A note on a function associated with the statistical limit superior*

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Abstract. *We investigate the connectedness and Baire classification of a function associated with the statistical limit superior.*

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1. Introduction

The concept of statistical convergence was first introduced by Fast [1] and after the papers of Šălát [7] and Fridy [2] it has become one of the most active areas of research in classical analysis. The concept of statistical limit superior was introduced by Fridy and Orhan in [4]. In this paper we consider a mapping σ which corresponds to each real sequence, its statistical limit superior and examine the connectedness and Baire classification of the function. The motivation of such a study comes from [6] and [9].

2. Preliminaries

We consider the set s of all real sequences $a = (a_n)_{n=1}^{\infty}$ endowed with Ferchet metric $d(a, b)$ given by

$$d(a,b) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|a_k - b_k|}{1 + |a_k - b_k|},$$

where $a = (a_n)_{n=1}^{\infty}$ and $b = (b_n)_{n=1}^{\infty}$. It is known that the metric space (s, d) is complete and has the power of continuum .

We now recall the following definitions.

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Let $A \subset N$. Put $d_n(A) = \frac{1}{n}\sum_{k=1}^{n}\chi_A(k)$, where χ_A is the characteristic function of A. The numbers $\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$ and $\overline{d}(A) = \limsup_{n\to\infty} d_n(A)$ are called the lower and upper asymptotic densities of A, respectively. If $\underline{d}(A) = \overline{d}(A)$, then $d(A) = \overline{d}(A)$ is called the asymptotic density of A .

Definition 1 [see [1]]. *A sequence $(x_k)_{k=1}^{\infty}$ of real numbers is said to be statistically convergent to $\xi \in R$, denoted by $st - lim\ x = \xi$ if for any $\epsilon > 0$, we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{n \in N; |x_n - \xi| \geq \epsilon\}$.*

For a real number sequence $x = (x_n)_{n=1}^{\infty}$, let B_x denote the set

$$B_x = \{b \in R; d(\{k \in N; x_k > b\}) \neq 0\}.$$

Throughout the paper $d\{K\} \neq 0$ means either $d\{K\} > 0$ or the natural density of K does not exist.

Definition 2 [see [4]]. *For any $x = (x_n)_{n=1}^{\infty} \in s$ the statistical limit superior of x is defined by*

$$\begin{aligned} st - limsup\, x &= sup\ B_x, \qquad &&if\ B_x \neq \Phi, \\ &= -\infty, &&if\ B_x = \Phi. \end{aligned}$$

It has been shown in [4] that if $\beta = st - limsup\ x$ is finite, then for every $\epsilon > 0$,

$$d\{k \in N; x_k > \beta - \epsilon\} \neq 0 \text{ and } d\{k \in N; x_k > \beta + \epsilon\} = 0. \tag{1}$$

Conversely, if (1) holds for every $\epsilon > 0$, then $\beta = st - limsup\ x$.

We now introduce a mapping σ on the set s in the following way. If $a = (a_n)_{n=1}^{\infty} \in s$, then

$$\sigma(a) = st - limsup\ a.$$

Clearly σ is a mapping from s to $[-\infty, \infty]$. If x, y etc. are members of s, we shall represent them generally by $x = (x_n)_{n=1}^{\infty}$, $y = (y_n)_{n=1}^{\infty}$ etc. Also N denotes the set of positive integers and R denotes the set of real numbers.

It can be easily seen by taking appropriate examples that the mapping σ is surjective but not injective. Further, given any real number t the set $\{x \in s; \sigma(x) = t\}$ has cardinality c where c is the power of continuum. In the next two sections we examine the function σ from the viewpoint of connectedness and then continuity and Baire classification.

3. Connected properties of the mapping σ

In this section we study about the connected properties of the mapping $\sigma : s \to [-\infty, \infty]$. We show that σ is not a connected mapping. While going to prove this result we got an interesting allied result which we prove first.

Theorem 1. *Given any connected set A in $(-\infty, \infty)$, there is a connected set B in s such that $\sigma(B) = A$.*

Proof. We first assume that A is a bounded closed interval.

Let $A = [a, b]$, $-\infty < a < b < \infty$. We can always find two sequences $x, y \in s$ such that $st - lim\ x = a$ and $st - lim\ y = b$. Then $\sigma(x) = a$, $\sigma(y) = b$. Now for $t \in [0, 1]$, consider $z(t) = (z_n)_{n=1}^{\infty} \in s$ where

$$z_n = tx_n + (1 - t)y_n \quad \text{for all } n \in N.$$

Then $st - lim\ z = ta + (1 - t)b$ and $\sigma(z(t)) = ta + (1 - t)b$. Now we define a mapping $g : [0, 1] \to s$ by $g(t) = z(t)$. Since the mapping g is continuous and every point of $[a, b]$ can be expressed in the form $ta + (1 - t)b$ for some $t \in [0, 1]$, we have $g([0, 1]) = B$ (say) is connected in s with $\sigma(B) = [a, b]$.

If A is a bounded open or half open interval, then the modification is evident.

Now let $A = [a, \infty)$, $-\infty < a < \infty$, and let m be the least positive integer greater than a. Therefore we can write

$$A = [a, m] \cup [a, m + 1] \cup \dots .$$

As in the preceding case we can find a nonempty connected set B_1 (say) in s (corresponding to statistically convergent sequences only) such that $\sigma(B_1) = [a, m]$. Again taking a member $x \in B_1$ with $\sigma(x) = m$ and another y from s (corresponding to a statistically convergent sequence) with $\sigma(y) = m + 1$, we can construct a connected set B_2' such that $\sigma(B_2') = [m, m+1]$ and $x \in B_1 \cap B_2'$. Let $B_2 = B_1 \cup B_2'$. Then B_2 is connected and $\sigma(B_2) = [a, m + 1]$. Repeating this process we obtain a sequence of nonempty connected sets $(B_i)_{i=1}^{\infty}$, in s with $\sigma(B_i) = [a, m + (i - 1)]$ and $B_i \subset B_{i+1}$ for all $i \in N$. Then clearly $B = \cup_{i=1}^{\infty} B_i$, is connected in s and $\sigma(B) = [a, \infty) = A$. Similarly, we can prove the result if A is any other type of an unbounded interval. Hence the proof. □

From the next theorem it follows that σ is not a connected mapping. To prove the theorem we first prove the lemma.

Lemma 1. *For $a \in (0, \infty)$, let $B_a = \{x \in s$; there exists j (even) such that $x_{j+1} = a, x_n = (-1)^{n-1}a$ for all $n > j + 1$, except a set $A \subset \{j + 1, j + 2, ...\}$ of density zero $\}$. Then B_a is a connected set in s and $\sigma(B_a) = \{a\}$.*

Proof. It is clear that B_a is nonvoid, because, for example the sequence $\alpha = \{x_1, x_2, a, -a, a, -a, ...\}$ is a member of B_a, where x_1, x_2 are real numbers. Let $x = (x_n)_{n=1}^{\infty} \in B_a$. By definition there exists j (even) such that $x_{j+1} = a$ and $x_n = (-1)^{n-1}a$ for all $n \in \{j + 1, j + 2, ...\} \setminus A$, where A is a set of density zero. Since the set of odd integers has density $\frac{1}{2}$ and it remains the same even if we delete a set of density zero, so evidently $\sigma(x) = st - limsup\ x = a$. (However, it should be noted that the superior limit of x may be different from a). Let $t \in [0, 1]$. We construct a sequence $b(t) = (b_n)_{n=1}^{\infty}$ as follows,

$$\begin{aligned} b_n &= tx_n + (-1)^{n-1}a, && for\ n \le j \ \ or\ n \in A, \\ &= a, && for\ n = j + 1, \\ &= (-1)^{n-1}a, && for\ n > j + 1, n \notin A. \end{aligned}$$

Then $b(t) \in B_a$ for each $t \in [0, 1]$. If we consider the mapping $g_x : [0, 1] \to s$ defined by $g_x(t) = b(t)$, then g_x is continuous on $[0, 1]$ and so $g_x[0, 1]$ is a connected set in s. Also $g_x[0, 1] \subset B_a$. This inclusion holds for each $x \in B_a$. So $\bigcup_{x \in B_a} g_x[0, 1] \subset B_a$.

Again,

$$\begin{aligned} g_x(0) = b(0) &= (-1)^{n-1}a, && for\ n \leq j \ \ or\ n \in A, \\ &= a, && for\ n = j+1, \\ &= (-1)^{n-1}a, && for\ n > j+1, n \notin A. \end{aligned}$$

So for each $x \in B_a$, the above sequence is in $g_x[0,1]$. Hence $\bigcap_{x \in B_a} g_x[0,1] \neq \phi$. So $\bigcup_{x \in B_a} g_x[0,1]$ is connected and the proof will be complete if we can show that $B_a \subset \bigcup_{x \in B_a} g_x[0,1]$.

Let $\alpha \in B_a$. Then $\alpha = (\alpha_n)_{n=1}^{\infty}$ is of the form

$$\begin{aligned} \alpha_n &= x_n, && for\ n \leq j(even), \\ &= a, && for\ n = j+1, \\ &= (-1)^{n-1}a, && for\ n > j+1, n \notin B, \\ &= y_n && for\ n \in B, \end{aligned}$$

where B is a subset of $\{n \in N; n > j+1\}$ which has natural density zero and x_n, y_n are real numbers. We show that $\alpha \in g_z[0,1]$ for some $z \in B_a$. For this let $z = (z_n)_{n=1}^{\infty}$ be defined by

$$\begin{aligned} z_n &= x_n + (-1)^n a, && for\ n \leq j(even), \\ &= a, && for\ n = j+1, \\ &= (-1)^{n-1}a, && for\ n > j+1, n \notin B, \\ &= y_n + (-1)^n a, && for\ n \in B. \end{aligned}$$

Then $z \in B_a$ and $g_z(1) = \alpha$. So $\alpha \in g_z[0,1]$. Thus $B_a = \bigcup_{x \in B_a} g_x[0,1]$ and so B_a is connected and $\sigma(B_a) = \{a\}$. This proves the theorem. □

Theorem 2. *Let A be an arbitrary nonvoid subset of* $(0,\infty)$. *Then there exists a connected set $B \subset s$ such that $\sigma(B) = A$.*

Proof. Let B_a, $a \in A$ has the same meaning as in the preceding lemma. Let $B = \bigcup_{a \in A} B_a$. Then $\sigma(B) = A$. We only show that B is connected, for this we prove that no two of the sets $\{B_a, a \in A\}$ are separated. Let $a_1, a_2 \in A$, $a_1 \neq a_2$. Let $x = (x_n)_{n=1}^{\infty} \in B_{a_1}$ and let $\epsilon > 0$ be given. Now from the definition of B_{a_1}, there is a j (even) such that

$$\begin{aligned} x_{j+1} &= a_1, \\ x_n &= (-1)^{n-1}a_1, \qquad for\ n > j+1, n \notin P, \end{aligned}$$

where $P \subset \{j+1, j+2, ...\}$ is a set of density zero. Choose an even $i \in N$ such that $\sum_{k>i} \frac{1}{2^k} < \epsilon$. Now we construct $y = (y_n)_{n=1}^{\infty} \in s$ as follows

$$\begin{aligned} y_k &= x_k, && for\ k \leq i(even), \\ &= a_2, && for\ k = i+1, \\ &= (-1)^{k-i-1}a_2, && for\ k > i+1, k \notin Q, \end{aligned}$$

where $Q \subset \{i+1, i+2, ...\}$ is a set of density zero. Then $y \in B_{a_2}$ and we have $d(x,y) < \epsilon$. This shows that every ϵ-ball of x contains a member of B_{a_2}, which implies that $x \in \overline{B_{a_2}}$. Thus $B_{a_1} \subset \overline{B_{a_2}}$. Similarly, $B_{a_2} \subset \overline{B_{a_1}}$. This completes the proof of the theorem. □

Remark 1. *It should be noted that the connected sets B_a or B in the preceding results are much larger than the corresponding sets which were used to prove the same property for infinite series* [6].

4. Baire classification of σ

We first prove the following result.

Lemma 2. *Let $a = (a_n)_{n=1}^{\infty} \in s$ and $\delta > 0$. Then $\sigma(B(a, \delta)) = [-\infty, \infty]$, where $B(a, \delta) = \{x \in s; d(a, x) < \delta\}$.*

The proof is parallel to the proof of lemma 3 [6] and so is omitted.

Before going to the next theorem we state, as a convention that any ray line $x > a$, where a is a real number, is to be treated as a neighbourhood of $+\infty$. similarly for $-\infty$.

Theorem 3. *The function σ is discontinuous everywhere in s.*

Proof. Let $a \in s$ and suppose that $-\infty < \sigma(a) < \infty$. Then by *Lemma 2*, given $\epsilon > 0$ there is no $\delta > 0$ such that $\sigma(B(a, \delta)) \subset (\sigma(a) - \epsilon, \sigma(a) + \epsilon)$. So σ is discontinuous at a . If $\sigma(a) = +\infty$ $(or - \infty)$, a similar argument shows that σ is discontinuous at a. □

Since σ is discontinuous everywhere in s it cannot belong to zero or the first Baire class. Next we show that σ belongs to the second Baire class when considered on a restricted domain.

Let $I \subset N$ be a set with natural density zero. Let $s(I)$ be the class of all real sequences $(a_n)_{n=1}^{\infty}$ for which

$$st - \lim_{n\to\infty} sup\ a_n = \lim_{n\to\infty} sup\ a_{k_n}.$$

where $(a_{k_n})_{n=1}^{\infty}$ is the subsequence of $(a_n)_{n=1}^{\infty}$ corresponding to the index set $N \setminus I$. For the next result we consider the function σ defined on $s(I)$ endowed with the Frechet metric.

Theorem 4. *The function σ is of the second Baire class on $s(I)$.*

Proof. Let $a = (a_n)_{n=1}^{\infty} \in s(I)$. Then $\sigma(a) = \lim\limits_{n\to\infty} sup\ a_{k_n}$, where $(a_{k_n})_{n=1}^{\infty}$ is the subsequence of $(a_n)_{n=1}^{\infty}$ corresponding to the index set $N \setminus I$. Let

$$\sigma_m(a) = sup\{a_n; n \in (\{m, m+1, ...\} \setminus I)\}, \quad m = 1, 2, 3, \ldots$$

Then $\sigma(a) = \lim\limits_{m\to\infty} \sigma_m(a)$.

We put further

$$\sigma_{m,k}(a) = max\{a_n; n \in (\{m, m+1, \ldots, m+k\} \setminus I)\}, \quad m, k = 1, 2, 3, \ldots,$$
$$= 0, \qquad if \quad (\{m, m+1, \ldots, m+k\} \setminus I) = \phi.$$

Then

$$\sigma(a) = \lim_{m\to\infty} \lim_{k\to\infty} \sigma_{m,k}(a). \tag{2}$$

We shall now show that for fixed m, k the function $\sigma_{m,k} : s(I) \to R$ is continuous at a, and the theorem then follows from (2).

If $\{m, m+1, \dots\} \setminus I = \phi$, then there is nothing to prove. Otherwise let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $\frac{\delta}{1-\delta} = \epsilon$ and let $b = (b_n)_{n=1}^{\infty} \in B\big(a, \frac{\delta}{2^{m+k}}\big)$. Then $d(a,b) < \frac{\delta}{2^{m+k}}$,

$$\text{i.e. } \frac{|a_j - b_j|}{1 + |a_j - b_j|} < \delta \quad \text{for } j = 1, 2, \dots, m+k,$$

which implies that

$$|a_j - b_j| < \frac{\delta}{1-\delta} = \epsilon \quad \text{for } j = 1, 2, ..., m+k,$$

and obviously

$$|a_l - b_l| < \epsilon \quad \text{for } l \in \{m, m+1, \dots, m+k\} \setminus I.$$

If $\sigma_{m,k}(a) = a_i$, where $i \in \{m, m+1, ..., m+k\} \setminus I$ and $\sigma_{m,k}(b) = b_p$ where $p \in \{m, m+1, ..., m+k\} \setminus I$, then because of a_i, b_p being maximum values we obtain

$$\sigma_{m,k}(b) = b_p \geq b_i \geq a_i - \epsilon = \sigma_{m,k}(a) - \epsilon$$

and

$$\sigma_{m,k}(a) = a_i \geq a_p \geq b_p - \epsilon = \sigma_{m,k}(b) - \epsilon.$$

Hence we get $|\sigma_{m,k}(a) - \sigma_{m,k}(b)| < \epsilon$. Hence the proof. □

Remark 2. *From the above theorem it immediately follows that the usual limsup function is exactly a function of the second Baire class on s. However, whether the function σ also belongs to the second Baire class or not when considered on the whole space s is not clear. Therefore the Baire classification of the function σ remains open.*

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