MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 151–170

Geodesic ball packings in $\mathbf{H}^2\times\mathbf{R}$ space for generalized Coxeter space groups

Jenő Szirmai[1,*]

[1] *Department of Geometry, Budapest University of Technology and Economics Institute of Mathematics, H-1 521 Budapest, Hungary*

Received April 22, 2010; accepted March 22, 2011

Abstract. After having investigated the geodesic ball packings in $\mathbf{S}^2\times\mathbf{R}$ space we consider the analogous problem in $\mathbf{H}^2\times\mathbf{R}$ space from among the eight Thurston geometries. In this paper, we determine the geodesic balls of $\mathbf{H}^2\times\mathbf{R}$ space and compute their volume, define the notion of the geodesic ball packing and its density. Moreover, we develop a procedure to determine the density of the geodesic ball packing for generalized Coxeter space groups of $\mathbf{H}^2\times\mathbf{R}$ and apply this algorithm to them. E. Molnár showed that the homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4,\boldsymbol{V}_4,\mathbf{R})$. In our work we will use this projective model of $\mathbf{H}^2\times\mathbf{R}$ geometry and in this manner the geodesic lines, geodesic spheres can be visualized on the Euclidean screen of computer.

AMS subject classifications: 52C17, 52C22, 53A35, 51M20

Key words: Thurston geometries, geodesic ball packing, tiling, space group

1. $\mathbf{H}^2\times\mathbf{R}$ space groups

$\mathbf{H}^2\times\mathbf{R}$ is one of the eight simply connected 3-dimensional maximal homogeneous Riemannian geometries. This Seifert fibre space is derived by the direct product of the hyperbolic plane $\mathbf{H}^2$ and the real line $\mathbf{R}$. The points are described by (P,p), where $P\in\mathbf{H}^2$ and $p\in\mathbf{R}$. The complete isometry group $Isom(\mathbf{H}^2\times\mathbf{R})$ of $\mathbf{H}^2\times\mathbf{R}$ can be derived by the direct product of the isometry group $Isom(\mathbf{H}^2)$ of the hyperbolic plane and the isometry group $Isom(\mathbf{R})$ of the real line as follows:

$$\begin{gathered}Isom(\mathbf{H}^2\times\mathbf{R}) := Isom(\mathbf{H}^2)\times Isom(\mathbf{R});\\ Isom(\mathbf{H}^2) := \{A\ :\ \mathbf{H}^2\mapsto\mathbf{H}^2\ :\ (P,p)\mapsto(PA,p)\}\ \text{for any fixed}\ p\in\mathbf{R}.\\ Isom(\mathbf{R}) := \{\rho\ :\ (P,p)\mapsto(P,\pm p+r)\},\ \text{for any fixed}\ P\in\mathbf{H}^2.\\ \text{here the "-" sign provides a reflection in the point}\ \frac{r}{2}\in\mathbf{R},\\ \text{by the "+" sign we get a translation of}\ \mathbf{R}.\end{gathered}\tag{1}$$

The structure of discontinuous acting, so finitely generated isometry group $\Gamma\subset Isom(\mathbf{H}^2\times\mathbf{R})$ is the following: $\Gamma := \langle(A_1\times\rho_1),\dots(A_n\times\rho_n)\rangle$, where $A_i\times\rho_i := A_i\times(R_i,r_i) := (g_i,r_i)$, $(i\in\{1,2,\dots n\}$ and $A_i\in Isom(\mathbf{H}^2)$, R_i is either the identity

*Corresponding author. *Email address:* `szirmai@math.bme.hu` (J. Szirmai)

map $\mathbf{1_R}$ of $\mathbf{R}$ or the point reflection $\overline{\mathbf{1}}_\mathbf{R}$. $g_i := A_i \times R_i$ is called the linear part of the transformation $(A_i \times \rho_i)$ and r_i is its translation part. The multiplication formula is the following:

$$(A_1 \times R_1, r_1) \circ (A_2 \times R_2, r_2) = ((A_1 A_2 \times R_1 R_2, r_1 R_2 + r_2). \qquad (2)$$

Definition 1. *L_Γ is a one-dimensional lattice on $\mathbf{R}$ fibres if there is a positive real number r such that*

$$L_\Gamma := \{kr : (P,p) \mapsto (P, p+kr),\ \forall P \in \mathbf{H}^2;\ \forall p \in \mathbf{R} \mid 0 < r \in \mathbf{R},\ k \in \mathbf{Z}\}.$$

Definition 2. *A group of isometries $\Gamma \subset Isom(\mathbf{H}^2 \times \mathbf{R})$ is called a space group if its linear parts form a cocompact (i.e. of a compact fundamental domain in $\mathbf{H}^2$) group Γ_0 called the point group of Γ; moreover, the translation parts to the identity of this point group are required to form a one-dimensional lattice L_Γ of $\mathbf{R}$.*

Remark 1. *It can be prove easily that such a space group Γ has a compact fundamental domain $\mathcal{F}_\Gamma$ in $\mathbf{H}^2 \times \mathbf{R}$.*

Definition 3. *$\mathbf{H}^2 \times \mathbf{R}$ space groups Γ_1 and Γ_2 are geometrically equivalent, called equivariant, if there is a "similarity" transformation $\Sigma := S \times \sigma$ ($S \in Hom(\mathbf{H}^2), \sigma \in Sim(\mathbf{R})$), such that $\Gamma_2 = \Sigma^{-1}\Gamma_1\Sigma$, where S is a piecewise linear (i.e. PL) homeomorphism of $\mathbf{H}^2$ which deforms the fundamental domain of Γ_1 into that of Γ_2. Here $\sigma(s,t) : p \to p \cdot s + t$ is a similarity of $\mathbf{R}$, i.e. multiplication by $0 \neq s \in \mathbf{R}$ and then addition by $t \in \mathbf{R}$ for every $p \in \mathbf{R}$.*

The equivariance class of a hyperbolic plane group or its orbifold [4] can be characterized by its *Macbeath-signature*. In 1967-69, Macbeath completed the classification of hyperbolic crystallographic plane groups, (for short NEC groups) [5]. He considered isometries containing orientation -preserving and -reversing transformations as well in the hyperbolic plane. His paper deals with NEC groups, but the Macbeath-signature economically characterizes the Euclidean and spherical plane groups, too. The signature of a plane group is the following

$$(\pm, g; [m_1, m_2, \dots, m_r];\ \{(n_{11}, n_{12}, \dots, n_{1s_1}), \dots, (n_{k1}, n_{k2}, \dots, n_{ks_k})\}), \qquad (3)$$

and, with the same notations, the combinatorial measure T of the fundamental polygon is expressed by:

$$T\kappa = \pi \Big\{ \sum_{l=1}^{r} \Big(\frac{2}{m_l} - 2 \Big) + \sum_{i=1}^{k} \Big(-2 + \sum_{j=s_1}^{s_i} \Big(-1 + \frac{1}{n_{ij}} \Big) \Big) + 2\chi \Big\}. \qquad (4)$$

Here $\chi = 2 - \alpha g$ ($\alpha = 1$ for $-$, $\alpha = 2$ for $+$, the sign $\pm$ refers to orientability), χ is the Euler characteristic of the surface with genus g, and κ will denote the Gaussian curvature of the realizing plane $\mathbf{S}^2$, $\mathbf{E}^2$ or $\mathbf{H}^2$, whenever $\kappa > 0$, $\kappa = 0$ or $\kappa < 0$, respectively. The genus g, proper periods m_l of r rotation centres and period-cycles $(n_{i1}, n_{i2}, \dots, n_{is_i})$ of dihedral corners on i^{th} one of the k boundary components, together with a marked fundamental polygon with side pairing generators and a corresponding group presentation determine a plane group up to a well-formulated equivariance for $\mathbf{S}^2$, $\mathbf{E}^2$ and $\mathbf{H}^2$, respectively [4, 5].

Similarly to the theorem proved by J.Z. Farkas in [2] for $\mathbf{S}^2\times\mathbf{R}$ space, we obtain the following:

Theorem 1. *Let Γ be an $\mathbf{H}^2\times\mathbf{R}$ space group, its point group Γ_0 belongs to one of the following three types:*

I. $\mathbf{G_{H^2}} \times \mathbf{1_R}$, *$\mathbf{1_R} : x \mapsto x$ is the identity of $\mathbf{R}$.*

II. $\mathbf{G_{H^2}} \times \langle\overline{\mathbf{1}}_\mathbf{R}\rangle$, *where $\overline{\mathbf{1}}_\mathbf{R} : x \mapsto -x+r$ is the $\frac{r}{2}$ reflection of $\mathbf{R}$ with some r and $\langle\overline{\mathbf{1}}_\mathbf{R}\rangle$ denotes its special linear group of two elements.*

III. If the hyperbolic group $\mathbf{G_{H^2}}$ contains a normal subgroup $\mathbf{G}$ of index two, then $\mathbf{G_{H^2}G} := \{\mathbf{G}\times\mathbf{1_R}\}\cup\{(\mathbf{G_{H^2}}\setminus\mathbf{G})\times\overline{\mathbf{1}}_\mathbf{R}\}$ *forms a point group.*

Here $\mathbf{G_{H^2}}$ is a group of hyperbolic isometries with a compact fundamental domain $\mathcal{F}_\Gamma$.

Proof. Types I and II come up and they are not equivalent with each other. Equivariance of the hyperbolic group component would be necessary, but then type I would be a normal subgroup in type II of index two, and this excludes the possibility of equivariance.

The groups of type III must be compared with the groups of type II, but it is clear that the equivarence is impossible.

The existence of further groups is excluded because if only $\mathbf{1_R}$ comes to the $\mathbf{R}$-component, then we obtain type I. When the $\mathbf{R}$-component of the point group Γ_0 includes the reflection $\overline{\mathbf{1}}_\mathbf{R}$, then $(A_i\times\overline{\mathbf{1}}_\mathbf{R})(A_j\times\overline{\mathbf{1}}_\mathbf{R})) = (A_iA_j\times\mathbf{1_R})$ shows that elements $g_k = (A_k\times\mathbf{1_R})$ form a normal subgroup of index two; consequently, Γ_0 lies in type II or in type III. □

Definition 4. *An $\mathbf{H}^2\times\mathbf{R}$ space group Γ is called a generalized Coxeter group if the generators $\mathbf{g}_i$, $(i=1,2,\dots m)$ of its point group Γ_0 are reflections and the possible translation parts of all the above generators are lattice translations, i.e. $\tau_i = 0,\ modL_\Gamma$ $(i=1,2,\dots m)$.*

In this paper we deal with "generalized Coxeter space groups" in an $\mathbf{H}^2\times\mathbf{R}$ space given by parameters $2\le p_1,p_2,p_3,\dots p_m\in\mathbb{N}$ where $\Sigma_{i=1}^m\frac{1}{p_i}<(m-2)$, $(m\ge 3)$:

1. $\Gamma^1_{(p_1,p_2,\dots p_m)}$ $(+,\ 0,\ [\ \ ]\ \{(p_1,p_2,\dots p_m)\})$,
   $\Gamma_0 = (\mathbf{g}_1,\mathbf{g}_2,\dots\mathbf{g}_m - \mathbf{g}_1^2,\mathbf{g}_2^2,\dots\mathbf{g}_m^2,(\mathbf{g}_1\mathbf{g}_2)^{p_1},\dots(\mathbf{g}_m\mathbf{g}_1)^{p_m})$,
2. $\Gamma^2_{(p_1,p_2,\dots p_m)}$ $\overline{(+,\ 0,\ [\ \ ]\ \{(p_1,p_2,\dots p_m)\})}$,
   $\Gamma_0 = (\mathbf{g}_1,\mathbf{g}_2,\dots\mathbf{g}_m,\mathbf{g}_{m+1} - \mathbf{g}_1^2,\mathbf{g}_2^2,\dots\ \mathbf{g}_m^2,\mathbf{g}_{m+1}^2,(\mathbf{g}_1\mathbf{g}_2)^{p_1},\dots(\mathbf{g}_m\mathbf{g}_1)^{p_m}$, $(\mathbf{g}_1\mathbf{g}_{m+1})^2$, $(\mathbf{g}_2\mathbf{g}_{m+1})^2,\dots(\mathbf{g}_m\mathbf{g}_{m+1})^2)$, that means that $\mathbf{g}_{m+1} = \overline{\mathbf{1}}_\mathbf{R}$.

For a fundamental domain of the above space groups we can combine a fundamental domain of a Coxeter group of the hyperbolic plan with a part of a real line segment r or $\frac{r}{2}$.

2. Geodesic curve of $\mathbf{H}^2\times\mathbf{R}$

In [6], E. Molnár has shown that homogeneous 3-spaces have a unified interpretation in the projective 3-sphere $\mathcal{PS}^3(\mathbf{V}^4, \boldsymbol{V}_4, \mathbf{R})$. In our work we shall use this projective model of $\mathbf{H}^2\times\mathbf{R}$ and the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$,$E_1^\infty(\mathbf{e}_1)$,$E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\}\subset\mathbf{V}^4$ with the unit point $E(\mathbf{e}=\mathbf{e}_0+\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3))$ which is distinguished by an origin E_0 and by ideal points of coordinate axes, respectively. Moreover, $\mathbf{y}=c\mathbf{x}$ with $0<c\in\mathbf{R}$ (or $c\in\mathbb{R}\setminus\{0\}$) defines a point $(\mathbf{x})=(\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}\subset\boldsymbol{V}_4$ describes simplex planes, especially the plane at infinity $(\boldsymbol{e}^0)=E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v}=\boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u})=(\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0=\mathbf{x}\boldsymbol{u}=\mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x})=(\mathbf{y})$ and plane $(\boldsymbol{u})=(\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus $\mathbf{H}^2\times\mathbf{R}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

The points of an $\mathbf{H}^2\times\mathbf{R}$ space, forming an open cone solid in the projective space $\mathcal{P}^3$, are the following:

$$\mathbf{H}^2\times\mathbf{R} := \big\{X(\mathbf{x}=x^i\mathbf{e}_i)\in\mathcal{P}^3 : -(x^1)^2+(x^2)^2+(x^3)^2<0<x^0,\ x^1\big\}.$$

In this context, E. Molnár [6] has derived the infinitesimal arc-length square at any point of $\mathbf{H}^2\times\mathbf{R}$ as follows

$$\begin{aligned}(ds)^2 = {} & \frac{1}{(-x^2+y^2+z^2)^2}\cdot[(x)^2+(y)^2+(z)^2](dx)^2+ \qquad (5)\\ & +2dxdy(-2xy)+2dxdz(-2xz)+[(x)^2+(y)^2-(z)^2](dy)^2+\\ & +2dydz(2yz)+[(x)^2-(y)^2+(z)^2](dz)^2. \qquad (6)\end{aligned}$$

This becomes simpler in the following special (cylindrical) coordinates (t,r,α), $(r\geq 0,\ -\pi<\alpha\leq\pi)$ with the fibre coordinate $t\in\mathbf{R}$. We describe points in our model by the following equations:

$$x^0=1,\quad x^1=e^t\cosh r,\quad x^2=e^t\sinh r\cos\alpha,\quad x^3=e^t\sinh r\sin\alpha. \qquad (7)$$

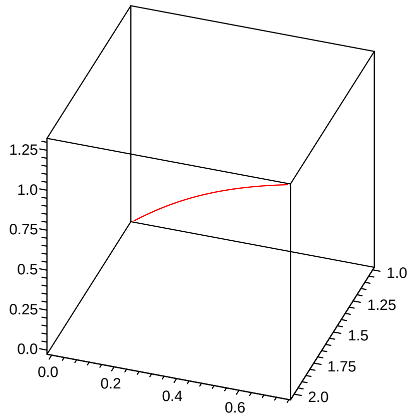
Then we have $x=\frac{x^1}{x^0}=x^1$, $y=\frac{x^2}{x^0}=x^2$, $z=\frac{x^3}{x^0}=x^3$, i.e. the usual Cartesian coordinates. By [6] we obtain that in this parametrization the infinitesimal arc-length square by (5) at any point of $\mathbf{H}^2\times\mathbf{R}$ is as follows

$$(ds)^2=(dt)^2+(dr)^2+\sinh^2 r(d\alpha)^2. \qquad (8)$$

Hence we get the symmetric metric tensor field g_{ij} on $\mathbf{H}^2\times\mathbf{R}$ by components:

$$g_{ij} := \begin{pmatrix}1&0&0\\0&1&0\\0&0&\sinh^2 r\end{pmatrix}. \qquad (9)$$

The geodesic curves of $\mathbf{H}^2\times\mathbf{R}$ are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the



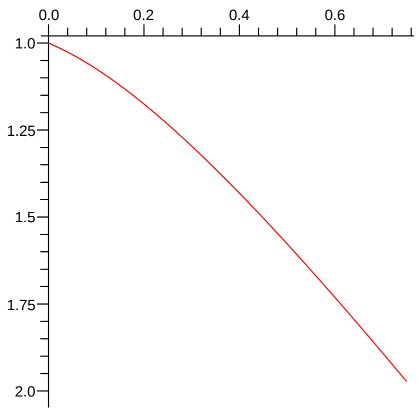


Figure 1: $\mathbf{H}^2\times\mathbf{R}$ *geodesic curve starting from* $(1,1,0,0)$, *i.e.* $t(0)=\phi(0)=\theta(0)=0$ *and with unit velocity by* $u=\frac{\pi}{10}, v=\frac{\pi}{3}$. *The arc-length parameter* $\tau=s$ *runs in* $[0,1]$

parametrized geodesic curves $\gamma(t(\tau), r(\tau), \alpha(\tau))$ in our model can be determined by the general theory of Riemann geometry:

By (9) the second order differential equation system of the $\mathbf{H}^2\times\mathbf{R}$ geodesic curve is the following [9]:

$$\ddot{\alpha}+2\coth(r)\,\dot{r}\dot{\alpha}=0,\ \ddot{r}-\sinh(r)\cosh(r)\dot{\alpha}^2=0,\ \ddot{t}=0, \tag{10}$$

from which we first get a line as a "geodesic hyperbola" on our model of $\mathbf{H}^2$ times a component on $\mathbf{R}$ each running with constant velocity c and ω, respectively:

$$t=c\cdot\tau,\quad \alpha=0,\quad r=\omega\cdot\tau,\quad c^2+\omega^2=1. \tag{11}$$

We can assume that the starting point of a geodesic curve is $(1,1,0,0)$, because we can transform a curve into an arbitrary starting point, moreover, unit velocity with "geographic" coordinates (u,v) can be assumed;

$$r(0)=\alpha(0)=t(0)=0;\quad \dot{t}(0)=\sin v,\ \dot{r}(0)=\cos v\cos u, \dot{\alpha}(0)=\cos v\sin u;$$
$$-\pi<u\leq\pi,\quad -\frac{\pi}{2}\leq v\leq\frac{\pi}{2}.$$

Then by (7), with $c=\sin v$, $\omega=\cos v$ we get the equation systems of a geodesic curve, visualized in Figure 1 in our Euclidean model [9]:

$$\begin{aligned}
x(\tau)&=e^{\tau\sin v}\cosh\left(\tau\cos v\right),\\
y(\tau)&=e^{\tau\sin v}\sinh\left(\tau\cos v\right)\cos u,\\
z(\tau)&=e^{\tau\sin v}\sinh\left(\tau\cos v\right)\sin u,\\
-\pi&<u\leq\pi,\quad -\frac{\pi}{2}\leq v\leq\frac{\pi}{2}.
\end{aligned}\tag{12}$$

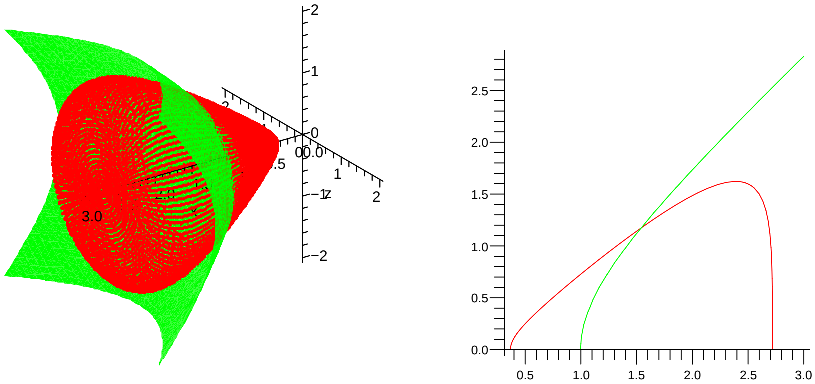


Figure 2: a, b

Remark 2. *Thus we have also harmonized the scales along the fibre lines and that of* $\mathbf{H}^2$.

Definition 5. *The distance* $d(P_1, P_2)$ *between points* P_1 *and* P_2 *is defined by the arc length of the geodesic curve from* P_1 *to* P_2.

Definition 6. *The geodesic sphere of radius* ρ *(denoted by* $S_{P_1}(\rho)$*) with the centre at the point* P_1 *is defined as the set of all points* P_2 *in the space with the condition* $d(P_1, P_2) = \rho$. *Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in the* $\mathbf{H}^2\!\times\!\mathbf{R}$ *space.*

Remark 3. *In this paper we consider only the usual spheres with a "proper centre", i.e.* $P_1 \in \mathbf{H}^2\!\times\!\mathbf{R}$. *If the centre of a "sphere" lies on the absolute quadric or lies out of our model, the notion of the "sphere" (similarly to the hyperbolic space) can be defined, but these cases will be studied in a forthcoming work.*

Definition 7. *The body of the geodesic sphere of centre* P_1 *and of radius* ρ *in* $\mathbf{H}^2\!\times\!\mathbf{R}$ *space is called a geodesic ball, denoted by* $B_{P_1}(\rho)$, *i.e.* $Q \in B_{P_1}(\rho)$ *iff* $0 \leq d(P_1, Q) \leq \rho$.

Remark 4. *Henceforth, typically we choose* $(1, 1, 0, 0)$ *as the centre of the sphere and its ball, by the homogeneity of* $\mathbf{H}^2\!\times\!\mathbf{R}$.

Figure 2. a shows a geodesic sphere of radius $\rho = 1$ with the centre at the point $(1, 1, 0, 0)$ and Figure 2. b shows its intersection with the $(x, +z)$ halfplane. From (12) it follows that $S(\rho)$ is a simply connected surface in $\mathbf{E}^3$ for $\rho > 0$.

2.1. The volume of a geodesic ball

Theorem 2.

$$\begin{aligned} Vol(B(\rho)) &= \int_V \frac{1}{(x^2-y^2-z^2)^{3/2}} \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z \\ &= \int_0^\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\pi}^{\pi} |\tau \cdot \sinh(\tau\cos(v))|\ \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}\tau \\ &= 2\pi \int_0^\rho \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\tau \cdot \sinh(\tau\cos(v))|\ \mathrm{d}v\ \mathrm{d}\tau. \end{aligned} \tag{13}$$

Proof. We get the above volume formula of the geodesic ball $B(\rho)$ of radius ρ by the usual method of the classical differential geometry. We have used the metric tensor g_{ij} and the Jacobian of (12) and we shall apply careful numerical Maple computation. We remark that a power series expansion (here that of function sinh by $\tau\cos(v)$) leads towards a useful comparison of ball volumes in other Thurston geometries as well. □

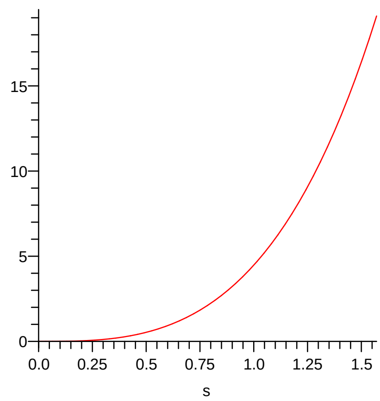


Figure 3: *The increasing function* $\rho \mapsto Vol(B(\rho))$

2.2. On $\mathbf{H}^2\times\mathbf{R}$ prism

In $\mathbf{H}^2\times\mathbf{R}$, a prism is the convex hull of two congruent p-gons ($p > 2$) in "parallel planes", (a "plane" is one sheet of concentric two sheeted hyperboloids in our model) related by translation along the radii joining their corresponding vertices that are the common perpendicular lines of the two "hyperboloid-planes". The prism is a polyhedron having at each vertex one hyperbolic p-gon and two "quadrangles". A $\mathbf{H}^2\times\mathbf{R}$ tiling can be generated from congruent prisms under a generalized Coxeter spcace group.

p-gonal faces of a prism are called cover-faces and other faces are side-faces. In these cases every face of each polyhedron meets only one face of another polyhedron. The midpoints of the side edges form a "hyperboloid plane" denoted by Π. It can be assumed that the plane Π is the *reference plane*: in our coordinate system (see (2.2)) it has the fibre coordinate $t = 0$.

Figure 4. b shows such a prism, a fundamental domain of the space group $\Gamma^1_{(3,7,2)}$ generated by the four reflections $\mathbf{g}_i \in Isom(\mathbf{H}^2)$ $(i = 1, 2, 3)$, $\mathbf{g}_4 \in \overline{\mathbf{1}}_{\mathbf{R}}$. The prism and its images by the above space group generate a tiling in $\mathbf{H}^2\times\mathbf{R}$ space. Figure 4. a shows the above prism with "reference plane" Π. By the properties of the $\mathbf{H}^2\times\mathbf{R}$

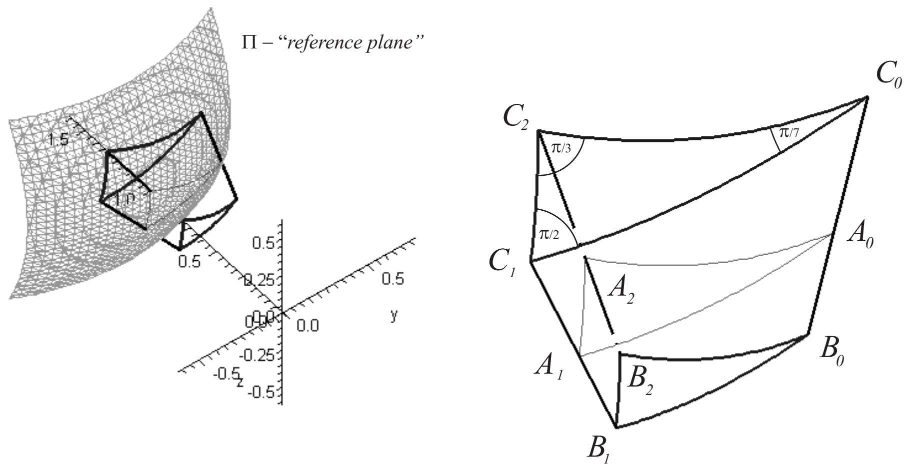


Figure 4: *a, b*

geometry (see Section 1) we obtain the following

Theorem 3. *The volume of an $\mathbf{H}^2\times\mathbf{R}$ p-gonal prism $\mathcal{P}_{B_0B_1B_2\dots B_{p-1}C_0C_1C_2\dots C_{p-1}}$ (see Fig. 4.a-b) can be computed by the following formula:*

$$Vol(\mathcal{P}) = \mathcal{A}\cdot h, \tag{14}$$

where $\mathcal{A}$ is the area of the hyperbolic p-gon $A_0A_1A_2\dots A_{p-1}$ in the reference plane Π and $h = B_0C_0$ is the height of the prism.

Remark 5. *It is clear that the orthogonal projection of the cover-faces on the plane Π form a hyperbolic Coxeter tiling where the tiles are hyperbolic p-gons.*

3. Ball packings

By Remark 1, it follows that an $\mathbf{H}^2\times\mathbf{R}$ space group Γ has a compact fundamental domain. Usually the shape of the fundamental domain of a crystallographic group of $\mathbf{H}^2$ is not determined uniquely but the area of the domain is finite and determined uniquely by the group on the base of its combinatorial measure (see formula (4)). Of course, the $\mathbf{R}$-component provides a parameter for the fundamental prism by its height. But this height will be chosen appropriate-optimally as a diameter of the optimal circle inscribed in the fundamental polygon in $\mathbf{H}^2$ to its prism in $\mathbf{H}^2\times\mathbf{R}$.

In the following, let Γ be a fixed generalized Coxeter group (see Definition 4) of $\mathbf{H}^2\times\mathbf{R}$. We will denote by $d(X, Y)$ the distance of two points X, Y by Definition 5.

Definition 8. *We say that the point set*

$$\mathcal{D}(K) = \{X \in \mathbf{H}^2\!\times\!\mathbf{R} \,:\, d(K,X) \leq d(K^{\mathbf{g}},X) \text{ for all } \mathbf{g} \in \Gamma\}$$

is the Dirichlet–Voronoi cell *(D-V cell) to* Γ *around the kernel point* $K \in \mathbf{H}^2\!\times\!\mathbf{R}$.

Definition 9. *We say that*

$$\Gamma_X = \{\mathbf{g} \in \Gamma \,:\, X^{\mathbf{g}} = X\}$$

is the stabilizer subgroup *of* $X \in \mathbf{H}^2\!\times\!\mathbf{R}$ *in* Γ.

Definition 10. *Assume that the stabilizer* $\Gamma_K = \mathbf{I}$, *i.e.* Γ *acts simply transitively on the orbit of a point* K. *Then let* $\mathcal{B}_K$ *denote the* greatest ball *of centre* K *inside the D-V cell* $\mathcal{D}(K)$*; moreover, let* $\rho(K)$ *denote the* radius *of* $\mathcal{B}_K$. *It is easy to see that*

$$\rho(K) = \min_{\mathbf{g}\in\Gamma\setminus\mathbf{I}} \frac{1}{2}\rho(K,K^{\mathbf{g}}).$$

Definition 11. *If the stabilizer* $\Gamma_K \neq \mathbf{I}$, *then* Γ *acts multiply transitively on the orbit of a point* K. *Then the greatest ball radius of* $\mathcal{B}_K$ *is*

$$\rho(K) = \min_{\mathbf{g}\in\Gamma\setminus\Gamma_K} \frac{1}{2}\rho(K,K^{\mathbf{g}}),$$

where K *belongs to a 0- 1- or 2-dimensional region (vertices, axes, reflection planes).*

In both cases, the Γ-images of $\mathcal{B}_K$ form a ball packing $\mathcal{B}_K^\Gamma$ with centre points $K^{\mathbf{G}}$.

Definition 12. *The* density *of ball packing* $\mathcal{B}_K^\Gamma$ *is*

$$\delta(K) = \frac{Vol(\mathcal{B}_K)}{Vol\mathcal{D}(K)}.$$

It is clear that the orbit K^Γ and the ball packing $\mathcal{B}_K^\Gamma$ have the same symmetry group; moreover, this group contains the starting crystallographic group Γ:

$$SymK^\Gamma = Sym\mathcal{B}_K^\Gamma \geq \Gamma.$$

Definition 13. *We say that the orbit* K^Γ *and the ball packing* $\mathcal{B}_K^\Gamma$ *is characteristic if* $SymK^\Gamma = \Gamma$, *else the orbit is not characteristic (n. char).*

Remark 6. *The Dirichlet-Voronoi cell belonging to a generalized Coxeter group* Γ *in the* $\mathbf{H}^2\!\times\!\mathbf{R}$ *space (see Definition 4) is a prism (see Figure 4).*

3.1. Simply transitive ball packings

Our problem is to find a point $K \in \mathbf{H}^2 \times \mathbf{R}$ and the orbit K^Γ for Γ such that $\Gamma_K = \mathbf{I}$ and the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal. In this case the ball packing $\mathcal{B}^\Gamma(K)$ is said to be *optimal.*

Our aim is to determine the maximal radius $\rho(K)$ of the balls, and the maximal density $\delta(K)$. The lattice of the considered space groups has a free parameter, we have to find the densest ball packing for fixed parameters $p(\Gamma)$, then we have to vary them to get the optimal ball packing

$$\delta(\Gamma) = \max_{K,\ p(\Gamma)} (\delta(K)). \tag{15}$$

Let Γ be a fixed generalized Coxeter group. The stabiliser of the possible kernel points is $\Gamma_K = \mathbf{I}$, i.e. we are looking for the optimal kernel point in a 3-dimensional region, inside a fundamental domain of Γ which is a prism with a free fibre parameter $p(\Gamma)$.

It can be assumed by the homogeneity of $\mathbf{H}^2 \times \mathbf{R}$, that the fibre coordinate of the center of the optimal ball is zero. It is clear that the optimal ball $\mathcal{B}_K$ has to touch all faces of the D-V cell to Γ around the kernel point K. Thus the height of the prism is $2\rho(K)$, where $\rho(K)$ is the radius of the inscribed circle of the hyperbolic p-gon $A_0A_1A_2, \dots A_{p-1}$. By Theorems 2 and 3, Definitions 10 and 12 and Theorem 4 of Z. Lučić and E. Molnár (cited from [4] as Lemma of V. S. Makarov), we have determined the data (radii, densities and volumes of optimal balls) of the optimal simply transitive ball packings to some generalized Coxeter group in the $\mathbf{H}^2 \times \mathbf{R}$ space which are summarized in Tables 1-11.

Theorem 4. *Among all convex polygons in $\mathbf{S}^2$ or in $\mathbf{H}^2$ with given angles $\alpha_1, \alpha_2, \dots \alpha_m$, $(m \geq 3)$ there exists up to an isometry respecting the order of angles, exactly one circumscribing a circle.*

Here $\alpha_i = \frac{\pi}{p_i}$ $(i = 1, \dots, m)$ will be the characteristic angles after Definition 4 for the given Coxeter group. The trigonometric formula

$$\cos \frac{\alpha_i}{2} = \cosh r \sin \frac{\beta_i}{2}$$

determines an angle β_i for fixed r in a barycentric rectangular triangle of the m-gon. Then $\Sigma_{i=1}^m \beta_i = 2\pi$ determines the optimal radius r of the circle inscribed into the m-gon. It turns out that r is independent of the cyclic order of α_i's, of course, but the isomorphism class of Γ depends on that order, in general.

3.2. Multiply transitive ball packings for "triangle groups"

Similarly to the simply transitive case we have to find a kernel point $K \in \mathbf{H}^2 \times \mathbf{R}$ and the orbit K^Γ for Γ such that the density $\delta(K)$ of the corresponding ball packing $\mathcal{B}^\Gamma(K)$ is maximal, but here $\Gamma_K \neq \mathbf{I}$. This ball packing is called $\mathcal{B}^\Gamma(K)$ *optimal*, too. In this multiply transitive case we are looking for the optimal kernel point K in different 0- 1- or 2-dimensional regions $\mathcal{L}$: Our aim is to determine the

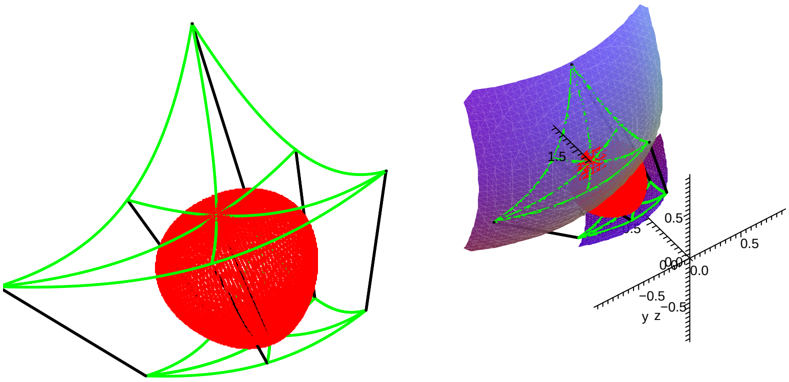


Figure 5: *Multiply transitive ball packing to $\mathbf{H}^2\times\mathbf{R}$ space groups $\Gamma_{(3,7,2)}$ where $K \in \mathcal{E}_{B_2C_2}$. a. Optimal ball with its D-V cell. b. Optimal ball and its D-V cell with parallel "hyperboloid planes" in the model*

maximal radius $\rho(K)$ of the balls, and the maximal density $\delta(K)$. Figure 5.a shows a fundamental domain (prism) of the space group $\Gamma^1_{(3,7,2)}$ generated by the four reflections $\mathbf{g}_i \in Isom\mathbf{H}^2$ $(i=1,2,3)$, $\mathbf{g}_4 \in \langle \overline{\mathbf{1}}_{\mathbf{R}} \rangle$:

1. K is an inner point of the "side faces": $K \in \mathcal{S}_{B_1C_1C_2B_2}$ (Table 12), $K \in \mathcal{S}_{B_0C_0C_2B_2}$ (Table 13), $K \in \mathcal{S}_{B_0C_0C_1B_1}$ (Table 14).
2. K is an inner point of the "edges of the cover faces": $K \in \mathcal{E}_{B_0B_1}$, $K \in \mathcal{E}_{B_1B_2}$, $K \in \mathcal{E}_{B_2B_0}$ or $K \in \mathcal{E}_{C_0C_1}$, $K \in \mathcal{E}_{C_1C_2}$, $K \in \mathcal{E}_{C_2C_0}$, but in these cases the results coincide with the correspondig results of case 1.
3. K is an inner point of the "side edges": $K \in \mathcal{E}_{B_0C_0}$ (Table 15), $K \in \mathcal{E}_{B_1C_1}$ (Table 16), $K \in \mathcal{E}_{B_2C_2}$ (Table 17).
4. K lies in the vertices B_0, B_1, B_2, C_0, C_1, C_2, but in these cases we have obtained the same results as in case 3.

Let Γ be a fixed generalized Coxeter group. The stabiliser of the possible kernel points is $\Gamma_K \neq \mathbf{I}$. The lattice of a considered space group can have a free parameter $p(\Gamma)$, then we have to find the densest ball packing for fixed parameters, and we have to vary them to get the optimal ball packing.

$$\delta(\Gamma) = \max_{K\in\mathcal{L},\ p(\Gamma)} (\delta(K)) \tag{16}$$

In this multiply transitive case it is sufficient to consider only problems 1 and 3 It can be assumed by the homogeneity of $\mathbf{H}^2\times\mathbf{R}$, that the fibre coordinate of the center of the optimal ball is zero. It is clear that the optimal ball $\mathcal{B}_K$ has to touch all faces of the D-V cell to Γ around the kernel point K. Thus the height of the prism is $2\rho(K)$, where $\rho(K)$ is the radius of the inscribed circle of a hyperbolic triangle. By Theorems 2 and 3, Definitions 10 and 12 and the projective interpretation of hyperbolic geometry we have determined the data (radii, densities and volumes of optimal balls) of the optimal multiply transitive ball packings to some generalized Coxeter group in $\mathbf{H}^2\times\mathbf{R}$ space. These are summarized in Tables 12-17. In Tables 1-17 we have denoted the space groups (given in Section 1.) by their parameters $(p_1, p_2, \dots p_m)$.

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3, 7, 2)$	≈ 0.10443046	≈ 0.00477404	≈ 0.30558225
$(3, 8, 2)$	≈ 0.13475670	≈ 0.01026279	≈ 0.29090180
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, 12, 2)$	≈ 0.17801237	≈ 0.02367862	≈ 0.25404325
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, 20, 2)$	≈ 0.19698480	≈ 0.03210037	≈ 0.22230571
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.20703226	≈ 0.03727721	≈ 0.17193994

Table 1: *Simply transitive cases*

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(4, 5, 2)$	≈ 0.15950373	≈ 0.01702698	≈ 0.33979495
$(4, 6, 2)$	≈ 0.20074534	≈ 0.03397751	≈ 0.32325666
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 12, 2)$	≈ 0.25967637	≈ 0.07367796	≈ 0.27094215
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 20, 2)$	≈ 0.27116548	≈ 0.08393054	≈ 0.24630642
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, p_2 \to \infty, 2)$	≈ 0.27750901	≈ 0.08998058	≈ 0.20642001

Table 2: *Simply transitive cases*

(p_1,p_2,p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(5,4,2)$	≈ 0.15950373	≈ 0.01702698	≈ 0.33979495
$(5,5,2)$	≈ 0.22080679	≈ 0.04524152	≈ 0.32609556
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(5,12,2)$	≈ 0.29573355	≈ 0.10897365	≈ 0.27067550
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(5,20,2)$	≈ 0.30511539	≈ 0.11972233	≈ 0.24979927
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(5,p_2\to\infty,2)$	≈ 0.31034450	≈ 0.12601089	≈ 0.21540854

Table 3: *Simply transitive cases*

(p_1,p_2,p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(p_1\to\infty,p_2\to\infty,2)$	≈ 0.36949897	0.21324491	≈ 0.18370271

Table 4: *Simply transitive cases*

(p_1,p_2,p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,4,3)$	≈ 0.21791658	≈ 0.04348445	≈ 0.38110542
$(3,5,3)$	≈ 0.26830879	≈ 0.08129745	≈ 0.36167909
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3,12,3)$	≈ 0.33362572	≈ 0.15670690	≈ 0.29902584
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3,20,3)$	≈ 0.34193509	0.16877298	≈ 0.27725648
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3,p_2\to\infty,3)$	≈ 0.34657359	≈ 0.17577239	≈ 0.24215675

Table 5: *Simply transitive cases*

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(4, 3, 3)$	≈ 0.21791658	≈ 0.04348445	≈ 0.38110542
$(4, 4, 3)$	≈ 0.30172805	≈ 0.11576317	≈ 0.36637522
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 12, 3)$	≈ 0.38650702	≈ 0.24427719	≈ 0.30176364
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 20, 3)$	≈ 0.39307307	0.25702692	≈ 0.28382721
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, p_2 \to \infty, 3)$	≈ 0.39675878	≈ 0.26437631	≈ 0.25452320

Table 6: *Simply transitive cases*

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(p_1 \to \infty, p_2 \to \infty, 3)$	≈ 0.46359870	0.42338117	≈ 0.21802221
$(p_1 \to \infty, p_2 \to \infty, 4)$	≈ 0.49977446	≈ 0.53165992	≈ 0.22574530
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(p_1 \to \infty, p_2 \to \infty, p_3 \to \infty)$	≈ 0.54930614	≈ 0.70836263	≈ 0.20523967

Table 7: *Simply transitive cases*

(p_1, p_2, p_3, p_4)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(4, 4, 4, 5)$	≈ 0.27518467	≈ 0.08773099	≈ 0.50739820
$(3, 4, 5, 5)$	≈ 0.15700198	≈ 0.01623744	≈ 0.49380312
$(4, 4, 4, 6)$	≈ 0.34869757	≈ 0.17904152	≈ 0.49031615
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 12, 3, 5)$	≈ 0.40164021	≈ 0.27432610	≈ 0.40764427
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(6, 6, 6, 6)$	≈ 0.65847895	1.23095533	≈ 0.44628448
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4, 4, 4, p_4 \to \infty)$	≈ 0.49110102	≈ 0.50416883	≈ 0.32677986
$(p_1 \to \infty, p_2 \to \infty,$ $p_3 \to \infty, p_4 \to \infty)$	≈ 0.88137359	≈ 3.01979277	≈ 0.27265109

Table 8: *Simply transitive cases*

(p_1,p_2,p_3,p_4,p_5)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,3,4,4,4)$	≈ 0.36949897	≈ 0.21324491	≈ 0.55110814
$(3,3,3,4,5)$	≈ 0.28377770	≈ 0.09623966	≈ 0.53975411
$(4,4,4,4,4)$	≈ 0.62686966	≈ 1.05919677	≈ 0.53783557
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4,12,3,5,6)$	≈ 0.78351114	≈ 2.09868039	≈ 0.45675624
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(6,6,6,6,6)$	≈ 0.93821462	3.66753885	≈ 0.46660987
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4,4,4,4,p_5\to\infty)$	≈ 0.77149663	≈ 2.00112887	≈ 0.41282041
$(p_1\to\infty,p_2\to\infty,$ $p_3\to\infty,p_4\to\infty,p_5\to\infty)$	≈ 1.12417722	≈ 6.47089771	≈ 0.30537159

Table 9: *Simply transitive cases*

$(p_1,p_2,p_3,p_4,p_5,p_6)$	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,3,3,3,3,4)$	≈ 0.38108663	≈ 0.23407853	≈ 0.58655570
$(3,3,3,3,4,4)$	≈ 0.53063753	≈ 0.63771121	≈ 0.57380916
$(4,4,4,4,4,4)$	≈ 0.88137359	≈ 3.01979277	≈ 0.54530219
$(4,4,4,4,4,5)$	≈ 0.91523420	≈ 3.39501468	≈ 0.53670651
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4,12,3,4,6,5)$	≈ 0.98663354	≈ 4.29151405	≈ 0.48297827
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(6,6,6,6,6,6)$	≈ 1.14621583	6.88161777	≈ 0.47776494
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(4,4,4,4,4,p_6\to\infty)$	≈ 0.97848676	≈ 4.18165765	≈ 0.45344268
$(p_1\to\infty,p_2\to\infty,$ $p_3\to\infty,p_4\to\infty,$ $p_5\to\infty,p_6\to\infty)$	≈ 1.31695790	≈ 10.73024209	≈ 0.32418862

Table 10: *Simply transitive cases*

$(p_1, p_2, \dots p_m)$	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,3,3,3,3,3,3)$	≈ 0.54527483	≈ 0.69267774	≈ 0.60653681
$(4,4,4,4,4,4,4)$	≈ 1.07040486	≈ 5.54276533	≈ 0.54942358
$(3,3,3,3,3,3,3,3)$	≈ 0.76428546	≈ 1.94411027	≈ 0.60726281
$(4,4,4,4,4,4,4,4)$	≈ 1.22422622	≈ 8.48710517	≈ 0.55168102
$(3,3,3,3,3,3,3,3,3)$	≈ 0.92753857	≈ 3.53909395	≈ 0.60726779
$(4,4,4,4,4,4,4,4,4)$	≈ 0.98964795	≈ 4.33267383	≈ 0.59723962

Table 11: *Simply transitive cases*

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,7,2)$	≈ 0.13146725	≈ 0.00952888	≈ 0.24224986
$(3,8,2)$	≈ 0.16885130	≈ 0.02020353	≈ 0.22851997
$(4,5,2)$	≈ 0.22080679	≈ 0.04524152	≈ 0.32609556
$(4,6,2)$	≈ 0.27465307	≈ 0.08722184	≈ 0.30325794
$(5,5,2)$	≈ 0.31755015	≈ 0.13503419	≈ 0.33839314
$(3,4,3)$	≈ 0.31648716	≈ 0.13367665	≈ 0.40333958
$(3,5,3)$	≈ 0.38359861	≈ 0.23876852	≈ 0.37149369
$(4,4,3)$	≈ 0.45572276	≈ 0.40197393	≈ 0.42115169
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.25541281	≈ 0.07009787	≈ 0.13103989
$(4, p_2 \to \infty, 2)$	≈ 0.36949897	≈ 0.21324491	≈ 0.18370271
$(3, p_2 \to \infty, 3)$	≈ 0.48121183	≈ 0.47401773	≈ 0.23516338
$(4, q \to \infty, 3)$	≈ 0.57466052	≈ 0.81258528	≈ 0.27005920
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(p_1 \to \infty, p_2 \to \infty, p_3 \to \infty)$	≈ 0.88137359	≈ 3.01979277	≈ 0.27265109

Table 12: *Multiply transitive cases,* $K \in \mathcal{S}_{B_1C_1C_2B_2}$

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3, 7, 2)$	≈ 0.18221438	≈ 0.02539789	≈ 0.46585899
$(3, 8, 2)$	≈ 0.23630516	≈ 0.05547848	≈ 0.44838670
$(4, 5, 2)$	≈ 0.27518467	≈ 0.08773099	≈ 0.50739820
$(4, 6, 2)$	≈ 0.34869757	≈ 0.17904152	≈ 0.49031615
$(5, 5, 2)$	≈ 0.38359861	≈ 0.23876852	≈ 0.49532492
$(3, 4, 3)$	≈ 0.33923144	≈ 0.16478092	≈ 0.46385497
$(3, 5, 3)$	≈ 0.42573116	≈ 0.32714332	≈ 0.45862108
$(4, 4, 3)$	≈ 0.48121183	≈ 0.47401773	≈ 0.47032675
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.36949897	≈ 0.21324491	≈ 0.27555407
$(4, p_2 \to \infty, 2)$	≈ 0.49110102	≈ 0.50416883	≈ 0.32677986
$(3, q \to \infty, 3)$	≈ 0.57141510	≈ 0.79869814	≈ 0.33368935
$(4, q \to \infty, 3)$	≈ 0.65847895	≈ 1.23095533	≈ 0.35702759
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(p_1 \to \infty, p_2 \to \infty, p_3 \to \infty)$	≈ 0.88137359	≈ 3.01979277	≈ 0.27265109

Table 13: *Multiply transitive cases,* $K \in \mathcal{S}_{B_0C_0C_2B_2}$

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3, 7, 2)$	≈ 0.16719759	≈ 0.01961498	≈ 0.39210043
$(3, 8, 2)$	≈ 0.21791658	≈ 0.04348445	≈ 0.38110542
$(4, 5, 2)$	≈ 0.23446232	≈ 0.05418749	≈ 0.36782909
$(4, 6, 2)$	≈ 0.30172805	≈ 0.11576317	≈ 0.36637522
$(5, 5, 2)$	≈ 0.31755015	≈ 0.13503419	≈ 0.33839314
$(3, 4, 3)$	≈ 0.33923144	≈ 0.16478092	≈ 0.46385497
$(3, 5, 3)$	≈ 0.42573116	≈ 0.32714332	≈ 0.45862108
$(4, 4, 3)$	≈ 0.45572276	≈ 0.40197393	≈ 0.42115169
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.34657359	≈ 0.17577239	≈ 0.24215674
$(4, p_2 \to \infty, 2)$	≈ 0.44068679	≈ 0.36315860	≈ 0.26231095
$(3, p_2 \to \infty, 3)$	≈ 0.57141510	≈ 0.79869814	≈ 0.33368935
$(4, p_2 \to \infty, 3)$	≈ 0.63504553	≈ 1.10193775	≈ 0.33140082
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(p_1 \to \infty, p_2 \to \infty, p_3 \to \infty)$	≈ 0.88137359	≈ 3.01979277	≈ 0.27265109

Table 14: *Multiply transitive cases,* $K \in \mathcal{S}_{B_0C_0C_1B_1}$

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,7,2)$	≈ 0.54527483	≈ 0.69267774	≈ 0.60653681
$(3,8,2)$	≈ 0.76428546	≈ 1.94411027	≈ 0.60726281
$(4,5,2)$	≈ 0.62686966	≈ 1.05919677	≈ 0.53783557
$(4,6,2)$	≈ 0.88137359	≈ 3.01979277	≈ 0.54530219
$(5,5,2)$	≈ 0.84248208	≈ 2.62573931	≈ 0.49603356
$(3,4,3)$	≈ 0.76428546	≈ 1.94411027	≈ 0.60726281
$(3,5,3)$	≈ 1.06127506	≈ 5.39521435	≈ 0.60682315
$(4,4,3)$	≈ 1.03171853	≈ 4.93669090	≈ 0.57115778
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, 2\cdot 10^7, 2)$	≈ 15.66651295	$\approx 1.93578975 \cdot 10^8$	≈ 0.29498326
$(4, 2\cdot 10^7, 2)$	≈ 16.00652388	$\approx 2.77505630 \cdot 10^8$	≈ 0.27349210
$(3, 2\cdot 10^7, 3)$	≈ 16.35966035	$\approx 3.96059077 \cdot 10^8$	≈ 0.28897958
$(4, 2\cdot 10^7, 3)$	≈ 16.54788653	$\approx 4.80961661 \cdot 10^8$	≈ 0.27754880
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2\cdot 10^7, 2\cdot 10^7, 2\cdot 10^7)$	≈ 17.05280712	$\approx 8.09527681 \cdot 10^8$	≈ 0.18888437

Table 15: *Multiply transitive cases,* $K \in \mathcal{E}_{B_0C_0}$

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3,7,2)$	≈ 0.24599664	≈ 0.06260769	≈ 0.42531231
$(3,8,2)$	≈ 0.31648716	≈ 0.13367665	≈ 0.40333958
$(4,5,2)$	≈ 0.38359861	≈ 0.23876852	≈ 0.49532492
$(4,6,2)$	≈ 0.48121183	≈ 0.47401773	≈ 0.47032675
$(5,5,2)$	≈ 0.53063753	≈ 0.63771121	≈ 0.47817430
$(3,4,3)$	≈ 0.64164528	≈ 1.13728811	≈ 0.56419031
$(3,5,3)$	≈ 0.78136831	≈ 2.08104580	≈ 0.52985372
$(4,4,3)$	≈ 0.88137359	≈ 3.01979277	≈ 0.54530219
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.48121183	≈ 0.47401773	≈ 0.23516338
$(4, p_2 \to \infty, 2)$	≈ 0.65847895	≈ 1.23095533	≈ 0.29752299
$(3, p_2 \to \infty, 3)$	≈ 0.98664696	≈ 4.29169668	≈ 0.34614445
$(4, p_2 \to \infty, 3)$	≈ 1.13440039	≈ 6.65916531	≈ 0.37370900
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2\cdot 10^7, 2\cdot 10^7, 2\cdot 10^7)$	≈ 17.05280712	$\approx 8.09527681 \cdot 10^8$	≈ 0.18888437

Table 16: *Multiply transitive cases,* $K \in \mathcal{E}_{B_1C_1}$

(p_1, p_2, p_3)	ρ	$Vol(\mathcal{B}_K(\rho))$	δ
$(3, 7, 2)$	≈ 0.28312815	≈ 0.09557797	≈ 0.37609098
$(3, 8, 2)$	≈ 0.36351992	≈ 0.20300040	≈ 0.35550752
$(4, 5, 2)$	≈ 0.53063753	≈ 0.63771121	≈ 0.47817430
$(4, 6, 2)$	≈ 0.65847895	≈ 1.23095533	≈ 0.44628448
$(5, 5, 2)$	≈ 0.84248208	≈ 2.62573931	≈ 0.49603356
$(3, 4, 3)$	≈ 0.64164528	≈ 1.13728811	≈ 0.56419031
$(3, 5, 3)$	≈ 0.78136831	≈ 2.08104580	≈ 0.52985372
$(4, 4, 3)$	≈ 1.03171853	≈ 4.93669090	≈ 0.57115778
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(3, p_2 \to \infty, 2)$	≈ 0.54930614	≈ 0.70836263	≈ 0.20523967
$(4, p_2 \to \infty, 2)$	≈ 0.88137359	≈ 3.01979277	≈ 0.27265109
$(3, p_2 \to \infty, 3)$	≈ 0.98664696	≈ 4.29169668	≈ 0.34614445
$(4, p_2 \to \infty, 3)$	≈ 1.30441812	≈ 10.40418124	≈ 0.38083116
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2 \cdot 10^7, 2 \cdot 10^7, 2 \cdot 10^7)$	≈ 17.05280712	$\approx 8.09527681 \cdot 10^8$	≈ 0.18888437

Table 17: *Multiply transitive cases,* $K \in \mathcal{E}_{B_2C_2}$

It is interesting to consider further locally densest (optimal) ball packings in the 3-dimensional Thurston geometries, because important information of the "crystal structures" are included by the local optimal ball arrangements. It is timely to rise the above question for further space groups in $\mathbf{H}^2\times\mathbf{R}$.

In this paper we have mentioned only some problems in discrete geometry of the $\mathbf{H}^2\times\mathbf{R}$ space, but we hope that from these it can be seen that our projective method suits to study and solve similar problems.

Acknowledgments

The author would like to thank the referees and Prof. Emil Molnár for the helpful suggestions.

References

[1] B. Divjak, Z. Erjavec, B. Szabolcs, B. Szilágyi, *Geodesics and geodesic spheres in* $\widetilde{\mathbf{SL}(2,\mathbf{R})}$ *geometry*, Math. Commun. **14**(2009), 413–424.

[2] J. Z. Farkas, *The classification of* $\mathbf{S}^2\times\mathbf{R}$ *space groups*, Beiträge Algebra Geom. **42**(2001), 235–250.

[3] J. Z. Farkas, E. Molnár, *Similarity and diffeomorphism classification of* $\mathbf{S}^2\times\mathbf{R}$ *manifolds*, Steps in Diff. Geometry (Proc. of Coll. on Diff. Geom, 25–30 July 2000, Debrecen, Hungary), 2001, 105–118.

[4] Z. Lučić, E. Molnár, *Fundamental domains for planar discontinuous groups and uniform tilings,* Geom. Dedicata **40**(1991), 123–143.

[5] A. M. Macbeath, *The classification of non-Euclidean plane crystallographic groups,* Can. J. Math. **19**(1967), 1192–1205.

[6] E. Molnár, *The projective interpretation of the eight 3-dimensional homogeneous geometries,* Beitrge Algebra Geom. **38**(1997), 261–288.

[7] E. Molnár, *On projective models of Thurston geometries, some relevant notes on Nil orbifolds and manifolds,* Sib. Elektron. Mat. Izv. **7**(2010), 491–498.

[8] E. Molnár, I. Prok, J. Szirmai, *Classification of tile-transitive 3-simplex tilings and their realizations in homogeneous spaces,* in: *Non-Euclidean Geometries, János Bolyai Memorial Volume*, (A. Prekopa and E. Molnár, Eds.), Mathematics and Its Applications **581**, Springer, 2006, 321–363.

[9] E. Molnár, B. Szilágyi, *Translation curves and their spheres in homogeneous geometries*, Publ. Math. Debrecen **78**(2011), 327–346.

[10] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. **15**(1983), 401–487 (Russian translation: Moscow "Mir", 1986).

[11] J. Szirmai, *The optimal ball and horoball packings of the Coxeter tilings in the hyperbolic 3-space*, Beiträge Algebra Geom. **46**(2005), 545–558.

[12] J. Szirmai, *The optimal ball and horoball packings to the Coxeter honeycombs in the hyperbolic d-space*, Beiträge Algebra Geom. **48**(2007), 35–47.

[13] J. Szirmai, *The densest geodesic ball packing by a type of* **Nil** *lattices*, Beiträge Algebra Geom. **48**(2007), 383–398.

[14] J. Szirmai, *The densest translation ball packing by fundamental lattices in* **Sol** *space*, Beiträge Algebra Geom. **51**(2010), 353–373.

[15] J. Szirmai, *Geodesic ball packing in* $\mathbf{S}^2\times\mathbf{R}$ *space for generalized Coxeter space groups*, Beiträge Algebra Geom. **52**(2011), 413–430

[16] W. P. Thurston, *Three-Dimensional Geometry and Topology*, (S. Levy, Ed.), Princeton University Press, Princeton, New Jersey, 1997.