

On an inequality of Grüss type

JOSIP PEČARIĆ* AND ŠIME UNGAR†

Abstract. *We prove an inequality of Grüss type for p -norm, which for $p = \infty$ gives an estimate similar to a result of Pachpatte [2].*

Key words: *Grüss' inequality, p -norm*

AMS subject classifications: 26D15, 26D10

Received June 25, 2006

Accepted August 25, 2006

1 Introduction

For differentiable functions $f, g: [a, b] \rightarrow \mathbb{R}$, $a \cdot b > 0$, Pachpatte has in [2] proved, using Pompeiu's mean value theorem [4], the following Grüss-like inequality:

$$\left| \int_a^b f(x)g(x) \, dx - \frac{1}{b^2-a^2} \left(\int_a^b f(x) \, dx \cdot \int_a^b xg(x) \, dx + \int_a^b g(x) \, dx \cdot \int_a^b xf(x) \, dx \right) \right| \leq \|f - \iota f'\|_\infty \int_a^b |g(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx + \|g - \iota g'\|_\infty \int_a^b |f(x)| \left| \frac{1}{2} - \frac{x}{a+b} \right| dx$$

where $\iota(t) = t$, $t \in [a, b]$.

We are going to prove an estimate for the left-hand side using the p -norm, $1 \leq p \leq \infty$, which will for $p = \infty$ give a result similar to the one in [2].

2 The main result

Before stating the main result, we will introduce some notation and state without proof an elementary lemma (see also [3] for some less obvious details of the proof):

Lemma 1. *For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, and $0 < a \leq x \leq b$, denote*

$$A(x, p) := \left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} \, du \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} \, du \right) dt \right)^{\frac{1}{q}}, \quad (1)$$

*Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 30, HR-10 000 Zagreb, Croatia, e-mail: pecaric@hazu.hr

†Department of Mathematics, University of Zagreb, Bijenička 30, HR-10 000 Zagreb, e-mail: ungar@math.hr

where for $p = 1$, i. e. $q = \infty$, the integrals are to be interpreted as the ∞ -norms, i. e. as maxima of the function $(u, t) \mapsto \frac{t}{u^2}$ on the corresponding domains of integration. Then

$$A(x, p) = \left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}}$$

for $1 < p, q < \infty, p, q \neq 2$;

$$A(x, 2) = \frac{1}{3} \left(\left(\ln \left(\frac{x}{a} \right) \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left(\ln \left(\frac{x}{b} \right) \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} = \lim_{p \rightarrow 2} A(x, p);$$

$$A(x, \infty) = \frac{a^2 + b^2}{2x} + x - (a + b) = \lim_{p \rightarrow \infty} A(x, p);$$

$$A(x, 1) = \frac{1}{a} + \frac{b}{x^2} = \lim_{p \rightarrow 1} A(x, p).$$

Now we state the main result

Theorem 2. Let the functions $f, g: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$, the following inequality holds:

$$\left| \int_a^b f(x)g(x) dx - \frac{1}{b^2 - a^2} \left(\int_a^b f(x) dx \cdot \int_a^b xg(x) dx + \int_a^b g(x) dx \cdot \int_a^b xf(x) dx \right) \right| \leq \frac{(b-a)^{\frac{1}{p}}}{b^2 - a^2} \left(\|f - \iota f'\|_p \int_a^b x |g(x)| A(x, p) dx + \|g - \iota g'\|_p \int_a^b x |f(x)| A(x, p) dx \right) \quad (2)$$

where $\iota(t) = t, t \in [a, b]$, and $A(x, p)$ is as in Lemma 1.

First we prove a lemma:

Lemma 3. Let the function $\varphi: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $a \cdot b > 0$. Then

$$t\varphi(x) - x\varphi(t) = xt \int_x^t (\varphi(u) - u\varphi'(u)) \frac{1}{u^2} du \quad (3)$$

for all $x, t \in [a, b]$.

Proof. Define $\Phi: [1/b, 1/a] \rightarrow \mathbb{R}$ by $\Phi(t) := t\varphi(\frac{1}{t})$. The function Φ is continuous and differentiable on $(1/b, 1/a)$, and for all $x_1, x_2 \in [1/b, 1/a]$ we have

$$\begin{aligned} \Phi(x_1) - \Phi(x_2) &= \int_{x_2}^{x_1} \Phi'(t) dt = \int_{x_2}^{x_1} \left(\varphi\left(\frac{1}{t}\right) - \frac{1}{t} \varphi'\left(\frac{1}{t}\right) \right) dt & \left[u := \frac{1}{t} \right] \\ &= - \int_{1/x_2}^{1/x_1} (\varphi(u) - u\varphi'(u)) \frac{1}{u^2} du. & (4) \end{aligned}$$

Denote $x_1 =: 1/x$ and $x_2 =: 1/t$. Then for all $x, t \in [a, b]$ from (4) we get

$$\frac{1}{x} \varphi(x) - \frac{1}{t} \varphi(t) = \int_x^t (\varphi(u) - u\varphi'(u)) \frac{1}{u^2} du$$

which gives (3) and proves the lemma. □

Proof of Theorem 2. By *Lemma 3* we have

$$\begin{aligned} t f(x) - x f(t) &= x t \int_x^t (f(u) - u f'(u)) \frac{1}{u^2} du \\ t g(x) - x g(t) &= x t \int_x^t (g(u) - u g'(u)) \frac{1}{u^2} du. \end{aligned}$$

Multiplying these identities by $g(x)$ and $f(x)$ respectively, and adding the results, gives

$$\begin{aligned} &2 t f(x) g(x) - x g(x) f(t) - x f(x) g(t) \\ &= x g(x) t \int_x^t (f(u) - u f'(u)) \frac{1}{u^2} du + x f(x) t \int_x^t (g(u) - u g'(u)) \frac{1}{u^2} du, \end{aligned}$$

which we integrate on t and obtain

$$\begin{aligned} &(b^2 - a^2) f(x) g(x) - x g(x) \int_a^b f(t) dt - x f(x) \int_a^b g(t) dt \\ &= x g(x) \int_a^b \left(\int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right) dt \\ &\quad + x f(x) \int_a^b \left(\int_x^t (g(u) - u g'(u)) \frac{t}{u^2} du \right) dt. \end{aligned}$$

Integrating the last identity on x and taking the absolute value, gives

$$\begin{aligned} &\left| (b^2 - a^2) \int_a^b f(x) g(x) dx - \int_a^b x g(x) dx \cdot \int_a^b f(t) dt - \int_a^b x f(x) dx \cdot \int_a^b g(t) dt \right| \\ &\leq \left| \int_a^b \left(x g(x) \int_a^b \left(\int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right) dt \right) dx \right| \\ &\quad + \left| \int_a^b \left(x f(x) \int_a^b \left(\int_x^t (g(u) - u g'(u)) \frac{t}{u^2} du \right) dt \right) dx \right|. \end{aligned} \quad (5)$$

For the first addend on the right-hand side of (5) we have

$$\begin{aligned} &\left| \int_a^b \left(x g(x) \int_a^b \left(\int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right) dt \right) dx \right| \\ &\leq \int_a^b \left(|x g(x)| \cdot \left| \int_a^b \left(\int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right) dt \right| \right) dx, \end{aligned} \quad (6)$$

and for the second factor under the integral on x (i. e. the outermost integral) in (6), we find

$$\begin{aligned} &\left| \int_a^b \left(\int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right) dt \right| \\ &\leq \int_a^b \left| \int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right| dt \\ &= \int_a^x \left| \int_t^x (f(u) - u f'(u)) \frac{t}{u^2} du \right| dt + \int_x^b \left| \int_x^t (f(u) - u f'(u)) \frac{t}{u^2} du \right| dt. \end{aligned} \quad (7)$$

Application of the Hölder's inequality shows that the last line in (7) is

$$\begin{aligned}
&\leq \left(\int_a^x \left(\int_t^x |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_x^b \left(\int_x^t |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
&\leq \left(\int_a^b \left(\int_a^b |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \right). \tag{8}
\end{aligned}$$

The first factor in (8) equals $(b-a)^{\frac{1}{p}} \|f - \iota f'\|_p$, and, by *Lemma 1*, the second factor equals $A(x, p)$. Plugging this into (6) shows that for the first addend on the right-hand side in (5) we have

$$\begin{aligned}
&\left| \int_a^b \left(xg(x) \int_a^b \left(\int_x^t (f(u) - uf'(u)) \frac{t}{u^2} du \right) dt \right) dx \right| \\
&\leq \int_a^b |xg(x)| \cdot (b-a)^{\frac{1}{p}} \|f - \iota f'\|_p \cdot A(x, p) dx \\
&= (b-a)^{\frac{1}{p}} \|f - \iota f'\|_p \cdot \int_a^b x |g(x)| A(x, p) dx. \tag{9}
\end{aligned}$$

An analogous inequality holds for the second addend in (5), so putting these two inequalities into (5) and dividing by $b^2 - a^2$ gives the required inequality (2), proving the theorem. \square

Inequality (2) has a particularly simple form in case $p = 1$, so we state it as a corollary:

Corollary 4. *Let the functions $f, g: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then the following inequality holds:*

$$\begin{aligned}
&\left| \int_a^b f(x)g(x) dx - \frac{1}{b^2 - a^2} \left(\int_a^b f(x) dx \cdot \int_a^b xg(x) dx + \int_a^b g(x) dx \cdot \int_a^b xf(x) dx \right) \right| \\
&\leq \frac{1}{a+b} \left(\|f - \iota f'\|_1 \int_a^b |g(x)| \left(\frac{x}{a} + \frac{b}{x} \right) dx + \|g - \iota g'\|_1 \int_a^b |f(x)| \left(\frac{x}{a} + \frac{b}{x} \right) dx \right)
\end{aligned}$$

where $\iota(t) = t$, $t \in [a, b]$.

References

- [1] G. GRÜSS, *Über das Maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x)g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx$* , Math. Z. **39**(1935), 215–226.
- [2] B. G. PACHPATTE, *On Grüss like integral inequalities via Pompeiu's mean value theorem*, Journal of Inequalities in Pure and Applied Mathematics **6**(1995), Article 82, 1–5.

- [3] J. PEČARIĆ, Š. UNGAR, *On an inequality of Ostrowski type*, Journal of Inequalities in Pure and Applied Mathematics, to appear.
- [4] D. POMPEIU, *Sur une proposition analogue au théorème des accroissements finis*, Mathematica (Cluj, Romania) **22**(1946), 143–146.