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On the degree of approximation of continuous functions by means of Fourier series*

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Abstract. We generalize some results on the degree of approximation of continuous functions by means of Fourier series, which were obtained by Chandra ([1, 2]) and Leindler ([4]). Some applications of the main results are given.

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1. Introduction

Let $f(x)$ be a $2\pi-$periodic continuous function. Denote by $S_n(f,x)$ the $n-$th partial sum of its Fourier series, $\omega(\delta) := \omega(f,\delta)$ the modulus of continuity of f. Let $A := (a_{nk})(k,n=0,1,\cdots)$ be a lower triangular infinite matrix of real numbers, that is, $a_{nk}=0$ for all $k>n$. The $A-$transform of $\{S_n(f,x)\}$ is given by

$$T_n(f) := T_n(f,x) := \sum_{k=0}^{n} a_{nk} S_k(f,x),\ n=0,1,\cdots.$$

The following theorems can be found in [1], [2]:
Theorem A. *Let (a_{nk}) satisfy the following conditions:*

$$a_{nk}\ge 0 \quad \text{and} \quad \sum_{k=0}^{n} a_{nk}=1, \tag{1}$$

$$a_{nk}\le a_{n,k+1},\ k=0,1,\cdots,n-1; n=0,1,\cdots. \tag{2}$$

Suppose $\omega(t)$ is such that

$$\int_u^{\pi} t^{-2}\omega(t)dt = O\left(H(u)\right), \quad (u\to 0+), \tag{3}$$

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where $H(u)\geq 0$ *and*

$$\int_0^t H(u)du = O\left(tH(t)\right), \quad (t\to 0+). \tag{4}$$

Then

$$\|T_n(f)-f\| = O\left(a_{nn}H(a_{nn})\right), \tag{5}$$

where $\|\cdot\|$ *denotes the supnorm.*
Theorem B. *Let (1), (2) and (3) hold. Then*

$$\|T_n(f)-f\| = O\Big(\omega(\pi/n)+a_{nn}H(\pi/n)\Big). \tag{6}$$

If, in addition, $\omega(t)$ *satisfies (4), then*

$$\|T_n(f)-f\| = O\Big(a_{nn}H(\pi/n)\Big). \tag{7}$$

Theorem C. *Let us assume that (1) and*

$$a_{nk}\geq a_{n,k+1},\ k=0,1,\cdots,n-1; n=0,1,\cdots \tag{8}$$

hold. Then

$$\|T_n(f)-f\| = O\Big(\omega(\pi/n)+\sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{r=0}^{k+1}a_{nr}\Big). \tag{9}$$

Theorem D. *Let (1), (3), (4) and (8) hold. Then*

$$\|T_n(f)-f\| = O\Big(a_{n0}H(a_{n0})\Big). \tag{10}$$

Recently, Leindler [4] has showed that the monotonic condition in (2) and (8) can be essentially relaxed. To state his results, we need some notions.

For a fixed n, $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ of nonnegative numbers tending to zero is called *rest bounded variation*, or briefly $\alpha_n\in RBVS$, if there is a constant $K(\alpha_n)$ only depending on α_n such that

$$\sum_{k=m}^{\infty}|\Delta a_{nk}| := \sum_{k=m}^{\infty}|a_{nk}-a_{n,k+1}|\leq K(\alpha_n)a_{nm} \tag{11}$$

for all natural numbers m.

For a fixed n, $\alpha_n = \{a_{nk}\}_{k=0}^{\infty}$ of nonnegative numbers tending to zero is called *Head bounded variation*, or briefly $\alpha_n\in HBVS$, if there is a constant $K(\alpha_n)$ only depending on α_n such that

$$\sum_{k=0}^{m-1}|\Delta a_{nk}|\leq K(\alpha_n)a_{n,m} \tag{12}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence α_n has only finite nonzero terms, and the last nonzero term is a_{nN}.

Leindler's main result in [4] can be read as follows:

Theorem E. *The statements of Theorem A, B, C and D hold with (12) in place of (2), and with (11) in place of (8), respectively; naturally maintaining all the other assumptions.*

It should be noted that in the previous theorems of Chandra and Leindler, a sequence of sequences $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ has appeared. Thus, it is natural to assume that $\{K(\alpha_n)\}_{n=0}^{\infty}$ is bounded, that is, there is an absolute constant K such that $0 \leq K(\alpha_n) \leq K$ for $n = 1, 2, \cdots$.

In the present paper, we further generalize Theorem E by establishing the following:

Theorem 1. *Let (1) hold. Suppose that $\omega(t)$ satisfies (3). Then*

$$\|T_n(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=0}^{n} |\Delta a_{nk}| H(\pi/n)\Big), \tag{13}$$

If, in addition, $\omega(t)$ satisfies (4), then

$$\|T_n(f) - f\| = O\left(\sum_{k=0}^{n} |\Delta a_{nk}| H\left(\sum_{k=0}^{n} |\Delta a_{nk}|\right)\right), \tag{14}$$

$$\|T_n(f) - f\| = O\left(\sum_{k=0}^{n} |\Delta a_{nk}| H\left(\pi/n\right)\right). \tag{15}$$

Theorem 2. *Let (1) hold. Then*

$$\|T_n(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k) \sum_{r=0}^{k+1} a_{nr} + \sum_{k=1}^{n} \omega(\pi/k) \sum_{r=k}^{n} |\Delta a_{nr}|\Big). \tag{16}$$

As an application of our results, we will show that Theorem 1 and Theorem 2 imply all the results of Theorem E, thus Theorem A-Theorem D. Also, we will give some generalizations of Theorem A–Theorem E by applying Theorem 1 and Theorem 2 to a more general class of sequences than $RBVS$.

2. Proofs of theorems

We need some Lemmas.

Lemma 1 (see [1]). *If (3)and (4) hold, then*

$$\int_0^{\pi/n} \omega(t)dt = O\big(n^{-2} H(\pi/n)\big). \tag{17}$$

Lemma 2 (see [1]). *If (3)and (4) hold, then*

$$\int_0^{r} t^{-1}\omega(t)dt = O\big(rH(r)\big), \quad r \to 0+\,. \tag{18}$$

Lemma 3. *For any lower triangular infinite matrix* (a_{nk}), $k, n = 0, 1, 2, \cdots$ *of nonnegative numbers, it holds uniformly in* $0 < t \le \pi$, *that*

$$\sum_{k=0}^{n} a_{nk} \sin\left(k + \frac{1}{2}\right) t = O\left(\sum_{r=0}^{\tau} a_{nr} + \frac{1}{t}\sum_{r=\tau}^{n} |\Delta a_{nr}|\right), \tag{19}$$

where τ *denotes the integer part of* π/t. *It also holds that*

$$\sum_{k=0}^{n} a_{nk} \sin\left(k + \frac{1}{2}\right) t = O\left(\frac{1}{t}\sum_{r=0}^{n} |\Delta a_{nr}|\right). \tag{20}$$

Proof. Since (a_{nk}) is a lower triangular infinite matrix, that is, $a_{nk} = 0$ for $k > n$, then

$$a_{nm} \le \sum_{k=m}^{n} |\Delta a_{nk}| \tag{21}$$

for $m = 0, 1, 2, \cdots, n$. It is elementary to deduce that for arbitrary $\lambda_n \ge 0$ and for $n \ge m \ge 0$,

$$\left|\sum_{k=m}^{n} \lambda_k \sin\left(k + \frac{1}{2}\right) t \sin\frac{t}{2}\right| \le \frac{1}{2}\left(\lambda_m + \sum_{k=m}^{n-1} |\Delta \lambda_k| + \lambda_n\right) \tag{22}$$

holds.

By (21) and (22), assuming that $n \ge \tau$, we have

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{nk} \sin\left(k + \frac{1}{2}\right) t\right| &\le \sum_{r=0}^{\tau} a_{nr} + \left|\sum_{r=\tau}^{n} a_{nk} \sin\left(k + \frac{1}{2}\right) t\right| \\ &\le \sum_{r=0}^{\tau} a_{nr} + O\left(\frac{1}{t}\left(a_{n\tau} + \sum_{r=\tau}^{n-1} |\Delta a_{nk}| + a_{nn}\right)\right) \\ &= O\left(\sum_{r=0}^{\tau} a_{nr} + \frac{1}{t}\sum_{r=\tau}^{n} |\Delta a_{nr}|\right),\end{aligned}$$

which completes (19).

Similarly, we have

$$\begin{aligned}\left|\sum_{k=0}^{n} a_{nk} \sin\left(k + \frac{1}{2}\right) t\right| &= O\left(\frac{1}{t}\left(a_{n0} + \sum_{r=0}^{n-1} |\Delta a_{nk}| + a_{nn}\right)\right) \\ &= O\left(\frac{1}{t}\sum_{r=0}^{n} |\Delta a_{nr}|\right),\end{aligned}$$

hence, (20) is finished. □

Proof of Theorem 1. We will follow the ideas of Chandra ([1,2]) and Leindler ([4]).

Write

$$\Phi_x(t) := \frac{1}{2}\left(f(x+t)+f(x-t)-2f(x)\right).$$

Then

$$T_n(f,x)-f(x) = \frac{2}{\pi}\int_0^\pi \left(\Phi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^n a_{nk}\sin\left(k+\frac{1}{2}\right)t\right)dt. \tag{23}$$

Proof of (13). By (23), we have

$$\|T_n(f)-f\| \le \frac{2}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right) := I_1+I_2. \tag{24}$$

By (1) and the inequality $|\sin t|\le t$, we have for $0\le t\le \pi/n$,

$$\left|\sum_{k=0}^n a_{nk}\sin\left(k+\frac{1}{2}\right)t\right| = O(nt).$$

Therefore,

$$I_1 = O(n)\int_0^{\pi/n}\omega(t)dt = O\big(\omega(\pi/n)\big). \tag{25}$$

By (20) and (3),

$$I_2 = O\left(\sum_{k=0}^n|\Delta a_{nk}|\right)\int_{\pi/n}^{\pi} t^{-2}\omega(t)dt = O\left(\sum_{k=0}^n|\Delta a_{nk}|H(\pi/n)\right). \tag{26}$$

We complete (13) by combining (24)-(26).

Proof of (14). By (23) again, and

$$\sum_{k=0}^n|\Delta a_{nk}| \le 2\sum_{k=0}^n a_{nk} = 2 < \pi,$$

we get

$$\|T_n(f)-f\| \le \frac{2}{\pi}\left(\int_0^{\sum_{k=0}^n|\Delta a_{nk}|}+\int_{\sum_{k=0}^n|\Delta a_{nk}|}^{\pi}\right) := J_1+J_2. \tag{27}$$

By (1), we have

$$\left|\sum_{k=0}^n a_{nk}\sin\left(k+\frac{1}{2}\right)t\right| \le 1.$$

Hence, by (18), we have

$$J_1 = O(1)\int_0^{\sum_{k=0}^{n}|\Delta a_{nk}|} t^{-1}\omega(t)dt = O\left(\sum_{k=0}^{n}|\Delta a_{nk}|H\left(\sum_{k=0}^{n}|\Delta a_{nk}|\right)\right). \quad (28)$$

By (20) and (3), we have

$$\begin{aligned} J_2 &= O\left(\sum_{k=0}^{n}|\Delta a_{nk}|\int_{\sum_{k=0}^{n}|\Delta a_{nk}|}^{\pi} t^{-2}\omega(t)dt\right) \\ &= O\left(\sum_{k=0}^{n}|\Delta a_{nk}|H\left(\sum_{k=0}^{n}|\Delta a_{nk}|\right)\right). \quad (29) \end{aligned}$$

We finish (14) by combining (27)-(29).

Proof of (15). Note that $a_{nk}=0$ for $k>n$, we deduce that

$$a_{nj} \le \sum_{k=j}^{n}|\Delta a_{nk}| \le \sum_{k=0}^{n}|\Delta a_{nk}|$$

for $j=0,1,\cdots,n$, which implies that

$$1=\sum_{j=0}^{n}a_{nj} \le (n+1)\sum_{k=0}^{n}|\Delta a_{nk}|,$$

or in other words,

$$\sum_{k=0}^{n}|\Delta a_{nk}| \ge \frac{1}{2n}.$$

Hence, by (17), we obtain that

$$I_1 = O\left(\frac{1}{n}H(\pi/n)\right) = O\left(\sum_{k=0}^{n}|\Delta a_{nk}|H(\pi/n)\right). \quad (30)$$

Altogether by (24), (26) and (30), (15) is proved. □

Proof of Theorem 2. By (19) and the monotonicity of $\omega(t)$, we deduce that (see (24) for I_2)

$$\begin{aligned} I_2 &= O(1)\int_{\pi/n}^{\pi} t^{-1}\omega(t)\left(\sum_{r=0}^{\tau}a_{nr}+\frac{1}{t}\sum_{r=\tau}^{n}|\Delta a_{nr}|\right)dt \\ &= O(1)\sum_{k=1}^{n-1}\int_{\pi/(k+1)}^{\pi/k} t^{-1}\omega(t)\left(\sum_{r=0}^{\tau}a_{nr}+\frac{1}{t}\sum_{r=\tau}^{n}|\Delta a_{nr}|\right)dt \\ &= O\left(\sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{r=0}^{k+1}a_{nr}+\sum_{k=1}^{n}\omega(\pi/k)\sum_{r=k}^{n}|\Delta a_{nr}|\right). \quad (31) \end{aligned}$$

Altogether by (24), (25) and (31), we obtain (16). □

3. Applications of Theorems

Application 1. We remark that Theorem 1 implies Theorem E, and thus Theorem A–Theorem D. In fact, if $\{a_{nk}\} \in HBVS$, then

$$\sum_{k=0}^{n} |\Delta a_{nk}| = \sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{nn} \leq (K(\alpha_n) + 1)\, a_{nn}.$$

Thus, (14), (13) and (15) imply (5), (6) and (7), respectively.

If $\{a_{nk}\} \in RBVS$, then

$$\begin{aligned}\sum_{k=0}^{n} |\Delta a_{nk}| &\leq \sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{nn} \\ &\leq 2\sum_{k=0}^{n-1} |\Delta a_{nk}| + a_{n0} \\ &\leq 2\,(K(\alpha_n) + 1)\, a_{n0}, \end{aligned} \tag{32}$$

hence, (14) implies (10). Also, we derive from (15) and (13) that

$$\|T_n(f) - f\| = O\left(a_{n0} H(\pi/n)\right),$$

and

$$\|T_n(f) - f\| = O\left(\omega(\pi/n) + a_{n0} H(\pi/n)\right),$$

which are new results not stated in Theorem A-Theorem E.

Finally, we prove that (16) implies (9) if $\{a_{nk}\} \in RBVS$. In fact, since $\{a_{nk}\} \in RBVS$, then, similarly to (32), we get

$$\sum_{r=k}^{n} |\Delta a_{nr}| \leq (2K(\alpha_n) + 1)\, a_{nk}. \tag{33}$$

By using the definition of $RBVS$, we have

$$a_{nk} \leq \sum_{r=j}^{k-1} |\Delta a_{nr}| + a_{nj} \leq (2K(\alpha_n) + 1)\, a_{nj}$$

for $j = [k/2] + 1, \cdots, k$, which implies that

$$a_{nk} = O\left(\frac{1}{k} \sum_{r=[k/2]+1}^{k} |a_{nr}|\right) = O\left(\frac{1}{k} \sum_{r=0}^{k+1} |a_{nr}|\right). \tag{34}$$

By (33) and (34), we have

$$\sum_{r=k}^{n} |\Delta a_{nr}| = O\left(\frac{1}{k} \sum_{r=0}^{k+1} |a_{nr}|\right),$$

which shows that (16) implies (9).

Application 2. We can apply theorems to some $A-$transform with $\{a_{nk}\}_{k=0}^{\infty}$ may have lacunary terms for $0 \leq k \leq n$, which is impossible for $\{a_{nk}\}_{k=0}^{\infty} \in HBVS$ or $\{a_{nk}\}_{k=0}^{\infty} \in RBVS$.

Application 3. Very recently, Leindler [5] has extended the definition of $RBVS$ to the so-called $\gamma RBVS$. In our case, we can state the definition of $\gamma RBVS$ as follows:

For a fixed n, let $\gamma_n := \{\gamma_{nk}\}_{k=0}^{\infty}$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ of real numbers has the property

$$\sum_{k=m}^{\infty} |\Delta a_{nk}| \leq K(\alpha_n)\gamma_{nm}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ a $\gamma RBVS$, briefly denoted by $\alpha_n \in \gamma RBVS$.

If $\gamma_n = \alpha_n$, then $\gamma RBVS \equiv RBVS$.

Similarly, we can introduce a new kind of sequences $\gamma HBVS$ as follows:

For a fixed n, let $\gamma_n := \{\gamma_{nk}\}_{k=0}^{\infty}$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ of real numbers has the property

$$\sum_{k=0}^{m-1} |\Delta a_{nk}| \leq K(\alpha_n)\gamma_{nm}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{nk}\}_{k=0}^{\infty}$ a $\gamma HBVS$, briefly denoted by $\alpha_n \in \gamma HBVS$.

By a discussion similar to Application 1, Theorem 1 and Theorem 2, we have the following generalizations of Theorem E:

Theorem 3. *Let (a_{nk}) satisfy (1). Suppose that $\omega(t)$ satisfies (3), then*

(i) If $\{a_{nk}\}_{k=0}^{\infty} \in \gamma HBVS$. Then

$$\|T_n(f) - f\| = O\Big(\omega(\pi/n) + \gamma_{nn} H(\pi/n)\Big).$$

If, in addition, $\omega(t)$ satisfies (4), then

$$\|T_n(f) - f\| = O\Big(\gamma_{nn} H(\gamma_n)\Big),$$

and

$$\|T_n(f) - f\| = O\Big(\gamma_{nn} H(\pi/n)\Big).$$

(ii) If $\{a_{nk}\}_{k=0}^{\infty} \in \gamma RBVS$, then

$$\|T_n(f) - f\| = O\Big(\omega(\pi/n) + \gamma_{n0} H(\pi/n)\Big).$$

If, in addition, $\omega(t)$ satisfies (4), then

$$\|T_n(f) - f\| = O\left(\gamma_{n0} H\left(\gamma_{n0}\right)\right),$$
$$\|T_n(f) - f\| = O\left(\gamma_{n0} H\left(\pi/n\right)\right).$$

Theorem 4. *If* (a_{nk}) *satisfies (1) and* $\{a_{nk}\}_{k=0}^{\infty} \in \gamma RBVS$*, then*

$$\|T_n(f) - f\| = O\Big(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k) \sum_{r=0}^{k+1} a_{nr} + \sum_{k=1}^{n} \omega(\pi/k)\gamma_{nk}\Big).$$

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References

[1] P. Chandra, *On the degree of approximation of a class of functions by means of Fourier series*, Acta Math. Hungar. **52**(1988), 199–205.
[2] P. Chandra, *A note on the degree of approximation of continuous functions*, Acta Math. Hungar. **62**(1993), 21–23.
[3] L. Leindler, *On the uniform convergence and boundedness of a certain class of sine series*, Anal. Math. **27**(2001), 279–285.
[4] L. Leindler, *On the degree of approximation of continuous functions*, Acta Math. Hungar. **104**(2004), 105–113.
[5] L. Leindler, *Embedding results regarding strong approximation of sine series*, Acta Sci. Math. (Szeged), **71**(2005), 91–103.