MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 195–204

A note on normal forms for the closed fragment of system IL

Vedran Čačić[1] and Mladen Vuković[1,*]

[1] *Department of Mathematics, University of Zagreb, Bijenička cesta 30, HR-10 000 Zagreb, Croatia*

Received October 14, 2010; accepted November 5, 2011

Abstract. In [8], P. Hájek and V. Švejdar determined normal forms for the system ILF, and showed that we can eliminate the modal operator $\rhd$ from IL–formulas. The normal form for the closed fragment of the interpretability logic IL is an open problem (see [13]). We prove that in some cases we can eliminate the modal operator $\rhd$. We give an example where it is impossible to eliminate $\rhd$.
AMS subject classifications: 03F45, 03F25
Key words: modal logic, interpretability logic, normal form

1. Introduction

K. Gödel proposed treating a provability predicate as a modal operator. S. Kripke and R. Montague have taken up the same idea later. The correct choice of axioms, based on Löb's theorem, was seriously considered in the mid–seventies by several logicians independently: G. Boolos, D. de Jongh, R. Magari, G. Sambin and R. Solovay.

Provability logic is a modal description of a natural provability predicate. The basic system of provability logic is system GL (Gödel, Löb). The system GL is a modal propositional logic. The axioms of system GL are all tautologies, $\Box(A \to B) \to (\Box A \to \Box B)$, and $\Box(\Box A \to A) \to \Box A$. The inference rules of GL are modus ponens and necessitation $A/\Box A$. R. Solovay proved arithmetical completeness of modal system GL w.r.t. Peano Arithmetic in 1976.

A modal GL–sentence is called *letterless* if it contains no sentence letters, equivalently if it is a member of the smallest class containing $\bot$, and containing $A \to B$ and $\Box A$ whenever it contains A and B. As always, $\Box^0 A = A$ and $\Box^{i+1} A = \Box\Box^i A$. We shall say that a letterless sentence C is *in normal form* if it is a truth–functional combination of sentences of the form $\Box^i\bot$. Now, we give the normal form theorem for a closed fragment of the system GL.

Theorem 1 (see [11])**.** *Let B be a letterless GL–sentence. Then there are numbers $n,\ k_0, \ldots,\ k_n,\ m_1,\ \ldots,\ m_{n+1}$ such that*

$$GL \vdash B \leftrightarrow \left(\Box^{k_0}\bot \vee \bigvee_{i=1}^{n} (\Box^{k_i}\bot \wedge \neg\Box^{m_i}\bot) \vee \neg\Box^{m_{n+1}}\bot\right),$$

*Corresponding author. *Email addresses:* `vedran.cacic@math.hr` (V. Čačić), `mladen.vukovic@math.hr` (M. Vuković)

where $0 \leq k_0 < m_1 < k_1 < \ldots < m_n < k_n < m_{n+1} \leq \omega$ *and* $\Box^\omega \bot = \top$. *Moreover, expect for the degenerate case in which* B *is provable, the representation is unique.*

It is important to emphasize that in [4] G. Boolos proved that there are no normal forms in GL for a formula that contains a propositional variable. We would like to mention that S. N. Artëmov in [1] and L. D. Beklemishev in [2] proved the normal form theorem for GL using only semantics.

The provability logic of Peano arithmetic, Zermelo–Fraenkel set theory, and Gödel–Bernays set theory is system GL. It means that provability logic GL cannot distinguish some properties as e.g. finite axiomatizability, reflexivity, etc. Some logicians have considered a modal description of other arithmetical properties, for example interpretability, Π_n–conservativity, interpolability, etc. We study interpretability. Modal logics for interpretability were first studied by P. Hájek (1981) and V. Švejdar (1983). In [12], A. Visser introduced the binary modal logic IL (interpretability logic). The interpretability logic IL results from the provability logic GL, by adding the binary modal operator $\rhd$. We are only interested in interpretability as a system of modal logic, i.e. interpretability logic. We introduce our notation and some basic facts following [13].

The interpretability logic is a modal logic which describes the relation $Int_T(A, B)$ of relative interpretability between arithmetical theories like $T+A$ and $T+B$, where A and B are formulas (for detailed definitions see e.g. [13] or [7]).

The language of the interpretability logic contains the propositional letters p_0, p_1, ..., the logical connectives $\neg$, $\wedge$, $\vee, \rightarrow$ $\leftrightarrow$, the unary modal operators $\Box$ and $\Diamond$, and the binary modal operator $\rhd$. We use $\bot$ for false and $\top$ for true. We read $\rhd$ as binding stronger than binary boolean connectives, and weaker than negation and unary modal operators. The interpretability logic IL contains all axioms of the system GL and the following axioms $\Box(A \rightarrow B) \rightarrow A \rhd B$, $(A \rhd B \wedge B \rhd C) \rightarrow A \rhd C$, $(A \rhd C \ \wedge \ B \rhd C) \rightarrow (A \vee B) \rhd C$, $A \rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$, and $\Diamond A \rhd A$. The deduction rules of IL are modus ponens and necessitation.

A modal IL–sentence is called *letterless* if it contains no sentence letters, equivalently if it is a member of the smallest class containing $\bot$ and containinig $A \rightarrow B$ and $A \rhd B$ whenever it contains A and B. It can be seen that all letterless IL–formulas can be expressed in *reduced language*, using only $\bot$, $\rightarrow$, $\rhd$ and parentheses. That follows from these equivalences:

$$\begin{aligned}
\neg A &\iff (A \rightarrow \bot) \\
\top &\iff \neg\bot \\
A \vee B &\iff \neg A \rightarrow B \\
A \wedge B &\iff \neg(A \rightarrow \neg B) \\
A \leftrightarrow B &\iff (A \rightarrow B) \wedge (B \rightarrow A) \\
\Box A &\iff \neg A \rhd \bot \\
\Diamond A &\iff \neg(A \rhd \bot)
\end{aligned}$$

Let T be an arithmetical theory. The arithmetical interpretation $*$ is a function from the set of IL–formulas to the set of sentences of theory T such that: $(\bot)^* = (0 = 1)$ $(A \rightarrow B)^* = A^* \rightarrow B^*$ and $(A \rhd B)^* = Int_T(A^*, B^*)$.

The system IL is natural from the modal point of view, but arithmetically incomplete. For example, IL does not prove the formula W, i.e. $(A \rhd B) \rightarrow (A \rhd (B \wedge \Box \neg A))$ (see [6]). But we have $T \vdash W^*$ for every reasonable arithmetical theory T and every arithmetical interpretation $*$ (see [12]).

Various extensions of IL are obtained by adding some new axioms. These new axioms are called the principles of interpretability. Principles we shall consider in this paper are among the following.

M	$A \rhd B \rightarrow (A \wedge \Box C) \rhd (B \wedge \Box C)$	(Montagna's principle)
P	$A \rhd B \rightarrow \Box(A \rhd B)$	(principle of persistency)
F	$(A \rhd \Diamond A) \rightarrow \Box(\neg A)$	(Feferman's principle)
$KW1$	$(A \rhd \Diamond \top) \rightarrow (\top \rhd (\neg A))$	(transposition principle)

If X is a principle of interpretability we will denote by ILX the logic that arises by adding the princple X to IL. In [8], P. Hájek and V. Švejdar proved the following normal form theorem for the letterless sentence of the interpretability logic ILF.

Theorem 2 (see [8])**.** *For each letterless IL–formula B there are numbers n, $i_1, \dots,$ i_n, $j_1, \dots, j_n$ such that*

$$ILF \vdash B \leftrightarrow \bigwedge_{k=1}^{n} (\Box^{i_k} \bot \rightarrow \Box^{j_k} \bot),$$

where $0 \leq j_n < i_n < \dots i_2 < j_1 < i_1 \leq \omega$.

We would like to emphasize that P. Hájek and V. Švejdar proved the theorem completely syntactically. The theorem is obviously true for systems ILM and ILP (ILF is a subsystem of ILM and ILP). But, we do not know a normal form for systems weaker than ILF. For example, the systems IL and $ILKW1$ are weaker than ILF. In [13] A. Visser mentioned the normal form for the system IL as an open problem.

In [9], J. Joosten emphasizes that we can eliminate modal operator $\rhd$ in the system ILF, i.e. we have $ILF \vdash A \rhd B \leftrightarrow \Box(A \rightarrow (B \vee \Diamond B))$, for any letterless IL–formulas A, B.

2. Elimination of modal operator $\rhd$ in some cases

We will give some cases where we can eliminate the modal operator $\rhd$. At the beginning we define Veltman models. The notion of a Veltman model is defined in [6].

Definition 1. *An ordered triple $\langle W, R, (S_w : w \in W)\rangle$ is called a Veltman frame if it satisfies the following conditions:*

a) $\langle W, R\rangle$ is a GL–frame, i.e. W is a non–empty set, and R is a transitive and reverse well-founded relation on W;

b) For every $w \in W$ *we have* $S_w \subseteq W[w] \times W[w]$*, where* $W[w] = \{u : wRu\}$*;*

c) The relation S_w *is reflexive and transitive on* $W[w]$ *for every* $w \in W$*;*

d) If $wRuRv$*, then* uS_wv*.*

An ordered quadruple $\langle W, R, (S_w : w \in W), \Vdash \rangle$ *is called a Veltman model if it satisfies the following conditions:*

1) $\langle W, R, (S_w : w \in W) \rangle$ *is a Veltman frame;*

2) $\Vdash$ *is a forcing relation. We emphasize only the definition*

$$w \Vdash A \rhd B \text{ if and only if } \forall u((wRu \ \& \ u \Vdash A) \Rightarrow \exists v(uS_wv \ \& \ v \Vdash B)).$$

We denote a Veltman model $\langle W, R, (S_w : w \in W), \Vdash \rangle$ shortly by W. In [6], D. de Jongh and F. Veltman proved completeness of the system IL w.r.t. Veltman semantics.

Let W be a Veltman model. A *terminal world* in model W is $w \in W$ such that there is no $v \in W$ with wRv. A terminal world is also called *a world of type 1*. We say that a *world is of type 2* if it is nonterminal, and all its R–successors are terminal.

Note that the analogous definition of a "world of type 3" as a world that is not terminal, and all its R-successors are of type 2, is not meaningful, because such a world cannot exist in any Veltman model: if w is such a world, then it must have an R-successor u of type 2, and so u must have an R-successor v of type 1. But then v is also an R-successor of w by transitivity, and it cannot be of type 2 since it is terminal.

The following lemma will be very useful.

Lemma 1. *Every world in every Veltman frame is either of type 1, of type 2, or it has some R–successor of type 2.*

Proof. Suppose that a world w_0 is neither of these types. That in particular means w_0 has a nonterminal R–successor w_1. Since w_1 is a R–successor of w_0, it must not be of type 2, and since it is nonterminal, w_1 must have a nonterminal R–successor also; let's denote it by w_2. Because R is transitive, w_2 is also a nonterminal R–successor of w_0, and must have a nonterminal R–successor w_3, which is also a R–successor of $w_0 \ldots$ Continuing in this way, we obtain a sequence of worlds $w_0Rw_1Rw_2Rw_3R \ldots$, and that is a contradiction since the relation R must be reversely well-founded. □

Corollary 1. *Let* $\langle W, R, (S_w : w \in W) \rangle$ *be a Veltman frame, and* $w \in W$ *nonterminal. There exist* $u, v \in W$ *such that the following three conditions hold:*

(i) wRu *or* $w = u$*,*

(ii) uRv*, and*

(iii) every v' *such that* vS_uv' *is terminal.*

Proof. By using Lemma 1 we have that the world w is or type 2 or it has some R–successor of type 2. If w is of type 2, set $u := w$, else set u to be some R–successor of w of type 2. Either way, u is of type 2, so it is nonterminal. Set v to be any of its R–successors. Since every v' such that vS_uv' must also be an R–successor of u, it must be terminal and the corollary is proved. □

In [3], one can find the definitions of local and global semantic consequences. In [10], M. Kracht considers modal consequence relations. Now, we repeat the definitions of local and global equivalence.

We will say that some formulas φ and ψ are *locally equivalent* if we have $W, w \Vdash \varphi$ if and only if $W, w \Vdash \psi$, for every Veltman model W and every world $w \in W$. We denote local equivalence by $\varphi \Longleftrightarrow \psi$.

Let W be a Veltman model and φ a formula. If we have $W, w \Vdash \varphi$ for every $w \in W$, then we will write $W \Vdash \varphi$. We will say that formulas φ and ψ are *globally equivalent* if we have $W \Vdash \varphi$ if and only if $W \Vdash \psi$, for every Veltman model W. We denote global equivalence by $\varphi \overset{g}{\Longleftrightarrow} \psi$.

Local equivalence is surely better because it is stronger, and it has the property that any subformula B of A can be replaced by a locally equivalent formula B', and we get $A' = A(B'|B)$ which is locally equivalent to A. Global equivalence generally does not have that property.

It is interesting to see that being locally equivalent to $\top$ and being globally equivalent to $\top$ means the same: being modally valid. In fact, local equivalence of φ and ψ can be stated as (global or local) equivalence of $\varphi \leftrightarrow \psi$ and $\top$.

Intuitively, using local equivalence we characterize *worlds* on which a formula holds, while using global equivalence we characterize *models* in which a formula holds (on every world).

Besides equivalence to $\top$, we will consider these global equivalence properties: property of being globally equivalent to $\bot$, which means a formula cannot hold in any (nonempty) model, and being globally equivalent to $\Box\bot$, which means a formula holds only in models with empty accessibility relation R.

Our results will be of type: every formula that has some general form, is locally or globally equivalent to some simpler formula, for some definition of simplicity. Usually that simple formula will be $\bot$, $\top$, $\Box\bot$ or $\Box\Box\bot$ (note that all of them are of the main "form" $\Box^n\bot$, for $n \in \{\omega, 0, 1, 2\}$ — in particular, none of them use binary modal operator $\rhd$).

Definition 2. *An* affirmative formula *is a formula that holds on every terminal world. A* negative formula *is the one that holds on no terminal world. We denote formulas by letters F, G, H; affirmative formulas by A, B C; and negative formulas by N, M, P. We also distinguish* cross formulas*, those are of the form $\neg(A \rhd N)$; we denote them by letters X, Y, Z.*

Lemma 2. *These kinds of formulas are given by the following productions in the grammar of our reduced language:*

$$\begin{aligned} F &::= A \mid N \\ A &::= F \rhd G \mid N \to F \mid F \to B \\ N &::= \bot \mid A \to M \end{aligned}$$

This is just a concise way of writing that every (letterless) formula of some kind is of one of enumerated forms. For example, the third claim says: every negative formula is either $\bot$, or an implication whose left-hand side is an affirmative formula, and the right-hand side is a negative formula.

Proof. By induction on complexity of formulas. Evidently, $\bot$ is a negative formula. Also, any formula of the form $F \rhd G$ is affirmative: if w is terminal, there is no v such that $w\ R\ v \Vdash F$ holds, so we have $w \Vdash F \rhd G$ vacuously.

The only case remaining to be considered is $F \to G$. By induction hypothesis on F, we know that F is either affirmative or negative. In the latter case ($F = N$), if w is terminal, we cannot have $w \Vdash N$, so we have $w \Vdash N \to F$ vacuously.

In the former case ($F = A$), by induction hypothesis on G we have that G is either affirmative or negative. If G is affirmative ($G = B$) and w is terminal, we have $w \Vdash B$ and so $w \Vdash F \to B$. If G is negative ($G = M$) and w terminal, we must have $w \Vdash A$, but we cannot have $w \Vdash M$, so $w \Vdash A \to M$ cannot hold, and so $A \to M$ is a negative formula. Since that exhausts all cases, the inductive step is proved. $\square$

As we have seen, every letterless formula is either affirmative or negative. So, all terminal worlds are modally equivalent w.r.t. letterless sentences.

Our main result is the following theorem.

Theorem 3. *Let F be an IL–formula, A an affirmative formula, N a negative formula, and X a cross formula. Then we have*

$$\begin{aligned} \bot \rhd F &\iff \top \\ F \rhd A &\iff \top \\ A \rhd \bot &\iff \Box\bot \\ N \rhd X &\iff \top \\ X \rhd \bot &\iff \Box\Box\bot \\ N &\overset{g}{\iff} \bot \\ X \to \bot &\overset{g}{\iff} \Box\bot \end{aligned}$$

Since there are no worlds on which $\bot$ holds, we have that formula $\bot \rhd F$ is valid, for each formula F. The other equivalences from the above theorem we will prove by the following propositions.

Proposition 1. *For every formula F and every affirmative formula A, formula $F \rhd A$ is valid.*

Proof. Let w be any world, and let v be any R–successor of w such that $v \Vdash F$.

v is terminal $\Rightarrow v \Vdash A$ (because A is an affirmative formula)
$\Rightarrow w \Vdash F \rhd A$ (by reflexivity od the relation S_w)

v is nonterminal $\Rightarrow \exists u(vRu$ and u is a terminal)
$\Rightarrow u \Vdash A$ (because A is an affirmative formula)
$\Rightarrow w \Vdash F \rhd A$ (by $wRvRu$ we have vS_wu)

$\square$

Proposition 2. *If A is an affirmative formula, then the formula $A \rhd \bot$ is locally equivalent to $\Box\bot$.*

Proof.

$$\begin{aligned} w \text{ is terminal} &\Rightarrow w \Vdash \Box\bot \text{ and } w \Vdash A \rhd \bot \\ w \text{ is nonterminal} &\Rightarrow \exists u(wRu \text{ and } u \text{ is terminal}) \text{ and } w \nVdash \Box\bot \\ &\Rightarrow u \Vdash A (\text{because } A \text{ is an affirmative formula}) \\ &\Rightarrow w \nVdash A \rhd \bot \end{aligned}$$

□

Proposition 3. *If N is a negative formula, and X is a cross formula, then $N \rhd X$ is valid.*

Proof. Let w be any world, and w' some R–successor on which N holds. That means w' is nonterminal, so by Corollary 1, there exist worlds u and v such that $w'Ru$ or $w' = u$, then uRv, and last, every S_u–successor of v is terminal. From the first property we have either $wRw'Ru$, so $w'S_wu$, or $wRw' = u$, so $w'S_wu$ by reflexivity, because they are the same. We must show $u \Vdash X$.

We know that X is of the form $\neg(A \rhd M)$, for some affirmative formula A and negative formula M. Suppose the contrary, that $u \Vdash A \rhd M$. Since uRv and v is terminal, therefore $v \Vdash A$, there would have to be a world v' such that vS_uv' and $v' \Vdash M$. But since all such v's are terminal, M cannot hold on any of them. Therefore, $u \Vdash X$, which combined with $w'S_wu$ gives the claim $w \Vdash N \rhd X$. Since w was arbitrary, $N \rhd X$ is valid. □

Lemma 3. *Any cross formula holds on every world of type 2.*

Proof. Let w be a world of type 2, and $X = \neg(A \rhd N)$ be a cross formula.

$$\begin{aligned} w \text{ is nonterminal} &\Rightarrow \exists u(wRu \text{ and } u \text{ is terminal}) \\ &\Rightarrow u \Vdash A \text{ (because } A \text{ is an affirmative formula)} \\ &\Rightarrow w \Vdash \neg(A \rhd N) \end{aligned}$$

If we suppose $w \Vdash A \rhd N$, there would have to be a world v such that uS_wv and $v \Vdash N$. However, from uS_wv we would have wRv, and so v would be terminal (since all R–successors of w are terminal). But then $v \Vdash N$ is a contradiction, since N cannot hold on terminal worlds. □

Proposition 4. *If X is a cross formula, then the formula $X \rhd \bot$ is locally equivalent to $\Box\Box\bot$.*

Proof.

$$\begin{aligned} &x \Vdash X \rhd \bot && \Leftrightarrow \text{By Lemma 3} \\ &w \text{ has no } R \text{ successors of type 2} && \Leftrightarrow \text{By Lemma 1} \\ &w \text{ is of type 1 or type 2} && \Leftrightarrow \\ &w \Vdash \Box\Box\bot \end{aligned}$$

□

Proposition 5. *Every negative formula is globally equivalent to* $\bot$.

Proof. Let N be a negative formula, and let w be any world.

$$\begin{aligned} w \text{ is terminal} &\Rightarrow w \nVdash N \text{ and } w \nVdash \bot \\ w \text{ is nonterminal} &\Rightarrow \exists u(wRu \text{ and } u \text{ is terminal}) \\ &\Rightarrow v \nVdash N \end{aligned}$$

□

Proposition 6. *The negation of any cross formula is globally equivalent to* $\Box\bot$.

Proof.

$$\begin{aligned} w \nVdash \Box\bot \Leftrightarrow\, & w \text{ is nonterminal} \\ \Rightarrow\, & w \text{ is of type 2 or } \exists v(wRv \text{ and } v \text{ is of type 2}) \text{ (by Lemma 1} \\ \Rightarrow\, & w \Vdash X \text{ or } v \Vdash X \end{aligned}$$

□

Example 1 (A calculation example)**.**

$$\begin{aligned} \big((A \rhd N) \to M\big) \rhd \bot \Longleftrightarrow & \big(\neg(A \rhd N) \vee M\big) \rhd \bot \\ \Longleftrightarrow & (X \vee M) \rhd \bot \Longleftrightarrow (X \rhd \bot) \wedge (M \rhd \bot) \Longleftrightarrow \Box\Box\bot \wedge \Box\neg M \\ \Longleftrightarrow & \Box(\Box\bot \wedge \neg M) \Longleftrightarrow \Box(\Box\bot \wedge B) \Longleftrightarrow \Box\Box\bot \end{aligned}$$

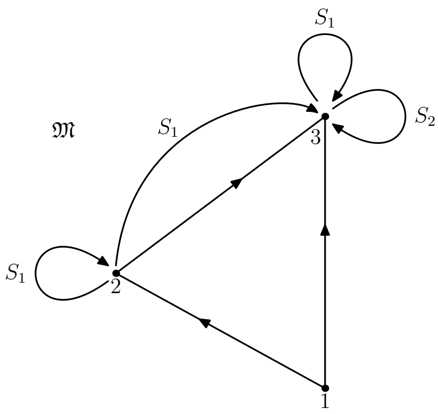
3. Non–eliminability of modal operator $\rhd$ in general setting

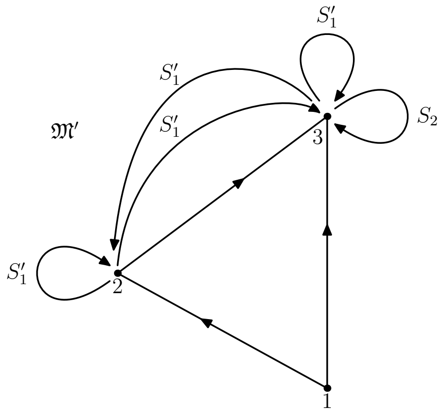
In this section, we give a formula from which it is impossible to eliminate $\rhd$. So, for the closed fragment of the interpretability logic IL there are no normal forms without using the modal operator $\rhd$. At the beginning we define two very simple Veltman frames.

$$\begin{array}{ll} W := \{1,2,3\} & W' := W \\ R := \{(1,2),\ (2,3),\ (1,3)\} & R' := R \\ S_1 := \{(2,2),\ (3,3),\ (2,3)\} & S_1' := \{(2,2),\ (3,3),\ (2,3),\ (3,2)\} \\ S_2 := \{(3,3)\} & S_2' := S_2 \\ S_3 := \emptyset & S_3' := S_3 \end{array}$$

So, we define two Veltman frames $\mathfrak{M}$ and $\mathfrak{M}'$, where $\mathfrak{M} = \langle W, R, (S_1, S_2, S_3)\rangle$ and $\mathfrak{M}' = \langle W', R', (S_1', S_2', S_3')\rangle$.

We illustrate the frames by the following pictures.





Let φ be the following formula

$$\varphi := \Diamond\Diamond\top \;\rightarrow\; (\top \rhd \Diamond\top)$$

Obviously we have $\mathfrak{M}, 1 \Vdash \Diamond\Diamond\top$, and $\mathfrak{M}, 1 \not\Vdash \top\rhd\Diamond\top$. So, $\mathfrak{M} \not\Vdash \varphi$. It is easy to see that $\mathfrak{M}' \Vdash \varphi$. Let us suppose that there is a GL–formula ψ such that $\psi \Longleftrightarrow \varphi$. We denote a Kripke frame (W, R) by $\mathfrak{N}$. If $\mathfrak{N} \Vdash \psi$, then obviously $\mathfrak{M} \Vdash \varphi$, a contradiction. If $\mathfrak{N} \not\Vdash \varphi$, then obviously $\mathfrak{M}' \not\Vdash \varphi$, and we get a contradiction again.

As Example 1 shows, we can simplify many other forms of IL–formulas, if we use axioms and local equivalences given in Theorem 3. One can consider those equivalences as “atoms” from which more complicated equivalences can be constructed, using the property that any subformula can be replaced by a locally equivalent one.

We think, although such a claim is still too vague to be strictly proved, that those equivalences listed in Theorem 3 are essentially the only cases where $\rhd$ as a main operator can be eliminated (and its operands are logically independent). However, we know that the modal operator $\rhd$ cannot always be eliminated, as shown in Section 3.

Acknowledgement

The authors would like to express their gratitude to anonymous referees for valuable comments and improvements of the article.

References

[1] S. N. Artëmov, *On modal logics axiomatizing provability*, Izv. Akad. Nauk SSSR Ser. Mat. **49**(1985), 1123–1154; English transl. in Math. SSSR–Izv. **27**(1985).

[2] L. D. Beklemishev, *Classification of Propositional Provability Logics*, Amer. Math. Ser. Transl. **192**(1999), 1–56.

[3] P. Blackburn, M. de Rijke, Y. Venema, *Modal Logic*, Cambridge University Press, Cambridge, 2001.

[4] G. Boolos, *On the nonexistence of certain normal forms in the logic of provability*, Jou. Sym. Logic **47**(1982), 638–640.

[5] G. Boolos, *The Logic of Provability*, Cambridge University Press, Cambridge, 1996.

[6] D. de Jongh, F. Veltman, *Provability Logics for Relative Interpretability*, In: *Mathematical Logic*, (P. P. Petkov, Ed.), Plenum Press, 1990, 31–42.

[7] D. de Jongh, G. Japaridze, *Provability logic*, in: *Handbook of proof theory*, (S. Buss, Ed.), North–Holland, 1998, 475–546.

[8] P. Hájek, V. Švejdar, *A Note on the Normal Form of Closed Formula of Interpretability Logic*, Studia Logica **50**(1991), 25–28.

[9] J. Joosten, *Closed Fragment of the Interpretability Logic of PRA with a Constant for* $I\Sigma_1$, Logic Group Preprint Series 218, Department of Philosophy, Utrecht University, 2003.

[10] M. Kracht, *Modal consequence relations, Handbook of Modal Logic*, Elsevier Science Publishers, New York, 2006.

[11] C. Smorynski, *Self-reference and modal logic*, Springer–Verlag, Berlin, 1985.

[12] A. Visser, *Interpretability logic*, In: *Mathematical Logic*, (P. P. Petkov, Ed.), Plenum Press, 1990, 175–210.

[13] A. Visser, *An overview of interpretability logic*, In: *Advances in modal logic, Vol. 1, Selected papers from the 1st international workshop (AiML'96)*, (K. Marcus, Ed.), CSLI Publications, 1998, 307–359.