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Estimation of the killing rate parameter in a diffusion model*

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Abstract. We consider a parameter estimation problem for a diffusion with killing, starting at a point in an open and bounded set. The infinitesimal killing rate function depends on a control variable and parameters. Values of the control variable are known while parameters have unknown values which have to be estimated from data. The minimum of three times: the maximum observation time, the first exit time from the open set, and the killing time, is observed. Instead of the maximum likelihood estimation method we propose and use the minimum χ^2-estimation method that is based on the conditional mean of the data observed before the maximum observation time is reached, and on the frequency of data that are equal to the maximum observation time. We prove that the estimator exists and is consistent and asymptotically normal. The method is illustrated by an example.

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1. Introduction

We consider the following problem. Let $X = (X_t, t \geq 0)$ be a diffusion in $\mathbb{R}^d$ given by the Itô stochastic differential equation (SDE):

$$X_t = X_0 + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s,\ t \geq 0. \tag{1}$$

Here $W = (W_t, t \geq 0)$ is the d-dimensional Wiener process, $b(x)$ is a drift function and $\sigma(x)$ is a diffusion coefficient function (see [13]). We assume that the diffusion X is randomly killed with the infinitesimal rate function $(t, x) \mapsto c(z; t, x, \kappa)$ which depends on an unknown parameter κ and a control variable z (being a function $z(s)$, $0 \leq s \leq T$). In other words, there exists a random time ζ, called the killing time, with the conditional distribution

$$\mathbb{P}(\zeta > t \mid \mathcal{F}^X) = e^{-\int_0^t c(z;s,X_s,\kappa)\, ds},\ t \geq 0, \tag{2}$$

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where $\mathcal{F}^X$ is a σ-algebra generated by $(X_s, s \geq 0)$. Let D be an open and bounded set in $\mathbb{R}^d$ such that X initially starts in D, i.e. $X_0 \in D$ with known distribution. Let $\tau_D = \inf\{t \geq 0 : X_t \in \mathbb{R}^d \setminus D\}$ be the first exit time of X from D and let η be the minimum of ζ and τ_D, i.e.

$$\eta = \zeta \wedge \tau_D. \tag{3}$$

It is obvious that the distribution of η depends on the parameter κ and the variable z. Our problem is to estimate the unknown parameter κ by using a sequence of independent observations of the random variable η resulting from different values of the control variable z, up to some prescribed deterministic time $T > 0$. The time T is called the maximum observation time. Hence we have a parameter estimation problem which is based on a sequence of censored samples that are observed with respect to given covariate values.

The method of estimation, we propose and discuss in this paper, belongs to the class of minimum χ^2 estimation methods. It is based on the appropriate comparison of the following model functionals with the corresponding sample ones: the conditional mean of η observed before T is reached, and the probability of the event $\{\eta \geq T\}$. We prove that the estimators exist, and that are consistent and asymptotically normal. In this method we use only functionals of η. Let us point out that functionals of η can be estimated more accurately by numerical methods than the density function of η. An alternative to the proposed method is the maximum likelihood method (ML). This method is often used. For its application we need the density of the variable η. We also compare the proposed method with ML.

The problem we consider here naturally appears up in a problem of search for a moving object as formulated by Mangel [11]. In Mangle's approach, the search problem is segmented into three parts, titled by moving object or target, searcher and detection. The target is modeled by a diffusion, the searcher is modeled by a deterministic path and the detection is a random variable depending on points of the Euclidian space and a parameter which is called the effort. An estimation of this parameter is the main object of our research.

The paper is organized in the following way. In Section 2, we state and discuss the conditions on the model given by SDE (1), the infinitesimal killing rate function (2) and the set D. The statistical structure is defined and the maximum likelihood estimation method is discussed in Section 3. The χ^2-estimation criterion is proposed and specified, and the main properties of the estimators are stated and proved in the same section. The method and its application to an estimation of search effort is illustrated by an example in Section 4.

2. Main assumptions

Let the diffusion X of (1) and the killing time ζ of (2) be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of all the possible values of the k-dimensional unknown parameter κ is denoted by $\mathcal{K}$. We assume that $\mathcal{K}$ is an open set in $\mathbb{R}^k$ with the closure $\bar{\mathcal{K}}$. Similarly, let $\mathcal{Z}$ denote the set of all possible values of the variable z. The partial differential operator with respect to κ is denoted by ∇_κ. We assume the following:

(A1) *The functions* $b : \mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^d \to M_d$ *are locally Lipschitz and have a linear growth (see [13]). Moreover, there exists a constant* $C > 0$ *such that for any* x, $|\det \sigma(x)| \geq C$.

(A2) *For each* $z \in \mathcal{Z}$, $(t, x, \kappa) \mapsto c(z; t, x, \kappa)$ *is a nonnegative, measurable and bounded function on* $[0, +\infty\rangle \times \mathbb{R}^d \times \mathcal{K}$. *For each* $t \geq 0$, $x \in \mathbb{R}^d$ *and* $z \in \mathcal{Z}$, $c(z; t, x, \cdot) \in C(\bar{\mathcal{K}}) \cap C^1(\mathcal{K})$. $\nabla_\kappa c(z; \cdot, \cdot, \cdot)$ *is a measurable and bounded function on* $[0, +\infty\rangle \times \mathbb{R}^d \times \mathcal{K}$ *for each* $z \in \mathcal{Z}$. *Moreover, for each* $\kappa \in \mathcal{K}$ *and* $z \in \mathcal{Z}$,

$$\mathbb{E}[e^{-\int_0^{+\infty} c(z;s,X_s,\kappa)\,ds}] < 1. \tag{4}$$

(A3) *The open, bounded and connected set* D *is a Lipshitz domain (see e.g. [1]).*

Due to (A1), the diffusion $X = (X_t, t \geq 0)$ in $\mathbb{R}^d$ exists, it is continuous, pathwise unique and it is a strong solution to SDE (1) (see [13]). Assumption (A2) enables a correct definition of ζ by using (2) and ensures that the probability of the event $\{\zeta = +\infty\}$ is strictly less then 1.

For given $z \in \mathcal{Z}$ and $\kappa \in \mathcal{K}$, let us denote by

$$F^{(z)}(t|\kappa) := 1 - \mathbb{P}_\kappa^{(z)}(\eta > t), \; t \geq 0 \tag{5}$$

the cumulative distribution function (cdf) of ζ [§]. The absolute continuity and strict monotonicity of (5) is granted by (A3) as proved in [7]. On the other hand, since τ_D is a stopping time, it follows from (2) that

$$\mathbb{P}_\kappa^{(z)}(\eta > t) = \mathbb{E}[\mathbb{1}_{\{\tau_D > t\}} \cdot e^{-\int_0^t c(z;s,X_s,\kappa)\,ds}]. \tag{6}$$

Let $\mathbb{E}_\kappa^{(z)}[\eta \mid \eta < T]$ and $\mathbb{V}\mathrm{ar}_\kappa^{(z)}[\eta \mid \eta < T]$ denote the conditional expectation and variance of η respectively, given $\{\eta < T\}$.

Theorem 1. *Let the conditions* $(A1) - (A3)$ *be satisfied. Then for any fixed* $T > 0$ *and* $z \in \mathcal{Z}$ *the following holds:*

(i) $\kappa \mapsto F^{(z)}(T|\kappa)$ *is continuous on* $\bar{\mathcal{K}}$, *continuously differentiable on* $\mathcal{K}$ *and*

$$\nabla_\kappa F^{(z)}(T|\kappa) = \mathbb{E}[\mathbb{1}_{\{\tau_D > T\}} \int_0^T \nabla_\kappa c(s, X_s, \kappa, z) ds \, e^{-\int_0^T c(z;s,X_s,\kappa)\,ds}]. \tag{7}$$

(ii) $\kappa \mapsto \mathbb{E}_\kappa^{(z)}[\eta \mid \eta < T]$ *is continuous on* $\bar{\mathcal{K}}$ *and continuously differentiable on* $\mathcal{K}$.

(iii) $\kappa \mapsto \mathbb{V}\mathrm{ar}_\kappa^{(z)}[\eta \mid \eta < T]$ *is strictly positive and continuous on* $\bar{\mathcal{K}}$.

Proof. Equation (7), and the statements about the continuity and differentiability of both, (5) and (7), with respect to κ, follow directly from (A2) and (6) by the dominated convergence theorem. Hence the statement (i) follows. By using (6), the Fubini theorem, and then the dominated convergence theorem one can conclude that the statement (ii) and the continuity of $\kappa \mapsto \mathbb{V}\mathrm{ar}_\kappa^{(z)}[\eta \mid \eta < T]$ follow from (A2) and (i). The strict positivity of the conditional variance follows from the fact that the cdf (5) is a strictly increasing function. □

[§]To emphasize that the distribution of η depends on κ and z, the probability of event $\{\eta > t\}$ is denoted by $\mathbb{P}_\kappa^{(z)}(\eta > t)$ and the expectation and variance with respect to $\mathbb{P}_\kappa^{(z)}$ are denoted by $\mathbb{E}_\kappa^{(z)}$ and $\mathbb{V}\mathrm{ar}_\kappa^{(z)}$, respectively.

3. Estimation methods

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_l\}$ consist of l different values of the control variable z. The data set consists of those outcomes of the random variable η which are recorded up to the maximum observational time T. Otherwise, the value of T is recorded. In other words, we record outcomes of the censored random variable $\eta^T = \eta \wedge T$. The cumulative distribution function of η^T (for given κ and z) has the form

$$F_T^{(z)}(t|\kappa) = \mathbb{1}_{\{t<T\}} F^{(z)}(t|\kappa) + \mathbb{1}_{\{t \geq T\}},\ t \in \mathbb{R}, \tag{8}$$

where $F^{(z)}(\cdot|\kappa)$ is the cdf of η given in (5). Thus the family of cdfs

$$\{\, F_T^{(z_i)}(\cdot|\kappa)\ :\ i = 1, 2, \ldots, l;\ \kappa \in \mathcal{K}\,\}, \tag{9}$$

is the statistical structure of η^T. Let

$$\eta_{i1}^T, \eta_{i2}^T, \ldots, \eta_{in_i}^T \tag{10}$$

be a random sample of size n_i from the distribution with cdf $F_T^{(z_i)}(\cdot|\kappa)$. A random sample from (9) (of size $\sum_{i=1}^l n_i$) consists of independent samples (10) for $i = 1, 2, \ldots, l$.

3.1. Maximum likelihood estimation

Let $f^{(z)}(\cdot|\kappa)$ be a density function of cdf $F^{(z)}(\cdot|\kappa)$. Then the function

$$t \mapsto f_T^{(z)}(t|\kappa) = \mathbb{1}_{\{t<T\}} f^{(z)}(t|\kappa) + \mathbb{1}_{\{t \geq T\}}(1 - F^{(z)}(T|\kappa)) \tag{11}$$

is the probability density function (pdf) of $F_T^{(z)}(\cdot|\kappa)$ with respect to the measure consisting of the Lebesgue measure on $\langle -\infty, T\rangle$ and the Dirac measure at T (see [9]). Let $I_{n_i}^{(i)}$ be the number of values in the sample (10) equal to T (i.e. the number of censored values), $1 \leq i \leq l$. The log-likelihood function (LLF) for (9) has the form

$$\ell(\kappa) = \sum_{i=1}^{l} \left(\sum_{\eta_{ij}^T < T} \log f^{(z_i)}(\eta_{ij}^T|\kappa) + I_{n_i}^{(i)} \log(1 - F^{(z_i)}(T|\kappa)) \right).$$

The corresponding Fisher information matrix is equal to: ¶

$$I(\kappa) = \sum_{i=1}^{l} \left(\int_0^T \frac{\nabla_\kappa^\tau \nabla_\kappa f^{(z_i)}(t|\kappa)}{f^{(z_i)}(t|\kappa)}\, dt + \frac{\nabla_\kappa^\tau \nabla_\kappa F^{(z_i)}(T|\kappa)}{1 - F^{(z_i)}(T|\kappa)} \right). \tag{12}$$

If the maximum likelihood estimator (MLE) exists, then it has the usual good properties such as the consistency, asymptotic efficiency and normality. More precisely, let us assume that (A4) holds (see the next subsection) and additionally that the

¶ $\nabla_\kappa^\tau \nabla_\kappa g$ denotes matrix $(\nabla_\kappa g)^\tau \cdot \nabla_\kappa g$.

killing rate function c is two-times differentiable with respect to κ and that the second derivative $\nabla_\kappa^2 c$ has the same properties as assumed for $\nabla_\kappa c$ by (A2). Then by using similar arguments as in the proof of Theorem 1 it can be proved that model (9) is regular enough so that MLE has the mentioned asymptotic properties by e.g. Theorems 17 and 18 in [6]. The proof is based on the following representation of the density of η that is obtained from (6) by using the Markov property and the dominated convergence theorem:

$$f^{(z)}(t|\kappa) = \mathbb{E}[\mathbb{1}_{\{\tau_D > t\}}(c(z;t,X_t,\kappa) + g(X_t,0+))e^{-\int_0^t c(z;s,X_s,\kappa)\,ds}].$$

Here $g(x,\cdot)$ is the density of τ_D in case $X_0 = x \in D$.

In order to solve the likelihood estimation problem, the density functions $f^{(z)}(\cdot|\kappa)$ must be available. Since $f^{(z)}(\cdot|\kappa)$ could be obtained in a closed form only in few cases, it is necessary to calculate it numerically. Two possibilities are at our disposal: distribution functions $F^{(z)}(\cdot|\kappa)$ could be estimated either by a Monte Carlo method or by using a deterministic numerical method that consists of solving an appropriate initial value problem and then by applying an appropriate integration formula to the obtained solution. In either cases, in order to estimate $f^{(z)}(t|\kappa)$, the numerical differentiation with respect to t must be used. Precisely, let $0 = t_0 < t_1 < \cdots < t_R = T$ be a subdivision of the time interval $[0,T]$, and let

$$N_{ir} = \#\{j : \eta_{ij}^T \in [t_{r-1}, t_r\rangle\},$$

be frequencies of the intervals $[t_{r-1}, t_r\rangle$, $r = 1, \ldots, R$, with respect to the data (10), for $i = 1, 2, \ldots, l$. Note that $\sum_{r=1}^R N_{ir} = n_i - I_{n_i}^{(i)}$ for each i. Then the criterion function for estimating the parameter κ, that is based on such an approximation, has the form:

$$\ell_R(\kappa) = \sum_{i=1}^{l} \left(\sum_{r=1}^{R} N_{ir} \log \Delta_r F^i + I_{n_i}^{(i)} \log(1 - F^{(z_i)}(T|\kappa)) \right), \tag{13}$$

where $\Delta_r F^i \equiv F^{(z_i)}(t_r|\kappa) - F^{(z_i)}(t_{r-1}|\kappa)$ for any i and r. A point of the maximum of the function ℓ_R, whenever it exists, will be called an approximate MLE. When the lengths $|t_r - t_{r-1}|$ of some of the intervals $[t_{r-1}, t_r\rangle$ with $N_{ir} \neq 0$, $r = 1, 2, \ldots, R$, are near zero, an instability of the numerical differentiation implies an instability of the criterion function $\ell_R(\kappa)$.

On the other hand, the function $\kappa \mapsto \ell_R(\kappa)$ of (13) is actually the exact log-likelihood function for the parameter κ with respect to the multinomial statistical model which is based on the interval frequencies: $(\{N_{ir} : r = 1, \ldots, R\} \cup \{I_{n_i}^{(i)}\},\ i = 1, \ldots, l)$. The corresponding Fisher information matrix is defined by expression:

$$I_R(\kappa) = \sum_{i=1}^{l} \left(\sum_{r=1}^{R} \frac{\nabla_\kappa^\tau \nabla_\kappa \Delta_r F^i}{\Delta_r F^i} + \frac{\nabla_\kappa^\tau \nabla_\kappa F^{(z_i)}(T|\kappa)}{1 - F^{(z_i)}(T|\kappa)} \right). \tag{14}$$

It is obvious that matrix (14) is an approximation of the information matrix $I(\kappa)$ of the model (9).

In the multinomial model the matrix $I_R(\kappa)$ of (14) is the inverse of the asymptotic covariance matrices of both estimators of κ, the first one being the approximate MLE, and the second one being the minimum χ^2-estimator based on Pearson's χ^2-criterion function:

$$H_n(\kappa) = \sum_{i=1}^{l} \left(\sum_{r=1}^{R} \frac{(N_{ir} - n_i \Delta_r F^i)^2}{n_i \Delta_r F^i} + \frac{(I_{n_i}^{(i)} - n_i(1 - F^{(z_i)}(T|\kappa)))^2}{n_i(1 - F^{(z_i)}(T|\kappa))} \right). \tag{15}$$

More precisely, according to Ferguson in [6] (Section 23), if some regularity conditions are satisfied (see Remark 1 in the next subsection), the mentioned two estimators have the same asymptotic distributions. Hence, we can say that these estimators are asymptotically nearly efficient. Note that if the lengths of the subdivision intervals are not small, both estimating functions, ℓ_R of (13) and H_n of (15), are stable in the previously described sense. In this case, both estimators are acceptable choices for estimating the parameter κ by using only cdfs $F^{(z)}(\cdot|\kappa)$, $z \in \mathcal{Z}$. However, in the case when the data set (10) is relatively small for each value of the control variable (in a way that the number of subdivision intervals R is small), an alternative minimum χ^2-estimation method could be more efficient. The purpose of this paper is to propose and analyze such an alternative. Moreover, the proposed method solves the parametric estimation problem based on a censored sample when the data are summarized in the following way. For each value of the control variable we are supplied with the sample conditional mean of the non-censored searching times and the number of the censored searching times.

3.2. Proposed minimum χ^2-estimation method

Let the following two statistics be available for each i:

$$X_{n_i}^{(i)} := \frac{\sum_{\eta_{ij}^T < T} \eta_{ij}^T}{n_i - I_{n_i}^{(i)}} \cdot \mathbb{1}_{\{I_{n_i}^{(i)} < n_i\}}, \quad Y_{n_i}^{(i)} := \frac{I_{n_i}^{(i)}}{n_i}, \tag{16}$$

let $n := \min_{1 \le i \le l} n_i$ and let $\mathbf{Z}_n$ be a $2l$-dimensional statistic defined by

$$\mathbf{Z}_n := (X_{n_1}^{(1)}, Y_{n_1}^{(1)}, X_{n_2}^{(2)}, Y_{n_2}^{(2)}, \ldots, X_{n_l}^{(l)}, Y_{n_l}^{(l)})^\tau. \tag{17}$$

Then by the strong law of large numbers (SLLN) (see [6]) we have:

$$\mathbf{Z}_n \xrightarrow{\text{a.s.}} \mathbf{A}(\kappa), \ n \to +\infty, \tag{18}$$

where

$$\mathbf{A}(\kappa) = (\mathbb{E}_\kappa^{(z_1)}[\eta \mid \eta < T], \mathbb{P}_\kappa^{(z_1)}(\eta \ge T), \ldots, \mathbb{E}_\kappa^{(z_l)}[\eta \mid \eta < T], \mathbb{P}_\kappa^{(z_l)}(\eta \ge T))^\tau. \tag{19}$$

Without loos of generality, we can assume $n = n_1 = \cdots = n_l$, i.e. that the subsamples (10) have the same sizes.

Theorem 2. *For $\kappa \in \mathcal{K}$, the following asymptotics is valid:*

$$\sqrt{n}(\mathbf{Z}_n - \mathbf{A}(\kappa)) \xrightarrow{\mathcal{D}} N(\mathbf{0}, \mathbf{\Sigma}(\kappa)), \ n \to +\infty,$$

where $\mathbf{\Sigma}(\kappa)$ *is a diagonal matrix of the form:*

$$\mathbf{\Sigma}(\kappa) = \mathrm{diag}\left(\frac{\mathbb{V}\mathrm{ar}_\kappa^{(z_1)}[\eta \mid \eta < T]}{\mathbb{P}_\kappa^{(z_1)}(\eta < T)}, \mathbb{P}_\kappa^{(z_1)}(\eta < T)\mathbb{P}_\kappa^{(z_1)}(\eta \geq T), \dots \right.$$
$$\left. \dots, \frac{\mathbb{V}\mathrm{ar}_\kappa^{(z_l)}[\eta \mid \eta < T]}{\mathbb{P}_\kappa^{(z_l)}(\eta < T)}, \mathbb{P}_\kappa^{(z_l)}(\eta < T)\mathbb{P}_\kappa^{(z_l)}(\eta \geq T)\right). \tag{20}$$

Proof. Because the samples (10) for $i = 1, 2, \dots, l$ are independent, it is sufficient to prove the theorem for $l = 1$. In this case the statement of the theorem is equivalent to the following statement (see [6]):

$$(\forall \mathbf{v} \in \mathbb{R}^2)\ (\sqrt{n}(\mathbf{Z}_n - \mathbf{A}(\kappa)) \mid \mathbf{v}) \xrightarrow{\mathcal{D}} N(0, (\mathbf{\Sigma}(\kappa)\mathbf{v} \mid \mathbf{v})),\ n \to +\infty. \tag{21}$$

Let $I_n \equiv I_n^{(1)}$, $X_n \equiv X_n^{(1)}$, $Y_n \equiv Y_n^{(1)}$, $\eta_j^T \equiv \eta_{ij}^T$ $(1 \leq j \leq n)$ and $e \equiv \mathbb{E}_\kappa^{(z_1)}[\eta \mid \eta < T]$, $v \equiv \mathbb{V}\mathrm{ar}_\kappa^{(z_1)}[\eta \mid \eta < T]$, $p \equiv \mathbb{P}_\kappa^{(z_1)}(\eta < T)$, $q = 1 - p$. Let $\mathbf{v} = (\alpha, \beta)^\tau$ be any vector in $\mathbb{R}^2$. Then for $n \in \mathbb{N}$ such that $I_n < n$, we have:

$$\begin{aligned}(\sqrt{n}(\mathbf{Z}_n - \mathbf{A}(\kappa)) \mid \mathbf{v}) &= \alpha\sqrt{n}(X_n - e) + \sqrt{n}\beta(Y_n - q) \\ &= \frac{1}{1 - \frac{I_n}{n}}\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\big(\alpha \mathbb{1}_{\langle 0,T\rangle}(\eta_j^T)(\eta_j^T - e) + \beta p(\mathbb{1}_{\{T\}}(\eta_j^T) - q)\big) \\ &\quad -\frac{1}{1 - \frac{I_n}{n}}\frac{\beta}{\sqrt{n}}(I_n - nq)(Y_n - q)\Big).\end{aligned} \tag{22}$$

Since $\mathbb{V}\mathrm{ar}[\alpha \mathbb{1}_{\langle 0,T\rangle}(\eta^T)(\eta^T - e) + \beta p(\mathbb{1}_{\{T\}}(\eta^T) - q)] = \alpha^2 vp + \beta^2 p^3 q < +\infty$, it follows that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\big(\alpha \mathbb{1}_{\langle 0,T\rangle}(\eta_j^T)(\eta_j^T - e) + \beta p(\mathbb{1}_{\{T\}}(\eta_j^T) - q)\big) \xrightarrow{\mathcal{D}} N(0, \alpha^2 vp + \beta^2 p^3 q) \tag{23}$$

by the central limit theorem (CLT) (see [6]). In addition, $\lim_n I_n/n = \lim_n Y_n = q$ a.s. by SLLN, and the random variables $(I_n - nq)/\sqrt{n}$ are bounded in probability by CLT. These facts imply that the random variable given by expression (22) converges to 0 in probability. Finally, SLLN and statement (23) imply (21). □

In accordance with Theorem 1, the functions $\kappa \mapsto \mathbf{A}(\kappa)$ and $\kappa \mapsto \mathbf{\Sigma}(\kappa)$ can be continuously extended to $\bar{\mathcal{K}}$, and the matrix $\mathbf{\Sigma}(\kappa)$ is regular for each $\kappa \in \bar{\mathcal{K}}$.

Now, we can define the minimum χ^2-criterion function. Let $\mathbf{M}(\kappa)$ be the matrix inverse of $\mathbf{\Sigma}(\kappa)$, $\kappa \in \bar{\mathcal{K}}$. Let for any $\kappa \in \bar{\mathcal{K}}$ the χ^2-function be defined by the expression

$$\begin{aligned}Q_n(\kappa) &:= n(\mathbf{M}(\kappa)(\mathbf{Z}_n - \mathbf{A}(\kappa)) \mid \mathbf{Z}_n - \mathbf{A}(\kappa)) \\ &= \sum_{i=1}^{l} n_i\left(\mathbb{P}_\kappa^{(z_i)}(\eta < T)\frac{(X_n^{(z_i)} - \mathbb{E}_\kappa^{(z_i)}[\eta \mid \eta < T])^2}{\mathbb{V}\mathrm{ar}_\kappa^{(z_i)}[\eta \mid \eta < T]} + \frac{(Y_n^{(z_i)} - \mathbb{P}_\kappa^{(z_i)}(\eta \geq T))^2}{\mathbb{P}_\kappa^{(z_i)}(\eta < T)\mathbb{P}_\kappa^{(z_i)}(\eta \geq T)}\right).\end{aligned} \tag{24}$$

Then a sequence of estimators $(\hat{\kappa}_n, n \in \mathbb{N})$ such that

$$Q_n(\hat{\kappa}_n) - \inf_{\kappa \in \mathcal{K}} Q_n(\kappa) \xrightarrow{\mathbb{P}} 0,\ n \to +\infty, \tag{25}$$

is called a sequence of *minimum* χ^2*-estimators*.

To prove the existence, consistency and asymptotic normality of the minimum χ^2-estimators with respect to (24) we need some technical assumptions. Let $\dot{\mathbf{A}}(\kappa)$ be the derivative of $\mathbf{A}(\kappa)$ with respect to κ.

(A4) *The parameter space $\mathcal{K}$ is a relatively compact set in $\mathbb{R}^k$.*

(A5) *The function $\kappa \mapsto \mathbf{A}(\kappa)$ is one-to-one on $\bar{\mathcal{K}}$ and for any $\kappa \in \mathcal{K}$ the rank of $\dot{\mathbf{A}}(\kappa)$ is equal to k.*

It is implicitly assumed in (A5) that the number l of values of the control variable satisfies the condition $2l \geq k$.

Theorem 3. *Let the assumptions* (A4) *and* (A5) *be satisfied and let $\kappa_0 \in \mathcal{K}$ be the true value of the parameter.*

(*i*) *For each sequence of the minimum χ^2-estimators $(\hat{\kappa}_n, n \in \mathbb{N})$ with respect to the χ^2-function* (24) *the following holds when $n \to +\infty$:*

$$\hat{\kappa}_n \xrightarrow{\mathbb{P}} \kappa_0, \tag{26}$$

$$\sqrt{n}(\hat{\kappa}_n - \kappa_0) \xrightarrow{\mathcal{D}} N(\mathbf{0}, (\dot{\mathbf{A}}(\kappa_0)^\tau \mathbf{M}(\kappa_0) \dot{\mathbf{A}}(\kappa_0))^{-1}), \tag{27}$$

$$Q_n(\hat{\kappa}_n) \xrightarrow{\mathcal{D}} \chi^2(2l - k). \tag{28}$$

(*ii*) *There exists a sequence of estimators $(\hat{\kappa}_n, n \in \mathbb{N})$ such that a.s.*

$$(\exists N \in \mathbb{N})(\forall n \geq N)\ \ Q_n(\hat{\kappa}_n) = \min_{\kappa \in \mathcal{K}} Q_n(\kappa). \tag{29}$$

The proof of Theorem 3 is based on the following lemma.

Lemma 1. *If the conditions* (A4) *and* (A5) *hold, then the function $\kappa \mapsto \mathbf{A}(\kappa)$ is bicontinuous in $\bar{\mathcal{K}}$ (see [6]) and all the diagonal entries of the function $\kappa \mapsto \mathbf{M}(\kappa)$ are bounded from below, and away from 0, uniformly with respect to $\kappa \in \bar{\mathcal{K}}$.*

Proof. The conditions (A4) and (A5) imply that the function $\kappa \mapsto \mathbf{A}(\kappa)$ is an immersion and one-to-one on the compact set $\bar{\mathcal{K}}$. Hence, by Theorem 4.5.3 in [3], it is a homeomorphism onto its image, implying the bicontinuity of $\mathbf{A} : \mathcal{K} \to \mathbb{R}^{2l}$. The second statement of Lemma is a consequence of the positivity and continuity of the diagonal entries on the compact set $\bar{\mathcal{K}}$. □

Proof of Theorem 3. The statements of part (*i*) are direct consequences of Theorem 2 and Lemma 1 by Theorems 23 and 24 in [6].

Let us prove statement (*ii*). Since $\bar{\mathcal{K}}$ is a compact set and the function Q_n depends continuously on both, the statistic $\mathbf{Z}_n$ and the parameter κ, there exists a random variable $\tilde{\kappa}_n$ with values in $\bar{\mathcal{K}}$ such that $Q_n(\tilde{\kappa}_n) = \min_{\kappa \in \bar{\mathcal{K}}} Q_n(\kappa)$ by e.g.

Theorem 6.10 in [14]. Now, in order to finish the proof of (ii) it is sufficient to prove that a.s. there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\tilde{\kappa}_n \in \mathcal{K}$. Let us assume that this is not true. Then there exists a subsequence $(\tilde{\kappa}_{n_m})$ of $(\tilde{\kappa}_n)$ at the border $\partial\mathcal{K}$ of $\mathcal{K}$ that converges to some point $\kappa_* \in \partial\mathcal{K}$ on an event of the positive probability. For any $\kappa \in \bar{\mathcal{K}}$, $\lim_n (Q_n(\kappa)/n) = (\mathbf{M}(\kappa)(\mathbf{A}(\kappa_0) - \mathbf{A}(\kappa))|(\mathbf{A}(\kappa_0) - \mathbf{A}(\kappa))) =: Q_0(\kappa)$ a.s. by SLLN. $Q_0(\kappa) = 0$ iff $\kappa = \kappa_0$ by (A5). Hence $\varepsilon := \inf Q_0|_{\partial\mathcal{K}} > 0$ since Q_0 is continuous and $\partial\mathcal{K}$ is a compact set. By the assumption and SLLN it follows that $\lim_m (Q_{n_m}(\tilde{\kappa}_{n_m})/n_m) = Q_0(\kappa_*)$ on an event of positive probability, since $\mathbf{A}$ and $\mathbf{M}$ are continuous on $\bar{\mathcal{K}}$. Moreover, on the same event we have that $Q_{n_m}(\kappa_0)/n_m \geq \min_{\kappa\in\bar{\mathcal{K}}} Q_{n_m}(\kappa)/n_m = Q_{n_m}(\tilde{\kappa}_{n_m})/n_m$ by the assumption. Hence $0 = \lim_m Q_{n_m}(\kappa_0)/n_m \geq \lim_m Q_{n_m}(\tilde{\kappa}_{n_m})/n_m = Q_0(\kappa_*) \geq \varepsilon$, which is a contradiction. □

Remark 1. *If we assume that the vector function*

$$\kappa \mapsto (\Delta_1 F^1, \Delta_2 F^1, .., \Delta_R F^1, 1 - F^{(z_1)}(T|\kappa), \dots, \Delta_1 F^l, \Delta_2 F^l, .., \Delta_R F^l, 1 - F^{(z_l)}(T|\kappa))^\tau$$

satisfies Assumption (*A*5)*, then the statements of Theorem 3 hold for a sequence of the minimum χ^2-estimators obtained by the Pearson's χ^2-criterion function (15).*

In order to compare the matrix $\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)$ and the information matrix $I(\kappa)$ of (12) we define the following objects. For a k-dimensional vector $\mathbf{v}$, the density $f^{(z_i)}(t|\kappa)$ and the function

$$g^{(z_i)}(t|\kappa) = (\nabla_\kappa \log f^{(z_i)}(t|\kappa)|\mathbf{v})$$

we consider $f^i(t) \equiv f^{(z_i)}(t|\kappa)$ and $g^i(t) \equiv g^{(z_i)}(t|\kappa)$, $i = 1, 2, \dots, l$. Then the inequality

$$\begin{aligned}(I(\kappa)\mathbf{v}|\mathbf{v}) &- (\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)\mathbf{v}|\mathbf{v}) \\ &= \sum_{i=1}^{l} \begin{vmatrix} \int_0^T f^i dt & \int_0^T t f^i dt & \int_0^T g^i f^i dt \\ \int_0^T t f^i dt & \int_0^T t^2 f^i dt & \int_0^T t g^i f^i dt \\ \int_0^T g^i f^i dt & \int_0^T t g^i f^i dt & \int_0^T (g^i)^2 f^i dt \end{vmatrix} \cdot \begin{vmatrix} \int_0^T f^i dt & \int_0^T t f^i dt \\ \int_0^T t f^i dt & \int_0^T t^2 f^i dt \end{vmatrix}^{-1} \geq 0 \end{aligned} \tag{30}$$

with an arbitrary $\mathbf{v}$ has the meaning $I(\kappa) \geq \dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)$. From Theorem 5.1 of [12] we get that $(\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa))^{-1} \geq (I(\kappa))^{-1}$, implying that the proposed method of estimation is not generally asymptotically efficient. In cases that the densities $f^{(z)}(\cdot|\kappa)$ satisfy the equations of the form $\nabla_\kappa \log f^{(z)} = \mathbf{a} + \mathbf{b}t$, for some vectors $\mathbf{a} = \mathbf{a}(\kappa, z)$ and $\mathbf{b} = \mathbf{b}(\kappa, z)$, inequality (30) turns into an equation meaning that the method is asymptotically efficient. For instance, this is fulfilled when the function c from Expression (2) does not depend on state x and time t, and $\tau_D = +\infty$ a.s.

When we have to estimate the statistical structure (9) numerically, it is more appropriate to compare $\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)$ with the approximate information matrix

$I_R(\kappa)$ of (14). Let $\mathbf{v}$ be any k-dimensional non-null vector. Then

$$\begin{aligned}&(I_R(\kappa)\mathbf{v}|\mathbf{v}) - (\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)\mathbf{v}|\mathbf{v})\\ &= \sum_{i=1}^{l} \frac{1}{F^i}\left(\sum_{r<s}\left(\sqrt{\frac{\Delta_s F^i}{\Delta_r F^i}}(\nabla_\kappa \Delta_r F^i|\mathbf{v}) - \sqrt{\frac{\Delta_r F^i}{\Delta_s F^i}}(\nabla_\kappa \Delta_s F^i|\mathbf{v})\right)^2\right.\\ &\quad \left. - \begin{vmatrix} \int_0^T f^i dt & \int_0^T t f^i dt \\ \int_0^T t f^i dt & \int_0^T t^2 f^i dt \end{vmatrix}^{-1} \cdot \begin{vmatrix} \int_0^T f^i dt & \int_0^T t f^i dt \\ \int_0^T g^i f^i dt & \int_0^T t g^i f^i dt \end{vmatrix}^2\right)\end{aligned}$$

by using the same notations as in (14) and (30). If there exists i such that $\mathbf{v}$ is not orthogonal to $\nabla_\kappa \log f^i(t)$ for all $t \in [0,T]$, then for $R = 1$ we have $(I_R(\kappa)\mathbf{v}|\mathbf{v}) - (\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa)\mathbf{v}|\mathbf{v}) < 0$, impying $(\dot{\mathbf{A}}(\kappa)^\tau \mathbf{M}(\kappa)\dot{\mathbf{A}}(\kappa))^{-1} < (I_1(\kappa))^{-1}$. Therefore, the proposed method is more efficient than the approximate maximum likelihood method when $R = 1$. We can expect that the same result holds when R (i.e. the number of points in time interval subdivision) is small. This does not have to be the case for larger R because $I_R(\kappa) \to I(\kappa)$ when $R \to \infty$ in a way that $\sup_r (t_r - t_{r-1}) \to 0$.

4. Example

Let a moving object be modeled by the two-dimensional scaled Brownian motion

$$X_t = X_0 + \sigma W_t,\ t > 0, \tag{31}$$

with $\sigma^2 = 0.5$ and the initial position $X_0 = x_0 = (0.5, 0.5)$. We consider this process until its first exit time from the open square $D = \langle 0,1\rangle \times \langle 0,1\rangle$. A searcher tries to detect the target X_t by using one of five different paths z_i ($i = 1,2,\ldots,5$) in D, defined by

$$z_i(t) = \begin{cases} x_{i0} + t v_i^{(1)}, & 0 \le t \le \alpha T, \\ z_i(\alpha T) + (t - \alpha T) v_i^{(2)}, & \alpha T < t \le T, \end{cases} \quad t \ge 0,$$

where $\alpha = 13/24$. The initial positions x_{i0} and the velocities $v_i^{(1)}$ and $v_i^{(2)}$ ($i = 1,2,\ldots,5$) are illustrated in Figure 1. The detection time is modeled by the killing time ζ with the infinitesimal killing rate (the rate function of detection) equal to:

$$c(z;t,x,\kappa) = 10\kappa \mathbb{1}_{\{|x-z(t)|<\sqrt{0.025}\}}.$$

This function depends on one-dimensional parameter κ which is called the search effort. We assume that the value of the parameter κ belongs to the interval $\mathcal{K} = \langle 0.25, 1.75\rangle$. The maximum observation (or search) time is $T = 1.2$.

In the example of this section the diffusion tensor is constant, drift is omitted, and the domain is a square. However, the function c is not a continuous function of t, thus supplying the example with an essential property of the proposed method.

Obviously, the model of the example satisfies Assumptions (A1), (A3), and (A4). Regarding Assumption (A2) the measurability, continuity and differentiability are

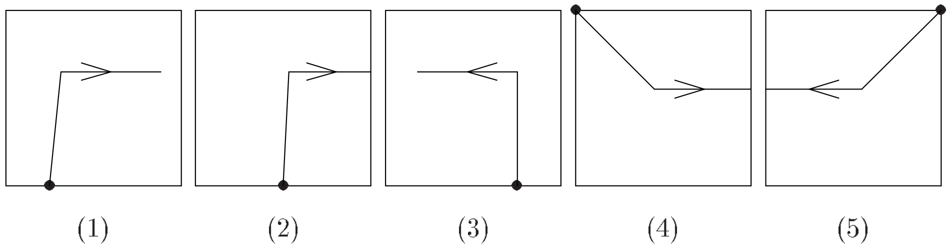

Figure 1: *The initial positions (a bullet) and the velocity directions of* z_i*,* $i = 1, 2, \ldots, 5$*, over* $D = \langle 0, 1\rangle \times \langle 0, 1\rangle$

also obvious. It remains to prove inequality (4) of Assumption (A2) for each $\kappa \in \mathcal{K}$ and $i = 1, \ldots, 5$. Let $\varepsilon = \sqrt{0.025}$. It turns out that (4) follows from the statement that for any $t > 0$, $\mathbb{P}(|X_t - z_i(t)| < \varepsilon) > 0$, being a consequence of the fact that X is a Brownian motion. Moreover, since for each $i = 1, \ldots, 5$, the function $\kappa \mapsto F^{(z_i)}(T|\kappa)$ is strictly increasing and differentiable on the interval $\mathcal{K}$, the model also satisfies Assumption (A5). The strict monotonicity of $F^{(z_i)}(T|\cdot)$ is a consequence of the strict positivity of $\frac{\partial}{\partial\kappa}F^{(z_i)}(T|\kappa)$ which can be proved by analyzing Expression (7) from Theorem 1. More precisely, if we assume that $\frac{\partial}{\partial\kappa}F^{(z)}(T|\kappa) = 0$ for some κ and $z = z_i$, then it follows from (7) that

$$0 = \mathbb{E}[\mathbb{1}_{\{\tau_D > T\}} \int_0^T \mathbb{1}_{\{|X_t - z(t)| < \varepsilon\}}\, dt] = \int_0^T \mathbb{P}(\tau_D > T, |X_t - z(t)| < \varepsilon)\, dt. \tag{32}$$

On the other hand, let $[t_0, t_1] \subset [0, T]$ be such an interval that $t_0 < t_1$ and the balls $K(z(t), \varepsilon) \subset D$ for all $t \in [t_0, t_1]$, and let z_0 be such a continuous function that $z_0(0) = x_0$, $z_0|_{[t_0,t_1]} = z|_{[t_0,t_1]}$ and $K(z_0(t), \varepsilon) \subset D$ for all $t \in [0, T]$. Notice that z is continuous. By the support theorem (see Theorem I.(6.6) in [1]) there exists a constant $c_0 > 0$ such that $\mathbb{P}(\sup_{0 \le t \le T} |X_t - z_0(t)| < \varepsilon) > c_0$. This implies that

$$\int_0^T \mathbb{P}(\tau_D > T, |X_t - z(t)| < \varepsilon)\, dt \ge \int_{t_0}^{t_1} \mathbb{P}(\sup_{0 \le t \le T} |X_t - z_0(t)| < \varepsilon) > c_0(t_1 - t_0) > 0$$

which contradicts (32).

Since the statistical structure (9) of this example cannot be obtained in a closed form, we have to estimate it numerically. For given z_i and κ, the process obtained by killing X at the detection time ζ, up to the first exit time τ_D from D, has a family of one-dimensional pdfs that are denoted by $p(t, \cdot)$, $t \ge 0$. The function $(t, x) \mapsto p(t, x)$ is a solution of the initial value problem for the 2^{nd}-order parabolic system:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\sigma^2}{2}\Delta + c(z_i; \cdot, \cdot, \kappa)\right) p(t, x) &= 0, \quad t > 0,\ x \in D, \\ p(t, x) &= 0, \quad t > 0,\ x \in \partial D, \\ p(0, x) &= \delta_{x_0}(x). \end{aligned} \tag{33}$$

Then $F^{(z_i)}(t|\kappa) = 1 - \int_D p(t, x)\, dx$ (see [7]). Numerical solutions to $p(t, x)$ are obtained by using the numerical procedure described in [10]. First, the problem

(33) is approximated on a grid by an ODE. We use homogenous and rectangular grids in D defined by the grid-steps $\Delta x_1 = \Delta x_2$. The refinement of grid-steps and time-steps is stopped at $\Delta x_1 = \Delta x_2 = 0.0125$ and $\Delta t = 1.25 \cdot 10^{-4}$. For this value of grid-step the first four digits of numerically evaluated $\mathbb{E}_\kappa^{(z_i)}[\eta|\eta < T]$ were unchanged when compared to the values obtained in the previous approximation. The obtained numerical solutions to (33) are used for numerical approximations of the cdf $F^{(z_i)}(\cdot|\kappa)$, and its functionals $\mathbb{E}_\kappa^{(z_i)}[\eta|\eta < T]$ and $\mathbb{V}\mathrm{ar}_\kappa^{(z_i)}[\eta|\eta < T]$.

A numerical approximation of the density function of η can be obtained by a numerical differentiation. In the case of a one-dimensional searching problem, numerical differentiation can be easily carried out and results illustrated. For this purpose we consider the set $D = \langle 0, 1\rangle \subset \mathbb{R}$, a scaled Brownian motion in $\mathbb{R}$ starting at $x = 0.5$, and an infinitesimal killing rate defined as in the two-dimensional case in terms of the indicator of an interval. Only one path $t \mapsto z(t)$ is considered. The corresponding graphs of cdf and pdf of η are illustrated in Figure 2. The figure demonstrates an instability of the process of numerical differentiation. Therefore, in the case analyzed here the maximum likelihood method of estimation can be inappropriate.

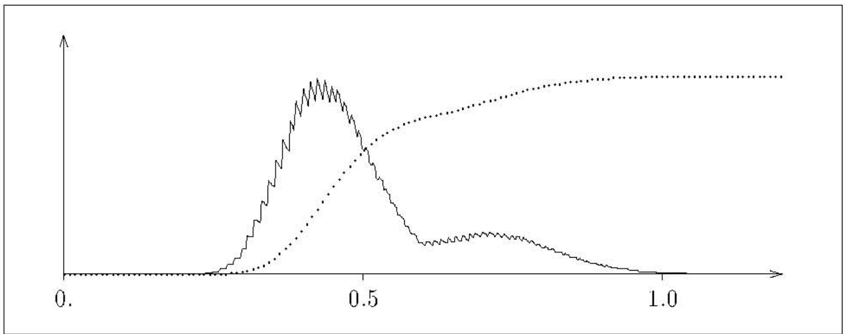

Figure 2: *Plot of estimated pdf (solid line) and cdf (dotted line) of η for $\kappa = 1$ and one-dimensional search*

Samples of the data (10) for the example are obtained by simulations. Outcomes of both random variables, ζ and τ_D, depend on the sample paths of the diffusion X until the observation time T. The simulation of the killing time ζ is based on the fact that $\zeta \le t$ if and only if $\int_0^t c(z; s, X_s, \kappa)\, ds \ge \xi$, where ξ is an exponentially distributed random variable with the expectation equal to 1, and independent of X (see e.g. [2]). The diffusion X is approximated by a process with discrete times, which is denoted by $X^\Delta = (X_k^\Delta; k = 0, 1, \dots K)$ and which is obtained by applying the Euler scheme to (31) with the time increment $\Delta = T/K = 0.00125$ (see e.g. [8]). The paths of X are approximated by the paths of this time-discrete process. Thus, an outcome of ζ is approximated by the first discrete time $t_k = k\Delta \in [0, T]$, which satisfies the condition $\Delta \sum_{i=0}^k c(z; i\Delta, X_i^\Delta, \kappa) \ge \xi$. An outcome of τ_D is approximated by the first discrete time $t_k = k\Delta \in [0, T]$ for which $X_k^\Delta \notin D$. We choose $\kappa = 1$ for the true value of the parameter.

n	$\hat{\text{std}}[\hat{\kappa}_n]$	$\hat{\text{bias}}[\hat{\kappa}_n]$ (95% CI)	$\hat{q}_{.025}$ (95% CI)	$\hat{q}_{.975}$ (95% CI)
20	0.23	-0.029 $(-0.043, -0.014)$	-0.4000 $(-0.4250, -0.3875)$	0.5000 $(0.5000, 0.5000)$
50	0.16	-0.005 $(-0.015, 0.005)$	-0.2625 $(-0.2875, -0.2625)$	0.3500 $(0.3125, 0.4000)$
100	0.12	-0.004 $(-0.011, 0.003)$	-0.2000 $(-0.2125, -0.1875)$	0.2375 $(0.2250, 0.2875)$
1000	0.04	0.002 $(-0.001, 0.004)$	-0.0625 $(-0.0750, -0.0625)$	0.0750 $(0.0750, 0.0875)$

Table 1: *Monte Carlo estimates of the distribution of* $\hat{\kappa}_n$ *for* $\kappa = 1$*. The bias* $\text{bias}[\hat{\kappa}_n]$*, the standard deviation* $\text{std}[\hat{\kappa}_n]$ *and the quantiles* $q_{.025}$ *and* $q_{.975}$ *of* $\hat{\kappa}_n - \kappa$*, are estimated. The sampling 95% confidence intervals of the bias and the quantiles are also calculated*

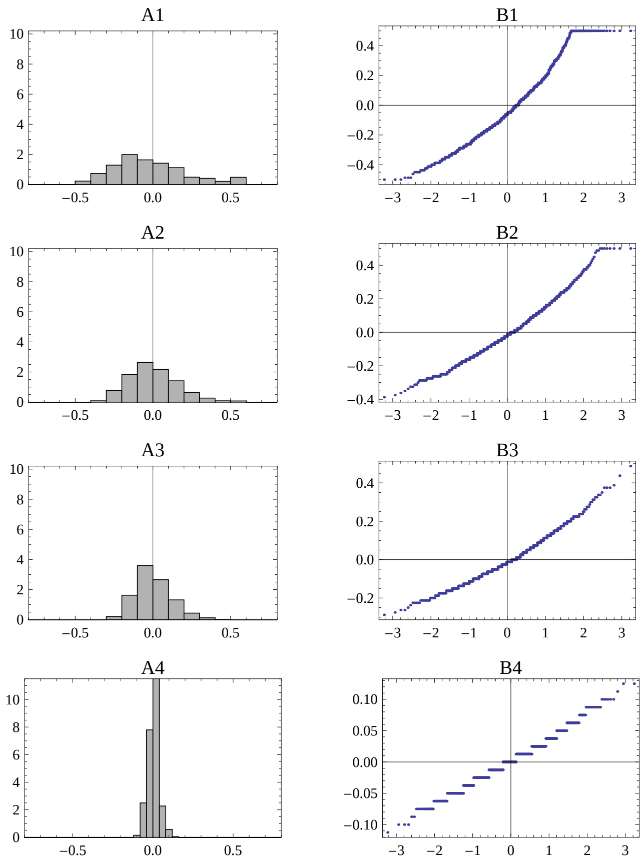

Figure 3: *Histograms (A1-4) and normal QQ-plots (B1-4) of the simulated distributions of* $\hat{\kappa}_n - \kappa$ *for* $\kappa = 1$ *and* $n = 20$ *(A1, B1);* $n = 50$ *(A2, B2);* $n = 100$ *(A3, B3);* $n = 1000$ *(A4, B4)*

In order to obtain an estimate of $\hat{\kappa}_n$ the χ^2-function (24) is minimized over a discrete set $\{0.25 + j \cdot 0.0125 \,:\, j = 0, 1, \dots, 120\} \subset \bar{\mathcal{K}}$ for the given sample of length $5n$ from (9).

$\hat{\kappa}_n$	$\widehat{\text{std}}[\hat{\kappa}_n^*]$	$\widehat{\text{bias}}[\hat{\kappa}_n^*]$	$\hat{q}^*_{.025}$	$\hat{q}^*_{.975}$	95% CI for κ
			$n = 20$		
0.7375	0.17	−0.014	$-0.2375^\ddagger$	$0.3750^\ddagger$	$(0.3625, 0.9750)^\star$
1.3000	0.25	$-0.105^\dagger$	$-0.5750^\ddagger$	$0.2000^\ddagger$	$(1.1000, 1.8750)^\star$
1.0875	0.25	$-0.045^\ddagger$	$-0.4500^\dagger$	$0.4125^\ddagger$	(0.6750, 1.5375)
0.8125	0.19	−0.019	$-0.3125^\ddagger$	$0.4625^\ddagger$	(0.3500, 1.1250)
0.8375	0.20	−0.017	$-0.3250^\ddagger$	0.5000	(0.3375, 1.1625)
			$n = 50$		
0.8625	0.13	−0.011	$-0.2375^\dagger$	$0.2875^\dagger$	(0.5750, 1.1000)
0.8000	0.12	−0.011	$-0.2125^\ddagger$	$0.2500^\ddagger$	(0.5500, 1.0125)
0.8500	0.13	0.002	$-0.2125^\ddagger$	0.3125	(0.5375, 1.0625)
0.7750	0.11	−0.003	$-0.1875^\ddagger$	$0.2500^\ddagger$	$(0.5250, 0.9625)^\star$
0.9750	0.17	−0.003	−0.2750	0.3875	(0.5875, 1.2500)
			$n = 100$		
0.8750	0.09	−0.000	$-0.1625^\ddagger$	$0.2125^\ddagger$	(0.6600, 1.0375)
1.0375	0.12	−0.001	−0.2125	0.2750	(0.7625, 1.2500)
0.9500	0.10	−0.005	−0.1875	$0.2000^\dagger$	(0.7500, 1.1375)
1.2250	0.14	−0.020	$-0.2750^\ddagger$	0.2750	(0.9500, 1.5000)
			$n = 1000$		
0.9875	0.04	0.002	−0.0625	0.0750	(0.9125, 1.0500)
0.9750	0.03	0.001	−0.0625	0.0750	(0.9000, 1.0375)
1.0375	0.04	0.002	−0.0625	0.0750	(0.9500, 1.1000)
0.9625	0.04	0.004	−0.0625	0.0750	(0.8875, 1.0250)

Table 2: *Monte Carlo estimates of the bootstrap distribution of* $\hat{\kappa}_n - \kappa$*. The numbers marked by* † *are not in the corresponding CI from Table 1; the same holds for the numbers marked by* ‡ *and, in addition, their sampling CI and the corresponding CI from Table 1 have an empty intersection. The CIs marked by* ⋆ *do not contain the true value* $\kappa = 1$

The distribution of $\hat{\kappa}_n$ is estimated by Monte Carlo methods. Precisely, for $n =$ 20, 50, 100 and 1000 we simulate samples of $\hat{\kappa}_n - \kappa$ of length $M = 1001$ by using the simulated samples (10) with the true value of the parameter $\kappa = 1$. Their histograms are presented in Figure 3. From such simulated data we estimated the bias, the standard deviation of $\hat{\kappa}_n$, as well as the 0.025 and the 0.975 quantiles of $\hat{\kappa}_n - \kappa$. The results are presented in Table 1. In order to get an insight into the speed of convergence of the estimators $\hat{\kappa}_n$ to the normal distribution, we show the normal QQ-plots of the simulated samples in Figure 3.

Since the true value of the parameter κ is generally unknown, the distribution of $\hat{\kappa}_n$ is usually approximated by the bootstrap methods. For $n =$ 20, 50, 100 and

1000 we chose several estimates $\hat{\kappa}_n$ of the unknown parameter. The estimates are obtained from independently simulated samples of the model (9) for $\kappa = 1$. For each chosen estimate we simulate a bootstrap sample of $\hat{\kappa}_n - \kappa$ of length $M = 1001$ by using the parametric bootstrap method (see [5]). From such simulated data we estimated approximations to the bias and the standard deviation of $\hat{\kappa}_n$, as well as the 95% confidence intervals (CI) of κ. Some of the results are presented in Table 2. From the comparison of results in Table 1 and Table 2 the following can be drawn. The bootstrap was satisfactory in estimating the standard deviation and the bias of $\hat{\kappa}_n$ for each chosen n. The bootstrap was also satisfactory in estimating the CI of κ for $n = 50$, 100 and 1000.

The sensitivity of the proposed minimum χ^2-estimation method with respect to the applied numerical method is discussed in [4].

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