or $2\gamma + \alpha = \pi$) contradict the facts $\alpha \geq \frac{\pi}{2}$ and $\alpha > \gamma$. Extending the configuration in Figure 23, with $\theta_6 = \delta$, we end up at a vertex whose angles arrangement is impossible, see Figure 31-I.

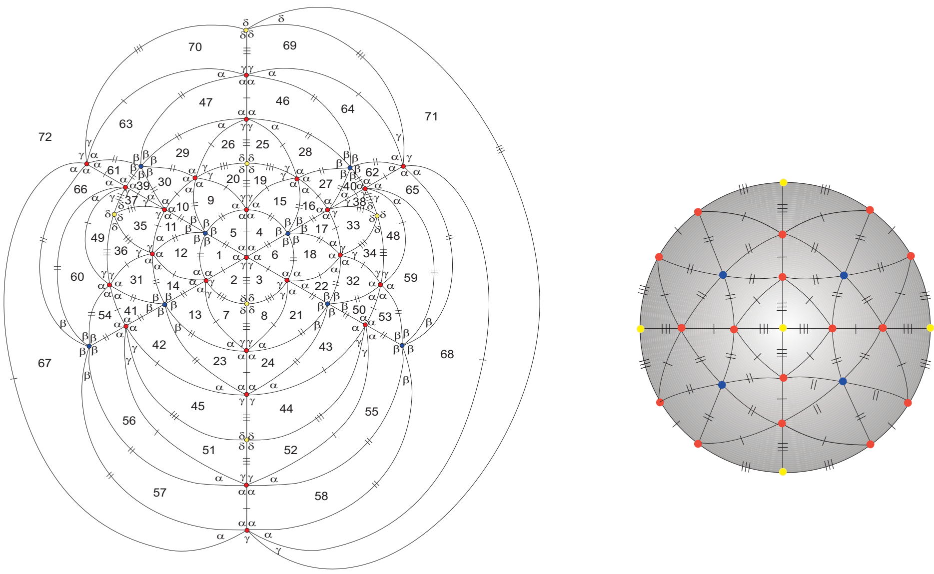

Figure 28: *2D and 3D representation of $\mathcal{G}$*

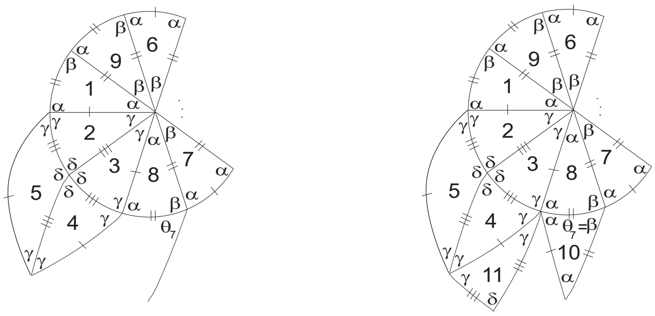

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Figure 29: *Local configuration*

If $\theta_6 = \gamma$ (Figure 31-II), then the sum containing the alternate angles δ and α is of the form $\alpha + \delta + \gamma = \pi$ or $\alpha + \delta + p\beta = \pi$, $p \geq 1$. However, the first case leads to $\alpha < \gamma$, which is an absurdity and the second case is impossible using the Elimination Lemma. □

Proposition 4. *The symmetry group of:*

i) $\mathcal{E}_\alpha$, $\frac{\pi}{2} < \alpha < \pi$, *is the dihedral group D_4, whose action on the set of vertices and faces gives rise to three classes of isogonality and two classes of isohedrality, respectively;*

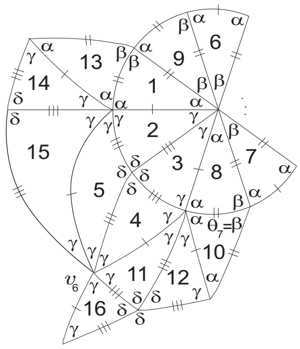
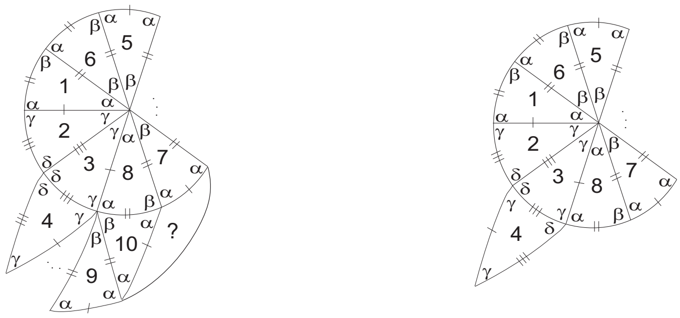


Figure 30: *Local configuration*



I II

Figure 31: *Local configuration*

ii) $\mathcal{L}_k, k \geq 3$ *is the group* $D_{2k} \times Z_2$*, partitioning the set of vertices and faces into three classes of isogonality and two classes of isohedrality, respectively;*

iii) $\mathcal{G}$ *is the group* $S_4 \times Z_2$*, whose action on the set of vertices and faces give rise to three classes of isogonality and two classes of isohedrality, respectively.*

Proof. For each $\alpha \in]\frac{\pi}{2}, \pi[$, $\mathcal{E}_\alpha$ has as a symmetry group the group generated by ρ^{xz}, the reflection on the coordinate plane xOz and $R^z_{\frac{\pi}{2}}$, the rotation through $\frac{\pi}{2}$ around the zz axis, that is the dihedral group of order 8, D_4. It acts on the set of vertices of $\mathcal{E}_\alpha$ giving rise to three classes of isogonality. The two vertices surrounded by $\beta = \delta$ angles form two of these classes. The remaining four vertices are all in the same isogonality class. In relation to the set of faces of $\mathcal{E}_\alpha$, D_4 acts partitioning it into two classes. Each class contains four congruent isosceles triangles.

The symmetries of $\mathcal{L}_k, k \geq 3$, fixing the vertex $(0,0,1)$ is the dihedral group D_{2k} of order $4k$ generated by ρ^{yz}, the reflection on the plane yOz and the rotation $R^z_{\frac{\pi}{k}}$. Besides, the reflection on the plane xOy, ρ^{xy}, sends $(0,0,1)$ to $(0,0,-1)$. Consequently the group of symmetries of $\mathcal{L}_k, k \geq 3$, is the group $D_{2k} \times Z_2$. There are three classes of isogonality determined by the action of this group, each one containing all vertices of the same type. As in the previous case, there are two classes of isohedrality, each one containing congruent tiles of the prototiles.

The f-tiling $\mathcal{G}$ has the same symmetries as an octahedron and so $S_4 \times Z_2$ is its symmetry group. It follows immediately that the action of $S_4 \times Z_2$ on the vertices and faces of $\mathcal{G}$ gives rise to three classes of isogonality and two classes of isohedrality. $\square$

Proposition 5. *If $\theta_1 = \delta$ (Figure 12), then $\Omega(T_1, T_2) = \varnothing$.*

Proof. If $\theta_1 = \delta$, in Figure 12, then $\alpha + \theta_1 < \pi$, since $\alpha + \theta_1 = \pi$ implies that the vertex is of valency four, which is incompatible with the edge sides.
Taking into account that $\alpha > \frac{\pi}{3}$ and $\delta > \frac{\pi}{3}$, then the sum $\alpha + \delta + \mu = \pi$, for some μ is of the form $\alpha + \delta + \gamma = \pi$ or $\alpha + \delta + \gamma + p\beta = \pi$, $p \geq 1$. Both cases are impossible, since $\alpha > \gamma$ and $2\gamma + \delta > \pi$. $\square$

2.3. $\beta > \alpha$ and $\gamma > \delta$

Suppose now that $\beta > \alpha$ and $\gamma > \delta$. Then, one has $\beta > \alpha > \gamma > \delta$ and also $\beta > \alpha > \gamma > \frac{\pi}{3}$.

Proposition 6. *If $\beta > \alpha$ and $\gamma > \delta$, then $\Omega(T_1, T_2) = \varnothing$.*

Proof. Assume that $\beta > \alpha$ and $\gamma > \delta$.
1. If $\theta_1 = \gamma$ in Figure 12, then $\alpha + \theta_1 \leq \pi$. Suppose first that $\alpha + \theta_1 = \pi$. Then, $\delta < \gamma < \frac{\pi}{2} < \alpha < \beta$ and by the adjacency condition (1), we conclude that $\cos\beta = \cos\delta$, which is an impossibility.

If $\alpha + \theta_1 < \pi$, then in order to satisfy the angle folding relation, the sum $\alpha + \theta_1$ is of the form $\alpha + \gamma + k\delta = \pi$, for some $k \geq 1$, since $\beta > \alpha > \gamma > \frac{\pi}{3}$. However, $\alpha + \gamma + k\delta > \gamma + \gamma + k\delta > \pi$, which is an absurdity.
2. If $\theta_1 = \delta$ (Figure 12), then $\alpha + \theta_1 < \pi$ due to incompatibility of the edge sides. As $\beta > \alpha > \gamma > \delta$ and $2\gamma + \delta > \pi$, then the sum $\alpha + \theta_1$ must be of the form $\alpha + k\delta = \pi$, for some $k \geq 1$. However, by the Elimination Lemma, the sequence of alternate angles $(\alpha, \delta, \delta, ..., \delta)$ is impossible. $\square$

2.4. $\beta > \alpha$ and $\delta > \gamma$

Assume finally that $\beta > \alpha$ and $\delta > \gamma$.

Proposition 7. *If $\theta_1 = \gamma$ (Figure 12), then $\Omega(T_1, T_2) = \varnothing$.*

Proof. Assume that $\theta_1 = \gamma$ in Figure 12, then $\alpha + \theta_1 = \pi$ or $\alpha + \theta_1 < \pi$. We will study both cases separately.
1. If $\alpha + \gamma = \pi$, then $\beta + \delta > \pi$, $\alpha > \frac{\pi}{2} > \gamma$ and by the adjacency condition (1), we conclude that $\beta = \delta > \frac{\pi}{2}$. Adding some new cells to the local configuration, we get a vertex surrounded by the sequence β, β, α and the sum of the alternate angles β and α does not satisfy the angle folding relation.
2. Suppose that $\alpha + \theta_1 < \pi$, with $\theta_1 = \gamma$. A decision about the angle $\theta_8 \in \{\gamma, \delta\}$ must be made, see Figure 33.
2.1. If $\theta_8 = \gamma$ and $\delta + \theta_8 = \pi$, then $\delta > \frac{\pi}{2} > \gamma$. By the adjacency condition (1), we get $\beta > \frac{\pi}{2}$ and consequently the sum $\alpha + \delta$ at vertex v_2 satisfies $\alpha + \delta + k\alpha = \pi$, $k \geq 1$. Since $2\gamma + \delta > \pi$, one has $\alpha + \delta + k\alpha = \pi < 2\gamma + \delta$ and consequently, $2\alpha \leq \alpha + k\alpha < 2\gamma$, which is an absurdity.

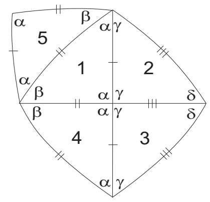


Figure 32: *Local configuration*

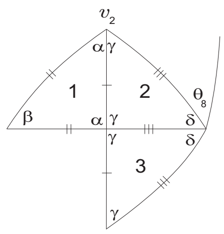


Figure 33: *Local configuration*

2.2. If $\theta_8 = \gamma$ and $\delta + \theta_8 < \pi$, then $\gamma < \frac{\pi}{2}$ and in order to satisfy the angle folding relation, $\delta + \gamma + k\alpha = \pi,\ k \geq 1$ or $\delta + \gamma + \beta = \pi$ or $\delta + \gamma + \alpha + \beta = \pi$. In case $\delta+\gamma+k\alpha = \pi,\ k \geq 1$, then $\alpha < \gamma$, which is impossible, since $\alpha > \gamma$. If $\delta+\gamma+\beta = \pi$ or $\delta + \gamma + \alpha + \beta = \pi$, then $\gamma > \beta > \frac{\pi}{3}$, contradicting the sums.

2.3. If $\theta_8 = \delta$, then $2\delta \leq \pi$.

2.3.1. Assume first that $\delta = \frac{\pi}{2}$. Then, $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ and, by the adjacency condition (1), $\alpha < \frac{\pi}{2}$ and $\beta \neq \frac{\pi}{2}$. The configuration in Figure 33 expands to the one shown in Figure 34-I, with $\theta_9 \in \{\alpha, \gamma\}$.

In case $\theta_9 = \alpha$, then angle $\theta_{10} = \beta$ in Figure 34-II, otherwise we would have $2\alpha + \gamma = \pi$ and so $\frac{\pi}{3} < \alpha < \frac{3\pi}{8} < \frac{\pi}{2}$. Consequently $\beta > \frac{\pi}{2}$, from the adjacency condition (1), and the configuration lead us to a vertex surrounded by angles α, β, β, whose sum $\alpha + \beta + \mu$ violates the angle folding relation for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$. Therefore, $\alpha + \gamma + \beta = \pi$, since the sum $\alpha + \gamma + \beta + \mu$ does not satisfy the angle folding relation, for any $\mu \in \{\alpha, \beta, \gamma, \delta\}$.

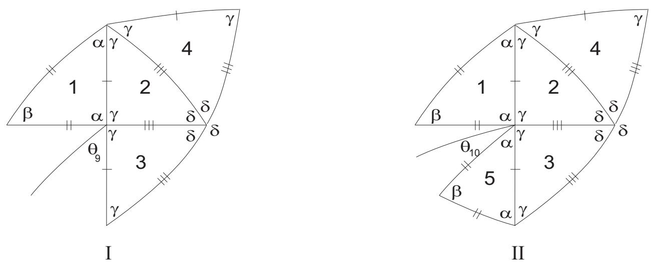


Figure 34: *Local configuration*

The configuration in Figure 34-II now takes the form in Figure 35.

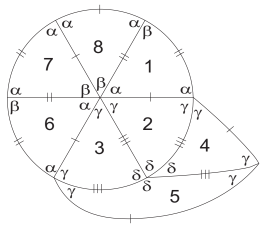


Figure 35: *Local configuration*

Note that all the sums of alternate angles α and γ satisfy $\alpha+\gamma+\beta=\pi$, since the other possibilities ($\alpha+2\gamma=\pi$ or $2\alpha+\gamma=\pi$) contradict the angle folding relation and the fact that $\alpha<\beta$. So the above configuration is now the one shown below:

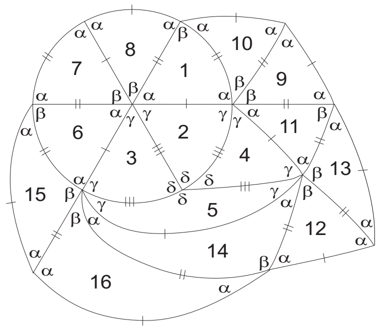


Figure 36: *Local configuration*

In the above configuration, the sum containing two alternate angles α is $3\alpha=\pi$ or $2\alpha+\gamma=\pi$, which is an impossibility, since in the first case we get $3\alpha=\pi=2\alpha+\beta$ which is impossible and in the second case, $\alpha=\beta$.

2.3.2. If $\gamma<\delta<\frac{\pi}{2}$, then the sum containing 2δ must satisfy $2\delta+\gamma=\pi$ or $2\delta+\gamma+s\alpha=\pi,\ s\geq 1$. In both cases, from $2\gamma+\delta>\pi$, we get $\delta<\gamma$, which is an absurdity. □

Proposition 8. *If $\theta_1=\delta$ (Figure 12), then $\Omega(T_1,T_2)=\varnothing$.*

Proof. If $\theta_1=\delta$, then $\alpha+\theta_1<\pi$, since the sum $\alpha+\theta_1=\pi$ is incompatible with the length of the sides. A decision must be made about the angle $\theta_{11}\in\{\gamma,\delta\}$ in Figure 37.

1. If $\theta_{11}=\gamma$, then $\gamma<\frac{\pi}{2}$. The angle labelled $\theta_{12}=\alpha$, otherwise we would have $\alpha+\delta+\gamma\leq\pi$ and from $2\gamma+\delta>\pi$, we conclude that $\alpha<\gamma$, which is an absurdity. Since $\theta_{12}=\alpha$, then $2\alpha+\delta\leq\pi$. But taking into account that $\alpha>\gamma$ and $2\gamma+\delta>\pi$, then $2\alpha+\delta>\pi$ contradicting $2\alpha+\delta\leq\pi$.

2. If $\theta_{11}=\delta$, then the vertex is of valency greater than four, due to the incompatibility of the length of the sides. Accordingly, $\alpha+\gamma<\alpha+\delta<\pi,\ \gamma+\delta<\pi, \gamma<\frac{\pi}{2}$

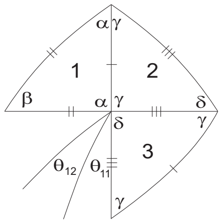


Figure 37: *Local configuration*

and so vertices of valency four are surrounded exclusively by angles α or by angles β or by angles δ or by alternate angles α and β. Either way, $\beta \geq \frac{\pi}{2}$ or $\delta = \frac{\pi}{2}$.

2.1. Assuming that $\beta = \frac{\pi}{2}$, then $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$ and consequently, the sum $\alpha + \delta + \mu$ does not satisfy the angle folding relation, for any $\mu \in \{\alpha, \beta, \gamma\, \delta\}$, since $\alpha > \gamma$, $\delta > \gamma$ and $2\gamma + \delta > \pi$.

2.2. If $\beta > \frac{\pi}{2}$, then vertices of valency four are surrounded by alternate angles α and β or exclusively by angles δ. If $\alpha + \beta = \pi$ and taking into account that $\alpha + \gamma < \alpha + \delta < \pi$, one has $\gamma < \delta < \beta$. By the order relation between the angles $\alpha > \gamma$ and $2\gamma + \delta > \pi$, the sum containing the alternate angles γ and δ does not satisfy the angle folding relation.

2.3. If $\delta = \frac{\pi}{2}$, then $\frac{\pi}{4} < \gamma < \frac{\pi}{2}$ and by compatibility of the edge sides and taking into account that $\alpha > \gamma$ and $2\gamma + \delta > \pi$, the sum containing alternate angles γ and δ violates the angle folding relation. □

All partial results obtained are summarized in the next theorem:

Theorem 1. *If $\beta > \alpha$, then $\Omega(T_1, T_2) = \varnothing$ and if $\alpha > \beta$, then $\Omega(T_1, T_2)$ is composed of 2-parameter family of dihedral f-tilings denoted by $\mathcal{F}_{k,\alpha}$, $k \geq 3$, $\alpha \in]\frac{\pi}{2}, \frac{2\pi}{3}[$, one parameter continuous family $\mathcal{E}_\alpha$, $\frac{\pi}{2} < \alpha < \pi$, one discrete family of isolated dihedral f-tilings $\mathcal{L}_k$, $k \geq 3$, and one isolated dihedral f-tiling $\mathcal{G}$ such that the sums of the alternate angles around vertices are of the form:*

$$\begin{aligned}
\alpha + \gamma = \pi \textit{ and } \beta = \delta = \frac{\pi}{k},\ \alpha \in]\frac{\pi}{2}, \frac{2\pi}{3}[,\ k \geq 3 \textit{ for } \mathcal{F}_{k,\alpha};\\
\alpha + \gamma = \pi \textit{ and } \beta = \delta = \frac{\pi}{2} \textit{ for } \mathcal{E}_\alpha;\\
\alpha + 2\gamma = \pi,\ \delta = \frac{\pi}{2} \textit{ and } \beta = \frac{\pi}{k},\ k \geq 3 \textit{ for } \mathcal{L}_k;\\
2\alpha + \gamma = \pi;\ \alpha = \arccos\frac{\sqrt{6}}{6},\ \delta = \frac{\pi}{2} \textit{ and } \beta = \frac{\pi}{3} \textit{ for } \mathcal{G},\ \textit{respectively.}
\end{aligned}$$

3D representations of a member of $\mathcal{F}_{3,\alpha}$, $\mathcal{E}_\alpha$, $\mathcal{L}_3$, and $\mathcal{G}$ are given in Figures 15, 22, 27 and 28, respectively.

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