Math. Commun. **17**(2012), 187–193

Normality of adjointable module maps

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Received September 21, 2010; accepted April 29, 2011

Abstract. Normality of bounded and unbounded adjointable operators is discussed. If T is an adjointable operator on a Hilbert C*-module which has polar decomposition, then T is normal if and only if there exists a unitary operator $\mathcal{U}$ which commutes with T and T^* such that $T = \mathcal{U}T^*$. Kaplansky's theorem for normality of the product of bounded operators is also reformulated in the framework of Hilbert C*-modules.

AMS subject classifications: Primary 46L08; Secondary 47A05, 46C05

Key words: Hilbert C*-modules, polar decomposition, normal operators, C*-algebra of compact operators, unbounded operators

1. Introduction

Normal operators may be regarded as a generalization of a selfadjoint operator T in which T^* need not be exactly T but commutes with T. They form an attractive and important class of operators which play a vital role in operator theory, especially in spectral theory. In this note we will study bounded and unbounded normal module maps between Hilbert C*-modules which have polar decomposition. Indeed, for an adjointable operator T between Hilbert C*-modules which has polar decomposition, we demonstrate that T is normal if and only if there exists a unitary operator $\mathcal{U}$ such that $T = \mathcal{U}T^*$. In this situation, $\mathcal{U}T \subseteq T\mathcal{U}$ and $\mathcal{U}T^* \subseteq T^*\mathcal{U}$ (compare [11, Problem 13, page 109]). The results are interesting even in the case of Hilbert spaces.

Suppose T, S are bounded adjointable operators between Hilbert C*-modules. Suppose T has polar decomposition and T and TS are normal operators. Then we show that ST is a normal operator if and only if S commutes with $|T|$. This fact has been proved by Kaplansky [12] in the case of Hilbert spaces.

Throughout the present paper we assume $\mathcal{A}$ to be an arbitrary C*-algebra. We deal with bounded and unbounded operators at the same time, so we denote bounded operators by capital letters and unbounded operators by small letters. We use the notations $Dom(.)$, $Ker(.)$ and $Ran(.)$ for domain, kernel and range of operators, respectively.

Hilbert C*-modules are essentially objects like Hilbert spaces, except that the inner product, instead of being complex-valued, takes its values in a C*-algebra. Although Hilbert C*-modules behave like Hilbert spaces in some ways, some fundamental Hilbert space properties like Pythagoras' equality, self-duality, and even

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decomposition into orthogonal complements do not hold. A (right) *pre-Hilbert C*-module* over a C*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module X equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : X\times X\to\mathcal{A}\,,\ (x,y)\mapsto\langle x,y\rangle$, which is $\mathcal{A}$-linear in the second variable y and has the properties:

$$\langle x,y\rangle=\langle y,x\rangle^*,\quad \langle x,x\rangle\geq 0 \ \text{ with equality only when } x=0.$$

A pre-Hilbert $\mathcal{A}$-module X is called a *Hilbert $\mathcal{A}$-module* if X is a Banach space with respect to the norm $\|x\|=\|\langle x,x\rangle\|^{1/2}$. A Hilbert $\mathcal{A}$-submodule W of a Hilbert $\mathcal{A}$-module X is an orthogonal summand if $W\oplus W^{\perp}=X$, where $W^{\perp}$ denotes the orthogonal complement of W in X. We denote by $\mathcal{L}(X)$ the C*-algebra of all adjointable operators on X, i.e., all $\mathcal{A}$-linear maps $T:X\to X$ such that there exists $T^*:X\to X$ with the property $\langle Tx,y\rangle=\langle x,T^*y\rangle$ for all $x,y\in X$. A bounded adjointable operator $\mathcal{V}\in\mathcal{L}(X)$ is called a *partial isometry* if $\mathcal{V}\mathcal{V}^*\mathcal{V}=\mathcal{V}$, see [16] for some equivalent conditions. For the basic theory of Hilbert C*-modules we refer to the books [14, 19] and the papers [4, 6].

An unbounded regular operator on a Hilbert C*-module is an analogue of a closed operator on a Hilbert space. Let us quickly recall the definition. A densely defined closed $\mathcal{A}$-linear map $t: Dom(t)\subseteq X\to X$ is called *regular* if it is adjointable and the operator $1+t^*t$ has a dense range. Indeed, a densely defined operator t with a densely defined adjoint operator t^* is regular if and only if its graph is orthogonally complemented in $X\oplus X$ (see e.g. [7, 14]). We denote the set of all regular operators on X by $\mathcal{R}(X)$. If t is regular, then t^* is regular and $t=t^{**}$; moreover, t^*t is regular and selfadjoint. Define $Q_t=(1+t^*t)^{-1/2}$ and $F_t=tQ_t$, then $Ran(Q_t)=Dom(t)$, $0\leq Q_t=(1-F_t^*F_t)^{1/2}\leq 1$ in $\mathcal{L}(X)$ and $F_t\in\mathcal{L}(X)$ [14, (10.4)]. The bounded operator F_t is called the bounded transform of a regular operator t. According to [14, Theorem 10.4], the map $t\to F_t$ defines an adjoint-preserving bijection

$$\mathcal{R}(X)\to\{F\in\mathcal{L}(X):\|F\|\leq 1 \ \text{ and } \ Ran(1-F^*F) \text{ is dense in } X\}.$$

Very often there are interesting relationships between regular operators and their bounded transforms. In fact, for a regular operator t, some properties transfer to its bounded transform F_t, and vice versa. Suppose $t\in\mathcal{R}(X)$ is a regular operator, then t is called *normal* iff $Dom(t)=Dom(t^*)$ and $\langle tx,tx\rangle=\langle t^*x,t^*x\rangle$ for all $x\in Dom(t)$. t is called *selfadjoint* iff $t^*=t$ and t is called *positive* iff t is normal and $\langle tx,x\rangle\geq 0$ for all $x\in Dom(t)$. In particular, a regular operator t is normal (resp., selfadjoint, positive) iff its bounded transform F_t is normal (resp., selfadjoint, positive). Moreover, both t and F_t have the same range and the same kernel. If $t\in\mathcal{R}(X)$, then $Ker(t)=Ker(|t|)$ and $\overline{Ran(t^*)}=\overline{Ran(|t|)}$, cf. [13]. If $t\in\mathcal{R}(X)$ is a normal operator, then $Ker(t)=Ker(t^*)$ and $\overline{Ran(t)}=\overline{Ran(t^*)}$.

A bounded adjointable operator T has polar decomposition if and only if $\overline{Ran(T)}$ and $\overline{Ran(|T|)}$ are orthogonal direct summands [19, Theorem 15.3.7]. The result has been generalized in Theorem 3.1 of [8] for regular operators. Indeed, for $t\in\mathcal{R}(X)$ the following conditions are equivalent:

- t has a unique polar decomposition $t=\mathcal{V}|t|$, where $\mathcal{V}\in\mathcal{L}(X)$ is a partial isometry for which $Ker(\mathcal{V})=Ker(t)$.
- $X=Ker(|t|)\oplus\overline{Ran(|t|)}$ and $X=Ker(t^*)\oplus\overline{Ran(t)}$.

- The adjoint operator t^* has polar decomposition $t^* = \mathcal{V}^*|t^*|$.
- The bounded transform F_t has polar decomposition $F_t = \mathcal{V}|F_t|$.

In this situation, $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$; moreover, we have $Ker(\mathcal{V}^*) = Ker(t^*)$, $Ran(\mathcal{V}) = \overline{Ran(t)}$ and $Ran(\mathcal{V}^*) = \overline{Ran(t^*)}$. That is, $\mathcal{V}\mathcal{V}^*$ and $\mathcal{V}^*\mathcal{V}$ are orthogonal projections onto the submodules $\overline{Ran(t)}$ and $\overline{Ran(t^*)}$, respectively.

The above facts and Proposition 1.2 of [7] show that each regular operator with a closed range has polar decomposition.

Recall that an arbitrary C*-algebra of compact operators $\mathcal{A}$ is a c_0-direct sum of elementary C*-algebras $\mathcal{K}(H_i)$ of all compact operators acting on Hilbert spaces H_i, $i \in I$, cf. [2, Theorem 1.4.5]. Generic properties of Hilbert C*-modules over C*-algebras of compact operators have been studied systematically in [1, 3, 7, 8, 10] and references therein. If $\mathcal{A}$ is a C*-algebra of compact operators, then for every Hilbert $\mathcal{A}$-module X, every densely defined closed operator $t : Dom(t) \subseteq X \to X$ is automatically regular and has polar decomposition, cf. [7, 8, 10].

The stated results also hold for bounded adjointable operators, since $\mathcal{L}(X)$ is a subset of $\mathcal{R}(X)$. From a topological point of view the space $\mathcal{R}(X)$ is studied in [15, 17, 18].

2. Normality

Proposition 1. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$ and $S \in \mathcal{L}(X)$ is an arbitrary operator which commutes with T and T^*. Then $\mathcal{V}$ and $|T|$ commute with S and S^*.*

Proof. It follows from the hypothesis that $(T^*T)S = S(T^*T)$ which implies $|T|S = S|T|$, or equivalently $|T|S^* = S^*|T|$. Using the commutativity of S with T and $|T|$, we get

$$(S\mathcal{V} - \mathcal{V}S)|T| = S\mathcal{V}|T| - \mathcal{V}|T|S = ST - TS = 0.$$

That is, $S\mathcal{V} - \mathcal{V}S$ acts as a zero operator on $\overline{Ran(|T|)}$. If $x \in Ker(|T|) = Ker(\mathcal{V})$, then $|T|x = \mathcal{V}x = 0$, consequently $|T|Sx = S|T|x = 0$. Then $Sx \in Ker(|T|) = Ker(\mathcal{V})$, therefore, $S\mathcal{V} - \mathcal{V}S$ acts as a zero operator on $Ker(|T|)$, too. We obtain

$$S\mathcal{V} - \mathcal{V}S = 0 \quad \text{on} \quad X = Ker(|T|) \oplus \overline{Ran(|T|)}.$$

The statement $S^*\mathcal{V} - \mathcal{V}S^* = 0$ on $X = Ker(|T|) \oplus \overline{Ran(|T|)}$ can be deduced from the commutativity of S with T^* and $|T|$ in the same way. □

Corollary 1. *Suppose $T \in \mathcal{L}(X)$ is a normal operator which admits the polar decomposition $T = \mathcal{V}|T|$, Then $\mathcal{V}$ and $|T|$ commute with the operators $T, T^*, \mathcal{V}$ and $\mathcal{V}^*$. In particular, $\mathcal{V}$ is a unitary operator on $\overline{Ran(T)} = \overline{Ran(T^*)}$.*

The results follow from Proposition 1, Proposition 3.7 of [14] and the fact that $\mathcal{V}\mathcal{V}^*T = \mathcal{V}^*\mathcal{V}T = T$.

Theorem 1. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ commuting with $|T|$ such that $T = \mathcal{U}|T|$. In this situation, $\mathcal{U}$ also commutes with T and T^*.*

Proof. If T is a normal operator, then $Ker(T) = Ker(T^*)$ and $\overline{Ran(T)} = \overline{Ran(T^*)}$. For every $x \in X = Ker(T) \oplus \overline{Ran(T^*)}$ we define

$$\mathcal{U}x \ = \ \begin{cases} x & \text{if } x \in Ker(T) \\ \mathcal{V}x & \text{if } x \in \overline{Ran(T^*)}, \end{cases}$$

$$\mathcal{W}x = \begin{cases} x & \text{if } x \in Ker(T^*) \\ \mathcal{V}^*x & \text{if } x \in \overline{Ran(T)}. \end{cases}$$

Then $\langle \mathcal{U}x, y\rangle = \langle x, \mathcal{W}y\rangle$ for all $x, y \in X$, that is, $\mathcal{W} = \mathcal{U}^*$. For each $x = x_1 + x_2 \in X$ with $x_1 \in Ker(T)$ and $x_2 \in \overline{Ran(T^*)}$ we have

$$\mathcal{U}\mathcal{U}^*x = \mathcal{U}(x_1 + \mathcal{V}^*x_2) = x_1 + \mathcal{V}\mathcal{V}^*x_2 = x.$$

Hence, $\mathcal{U}\mathcal{U}^* = 1$ on X. We also have $\mathcal{U}^*\mathcal{U} = 1$ and $T = \mathcal{U}|T|$ on X. Commutativity of $\mathcal{U}$ with T, T^* and $|T|$ follows from the commutativity of $\mathcal{V}$ with T, T^* and $|T|$.

Conversely, suppose $T = \mathcal{U}|T|$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ which commutes with $|T|$. Then $T^* = |T|\mathcal{U}^*$ and so $TT^* = \mathcal{U}\,|T|\,|T|\mathcal{U}^* = |T|\mathcal{U}\,|T|\mathcal{U}^* = T^*T$. □

Corollary 2. *Suppose $T \in \mathcal{L}(X)$ admits the polar decomposition $T = \mathcal{V}|T|$. Then T is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $T = \mathcal{U}T^*$. In this situation, $\mathcal{U}$ commutes with T and T^*.*

Proof. If T is a normal operator, then $|T| = |T^*| = \mathcal{V}T^*$ and so $T = \mathcal{V}|T| = \mathcal{V}|T^*| = \mathcal{V}^2T^*$. For $x \in X$ we define

$$\mathcal{U}x \ = \ \begin{cases} x & \text{if } x \in Ker(T) \\ \mathcal{V}^2x & \text{if } x \in \overline{Ran(T^*)}. \end{cases}$$

Then, as in the proof of Theorem 1,

$$\mathcal{U}^*x = \begin{cases} x & \text{if } x \in Ker(T^*) \\ \mathcal{V}^{*\,2}x & \text{if } x \in \overline{Ran(T)}, \end{cases}$$

which implies $\mathcal{U}$ is unitary and $T = \mathcal{U}T^*$. Commutativity of $\mathcal{U}$ with T and T^* follows from the commutativity of $\mathcal{V}$ with T and T^*.

Conversely, suppose $T = \mathcal{U}T^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then $T^* = (\mathcal{U}T^*)^* = T\mathcal{U}^*$ and so $T^*T = T\mathcal{U}^*\mathcal{U}T^* = TT^*$. □

If the normal operator $T \in \mathcal{L}(X)$ has closed range, one can find a shorter proof for the above result.

Theorem 2. *Suppose $t \in \mathcal{R}(X)$ admits the polar decomposition $t = \mathcal{V}|t|$. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\mathcal{U} = \mathcal{U}t$ and $t^*\mathcal{U} = \mathcal{U}t^*$ on $Dom(t) = Dom(t^*)$.*

Proof. Recall that t admits the polar decomposition $t = \mathcal{V}|t|$ if and only if its bounded transform F_t admits the polar decomposition $F_t = \mathcal{V}|F_t|$. Furthermore, t is a normal operator if and only if its bounded transform F_t is a normal operator.

If t is a normal operator, then there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $tQ_t = F_t = \mathcal{U}F_t^* = \mathcal{U}F_{t^*} = \mathcal{U}\,t^*Q_{t^*} = \mathcal{U}\,t^*Q_t$. Since $Q_t : X \to Ran(Q_t) = Dom(t)$ is invertible, we obtain $t = \mathcal{U}t^*$.

Conversely, suppose $t = \mathcal{U}t^*$ for a unitary operator $\mathcal{U} \in \mathcal{L}(X)$. Then, in view of Remark 2.1 of [8], we have $t^* = (\mathcal{U}t^*)^* = t^{**}\,\mathcal{U}^* = t\,\mathcal{U}^*$ on $Dom(t^*)$ and so $t^*t = t\,\mathcal{U}^*\,\mathcal{U}\,t^* = t\,t^*$.

According to Corollary 2 and the first paragraph of the proof, the unitary operator $\mathcal{U}$ commutes with F_t and F_t^*. Thus for every polynomial p we have $\mathcal{U}\,p(F_t^*F_t) = p(F_t^*F_t)\,\mathcal{U}$ and so for every continuous function $p \in \mathbf{C}[0,1]$ we have $\mathcal{U}\,p(F_t^*F_t) = p(F_t^*F_t)\mathcal{U}$. In particular, $\mathcal{U}\,(1 - F_t^*F_t)^{1/2} = (1 - F_t^*F_t)^{1/2}\,\mathcal{U}$ which implies $\mathcal{U}\,Q_t = Q_t\,\mathcal{U}$. This fact together with the equality $F_t\,\mathcal{U} = \mathcal{U}F_t$ imply that $t\,\mathcal{U}\,Q_t = tQ_t\,\mathcal{U} = \mathcal{U}\,tQ_t$. Again by invertibility of the map $Q_t : X \to Ran(Q_t) = Dom(t)$ we obtain $t\,\mathcal{U} = \mathcal{U}\,t$ on $Dom(t)$. To demonstrate the second equality we have $\mathcal{U}^*\,t = \mathcal{U}^*\,\mathcal{U}\,t^* = t^*$ which yields $t^*\,\mathcal{U} = (\mathcal{U}^*\,t)^* = t^{**} = t = \mathcal{U}\,t^*$ on $Dom(t^*)$. □

Corollary 3. *Suppose $t \in \mathcal{R}(X)$ has a closed range. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\,\mathcal{U} = \mathcal{U}\,t$ and $t^*\,\mathcal{U} = \mathcal{U}\,t^*$ on $Dom(t) = Dom(t^*)$.*

The proof follows immediately from Theorem 2, Proposition 1.2 of [7] and Theorem 3.1 of [8].

Corollary 4. *Suppose X is a Hilbert space (or a Hilbert C*-module over an arbitrary C*-algebra of compact operators) and $t : Dom(t) \subseteq X \to X$ is a densely defined closed operator. Then t is a normal operator if and only if there exists a unitary operator $\mathcal{U} \in \mathcal{L}(X)$ such that $t = \mathcal{U}t^*$. In this situation, $t\,\mathcal{U} = \mathcal{U}\,t$ on $Dom(t) = Dom(t^*)$.*

Consider two normal operators T and S on a Hilbert space. It is known that, in general, TS is not normal. Historical notes and several versions of the problem are investigated in [9]. Kaplansky showed that it may be possible that TS is normal while ST is not. Indeed, he showed that if T and TS are normal, then ST is normal if and only if S commutes with $|T|$, cf. [12]. We generalize his result for a bounded adjointable operator on Hilbert C*-modules. For this aim we need the Fuglede-Putnam theorem for bounded adjointable operators on Hilbert C*-modules. Using Theorem 4.1.4.1 of [5] for the unital C*-algebra $\mathcal{L}(X)$ we obtain:

Theorem 3 (Fuglede-Putnam)**.** *Assume that T, S and A are bounded adjointable operators in $\mathcal{L}(X)$. Suppose T and S are normal and $TA = AS$. Then $T^*A = AS^*$.*

Corollary 5. *Let $T, S \in \mathcal{L}(X)$ be such that T and TS are normal and T has a polar decomposition. ST is normal if and only if S commutes with $|T|$.*

Proof. Suppose ST and T are normal operators and $A = TS$ and $B = ST$. Then $AT = TB$. In view of Theorem 3, $A^*T = TB^*$, that is, $S^*T^*T = T\,T^*S^*$, and taking into account the normality of T, we find S^* commutes with T^*T. Therefore, $S^*|T| = |T|S^*$ and so S commutes with $|T|$ by taking an adjoint.

Conversely, suppose S commutes with $|T|$. Then the normal operator T has a representation $T = \mathcal{U}\,|T|$ in which $\mathcal{U} \in \mathcal{L}(X)$ is unitary and commutes with $|T|$. Therefore,

$$\mathcal{U}^*\,TS\,\mathcal{U} = \mathcal{U}^*\,\mathcal{U}\,|T|\,S\,\mathcal{U} = S|T|\mathcal{U} = S\,\mathcal{U}\,|T| = ST.$$

The operator ST is normal as an operator which is unitary equivalent with the normal operator TS. □

Acknowledgement

The author would like to thank the referee for his/her careful reading and useful comments.

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