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On a result related to transformations and summations of generalized hypergeometric series

Allen Richard Miller[1,†] and Richard Bruce Paris[2,*]

[1] *George Washington University, 1616 18th Street NW, No. 210, Washington, DC 20009-2525, USA*
[2] *University of Abertay Dundee, Dundee DD1 1HG, UK*

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Abstract. We deduce an explicit representation for the coefficients in a finite expansion of a certain class of generalized hypergeometric functions that contain multiple pairs of numeratorial and denominatorial parameters differing by positive integers. The expansion alluded to is given in terms of these coefficients and hypergeometric functions of lower order. Applications to Euler and Kummer-type transformations of a subclass of the generalized hypergeometric functions mentioned above together with an extension of the Karlsson-Minton summation formula are provided.

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1. Introduction

In [7], the authors have deduced transformation formulas of Euler and Kummer-type respectively for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$, where r pairs of numeratorial and denominatorial parameters differ by positive integers. In addition, in [5, 7], certain quadratic transformations for the former function as well as a generalization of the Karlsson-Minton summation theorem [3, 10] have been derived. All of the transformations mentioned above are extensions of previous results deduced in [4, 8, 6] and the latter extended summation formula [9] has been more efficiently derived in [7] in a simpler form.

In the sequel we denote sequences $a_1, \ldots, a_p$ simply by (a_p) and define the Pochhamer symbol, or shifted factorial, $(\alpha)_n$ for complex numbers α and integers n (positive, negative and zero) by

$$(\alpha)_n \equiv \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)},$$

where $\Gamma(\alpha)$ is the gamma function. Furthermore, we define products of Pochhammer symbols by

$$((a_p))_n \equiv (a_1)_n \ldots (a_p)_n,$$

[†]Passed away on 15 August 2010.
[*]Corresponding author. *Email addresses:* r.paris@abertay.ac.uk (R. B. Paris)

where when $p = 0$ the product is empty and reduces to unity. We shall adopt the notation $\left\{{n \atop k}\right\}$ for the Stirling numbers of the second kind as employed by Graham *et al.* [2, Section 6]. Recall that Stirling numbers of the second kind $\left\{{n \atop k}\right\}$ represent the number of ways to partition n objects into k nonempty subsets. Thus $\left\{{0 \atop 0}\right\} \equiv 1$ and $\left\{{n \atop 0}\right\} = 0$ when integer $n > 0$.

With this notation all of the results alluded to above are consequences of the following theorem whose proof is found in [7, Lemma 4]. This theorem enables an ${}_{r+s}F_{r+1}(x)$ hypergeometric function, where in the sequel $s = 1, 2$ and r pairs of numeratorial and denominatorial parameters differ by positive integers, to be expressed as a finite sum of ${}_sF_1(x)$ functions.

Theorem 1. *For a nonnegative integer s let (a_s) denote a parameter sequence containing s elements, where when $s = 0$ the sequence is empty. Let $(a_s + k)$ denote the sequence when k is added to each element of (a_s). Let $\mathcal{F}(x)$ denote the generalized hypergeometric function with r numeratorial and denominatorial parameters differing by positive integers (m_r), namely*

$$\mathcal{F}(x) \equiv {}_{r+s}F_{r+1}\left(\begin{matrix}(a_s), (f_r + m_r)\\ c, \quad (f_r)\end{matrix} \,\middle|\, x\right),$$

where convergence of the series representation for the latter occurs in an appropriate domain depending on the values of s and the elements of the parameter sequence (a_s). Then

$$\mathcal{F}(x) = \frac{1}{A_0} \sum_{k=0}^{m} x^k A_k \frac{((a_s))_k}{(c)_k} \, {}_sF_1\left(\begin{matrix}a_s + k\\ c + k\end{matrix} \,\middle|\, x\right),$$

where $m = m_1 + \cdots + m_r$, the coefficients A_k are defined by

$$A_k \equiv \sum_{j=k}^{m} \left\{ {j \atop k} \right\} \sigma_{m-j}, \tag{1}$$

and the σ_j $(0 \le j \le m)$ are generated by the relation

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j. \tag{2}$$

Although it is evident that

$$A_0 = (f_1)_{m_1} \cdots (f_r)_{m_r}, \qquad A_m = 1, \tag{3}$$

the coefficients A_k for $0 < k < m$ are otherwise defined implicitly by (1) and (2). It is the purpose of this brief communication to obtain in Section 2 an explicit representation (Theorem 2) for the coefficients A_k $(0 \le k \le m)$. Then in Section 3 we shall record a few of the salient results that are consequences of Theorems 1 and 2.

2. The coefficients A_k

We prove the following.

Theorem 2. *Suppose (f_r) is a nonempty sequence of complex numbers and (m_r) a sequence of positive integers such that $m \equiv m_1 + \cdots + m_r$. Suppose further that*

$$A_k \equiv \sum_{j=k}^{m} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \sigma_{m-j} \qquad (0 \le k \le m),$$

where the σ_j $(0 \le j \le m)$ are generated by the relation

$$(f_1 + x)_{m_1} \cdots (f_r + x)_{m_r} = \sum_{j=0}^{m} \sigma_{m-j} x^j .$$

Then

$$A_k = \frac{(-1)^k A_0}{k!} \, {}_{r+1}F_r \left(\begin{matrix} -k, \, (f_r + m_r) \\ (f_r) \end{matrix} \middle| 1 \right), \tag{4}$$

where $0 \le k \le m$ and A_0 is given by (3).

Proof. We define the monic polynomial $P(x)$ of degree m by

$$P(x) \equiv (f_1 + x)_{m_1} \ldots (f_r + x)_{m_r} = \sigma_m + \sigma_{m-1} x + \cdots + \sigma_1 x^{m-1} + \sigma_0 x^m,$$

where $\sigma_0 = 1$. It then follows that $\sigma_{m-j} = P^{(j)}(0)/j!$ $(0 \le j \le m)$ and consequently

$$A_k = \sum_{j=k}^{m} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \frac{P^{(j)}(0)}{j!}.$$

From [1, 24.1.1 (C)] and [1, 24.1.4 (C)], we have

$$\Delta^k P(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} P(x+j) = k! \sum_{j=k}^{m} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} \frac{P^{(j)}(x)}{j!},$$

where Δ denotes the forward difference operator used in numerical analysis. Hence

$$A_k = \frac{\Delta^k P(0)}{k!} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} P(j).$$

Now since

$$P(j) = (f_1 + j)_{m_1} \ldots (f_r + j)_{m_r}$$

and $(\alpha + j)_p = (\alpha)_p \, (\alpha + p)_j / (\alpha)_j$ for positive integers p, we obtain

$$A_k = \frac{(-1)^k}{k!} (f_1)_{m_1} \ldots (f_r)_{m_r} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(f_1 + m_1)_j}{(f_1)_j} \cdots \frac{(f_r + m_r)_j}{(f_r)_j},$$

or

$$\frac{A_k}{A_0} = \frac{(-1)^k}{k!} \sum_{j=0}^{k} \frac{(-k)_j}{j!} \frac{((f_r + m_r))_j}{((f_r))_j}$$

which evidently completes the proof. □

Corollary 1. *Let* (f_r) *and* (m_r) *be sequences as in Theorem 2. Then*

$$ {}_{r+1}F_r\left(\begin{matrix}-m, (f_r+m_r)\\ (f_r)\end{matrix}\,\middle|\,1\right) = \frac{(-1)^m\, m!}{(f_1)_{m_1}\cdots(f_r)_{m_r}}, \tag{5}$$

where $m = m_1 + \cdots + m_r$.

Proof. In (4) set $k = m$ and then use (3). □

We remark that Karlsson [3] proved (5) by employing other methods.

3. Transformation and summation formulas

Theorem 2 allows us to state in a somewhat simplified and more elegant form several results obtained in [7] by use of Theorem 1. Thus, for example, we have the following theorem that provides transformation formulas of Euler and Kummer-type respectively for the generalized hypergeometric functions ${}_{r+2}F_{r+1}(x)$ and ${}_{r+1}F_{r+1}(x)$, in which r pairs of numeratorial and denominatorial parameters differ by positive integers.

Theorem 3. *Let* (m_r) *be a nonempty sequence of positive integers such that* $m = m_1 + \cdots + m_r$. *Then if* $b \neq f_j$ $(1 \leq j \leq r)$, $(\lambda)_m \neq 0$, *where* $\lambda \equiv c - b - m$, *we have the transformation formulas*

$$ {}_{r+2}F_{r+1}\left(\begin{matrix}a, b, (f_r+m_r)\\ c, \quad (f_r)\end{matrix}\,\middle|\,x\right) = (1-x)^{-a}\,{}_{m+2}F_{m+1}\left(\begin{matrix}a, \lambda, (\xi_m+1)\\ c, \quad (\xi_m)\end{matrix}\,\middle|\,\frac{x}{x-1}\right),$$

where $|x| < 1$, $\mathrm{Re}\, x < \frac{1}{2}$, *and*

$$ {}_{r+1}F_{r+1}\left(\begin{matrix}b, (f_r+m_r)\\ c, \quad (f_r)\end{matrix}\,\middle|\,x\right) = e^x\,{}_{m+1}F_{m+1}\left(\begin{matrix}\lambda, (\xi_m+1)\\ c, \quad (\xi_m)\end{matrix}\,\middle|\,-x\right),$$

where $|x| < \infty$. *The* (ξ_m) *are the nonvanishing zeros of the associated parametric polynomial* $Q_m(t)$ *of degree* m *defined by*

$$Q_m(t) \equiv \sum_{k=0}^{m} A_k (b)_k (t)_k (\lambda - t)_{m-k},$$

where the A_k $(0 \leq k \leq m)$ *are given by (4).*

The Karlsson-Minton summation formula [3, 10], which is also a consequence of Theorem 1 as shown in [7], states that for $\mathrm{Re}\,(-a) > m - 1$

$$ {}_{r+1}F_r\left(\begin{matrix}a, b, (f_r+m_r)\\ b+1, \quad (f_r)\end{matrix}\,\middle|\,1\right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)}\,\frac{(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}}{(f_1)_{m_1}\cdots(f_r)_{m_r}}. \tag{6}$$

An elegant extension of this summation theorem may be obtained directly from Theorem 1 by setting $x = 1$ and $s = 2$ in the latter, employing the Gauss summation theorem for ${}_2F_1(1)$ and utilization of Theorem 2. Thus we obtain the following.

Theorem 4. *Suppose (m_r) is a sequence of positive integers such that $m = m_1 + \cdots + m_r$. Then, provided that $Re\,(c-a-b) > m$ we have*

$$
{}_{r+2}F_{r+1}\left(\begin{matrix} a, b, (f_r+m_r) \\ c, \quad (f_r) \end{matrix}\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} {}_{r+1}F_r\left(\begin{matrix} -k, (f_r+m_r) \\ (f_r) \end{matrix}\middle|\, 1\right) \frac{(a)_k (b)_k}{(1+a+b-c)_k\, k!}. \qquad (7)
$$

If in (7) we set $c = b$, the right-hand side of the latter vanishes and we have the following.

Corollary 2. *Suppose (m_r) is a sequence of positive integers such that $m = m_1 + \cdots + m_r$. Then*

$$
{}_{r+1}F_r\left(\begin{matrix} a, (f_r+m_r) \\ (f_r) \end{matrix}\middle|\, 1\right) = 0, \qquad Re\,(-a) > m. \qquad (8)
$$

This last result was also obtained by Karlsson [3] who used other methods.

If $c = b+1$, then (7) reduces to

$$
{}_{r+1}F_r\left(\begin{matrix} a, b, (f_r+m_r) \\ b+1, \quad (f_r) \end{matrix}\middle|\, 1\right) = \frac{\Gamma(1+b)\Gamma(1-a)}{\Gamma(1+b-a)} \sum_{k=0}^{m} {}_{r+1}F_r\left(\begin{matrix} -k, (f_r+m_r) \\ (f_r) \end{matrix}\middle|\, 1\right) \frac{(b)_k}{k!},
$$

where $\mathrm{Re}\,(-a) > m-1$. Then upon employing (6) and (8) we obtain the following.

Corollary 3. *Suppose (m_r) is a sequence of positive integers. Then*

$$
\sum_{k=0}^{\infty} {}_{r+1}F_r\left(\begin{matrix} -k, (f_r+m_r) \\ (f_r) \end{matrix}\middle|\, 1\right) \frac{(b)_k}{k!} = \frac{(f_1-b)_{m_1} \ldots (f_r-b)_{m_r}}{(f_1)_{m_1} \ldots (f_r)_{m_r}}.
$$

When $f_j = f$, $m_j = m$ $(1 \le j \le r)$, we find

$$
\sum_{k=0}^{\infty} {}_{r+1}F_r\left(\begin{matrix} -k, f+m, \ldots, f+m \\ f, \quad \ldots, \quad f \end{matrix}\middle|\, 1\right) \frac{(b)_k}{k!} = \left(\frac{(f-b)_m}{(f)_m}\right)^r,
$$

which is a generalization of the Vandermonde-Chu convolution theorem; see [11, pp. 30–31].

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