On the equivalence of Mann and Ishikawa iteration methods with errors

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Abstract. *We show that for several classes of mappings Mann and Ishikawa iteration procedures with errors in the sense of Xu [14] are equivalent. It is worth to mention here that, our results are the extensions or generalizations of some known recent results about equivalences.*

Key words: *Ishikawa and Mann iterations with errors, strongly pseudocontractive mappings, Lipschitzian mappings, Banach spaces*

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1. Introduction

Let X be a real Banach space and let X^* be its dual space. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = ||x|| \, ||f||, \; ||f|| = ||x||\},$$

where $\langle . \, , \, . \rangle$ denotes the generalized duality pairing. Further, let K be a nonempty subset of X and T a self-mapping of K. $F(T)$ and $D(T)$ are the set of fixed points and the domain of T, respectively.

Definition 1. *The mapping $T : K \to K$ is said to be strongly pseudocontractive if there exists $t > 1$ such that*

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\|, \tag{1.1}$$

for all $x, y \in K$ and $r > 0$. If $t = 1$ in (1.1), then T is called pseudocontractive.

Definition 2. *The mapping $T' : K \to K$ is said to be strictly hemicontractive if $F(T') \neq \phi$ and if there exists $t > 1$ such that*

$$\|x - q\| \leq \|(1 + r)(x - q) - rt(T'x - q)\|, \tag{1.2}$$

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for all $x \in K$, $q \in F(T')$ *and* $r > 0$.

Definition 3. *A mapping* $T'' : D(A) \to R(A)$ *in* X *is called accretive if the following inequality*

$$\|x - y\| \leq \|x - y + s(T''x - T''y)\|, \tag{1.3}$$

holds for each $x, \quad y \in D(T'')$, *and for all* $s \geq 0$. T'' *is pseudocontractive iff* $I - T''$ *is accretive, where* I *denotes the identity operator.*

Definition 4. *A mapping* $A : K \to K$ *is called strongly accretive if for each* $x, \quad y \in K$, *there exists* $j(x - y) \in J(x - y)$ *such that*

$$\langle Ax - Ay, \quad j(x - y) \rangle \geq k \|x - y\|^2, \tag{1.4}$$

for some constant $k > 0$. *Without loss of generality, we shall assume that* $k \in (0, 1)$.

Definition 5. *The mapping* $G : X \to X$ *is called Lipschitz if there exists a constant* $L > 0$ *such that*

$$||Gx - Gy|| \leq L||x - y||, \tag{1.5}$$

$\forall x, \ y \in D(G)$.

Let K be a nonempty convex subset of an arbitrary normed space X and let $T : K \to K$ be a selfmap of K.

In 1995, Liu [8] introduced iterative schemes with errors as follows:

Algorithm 1. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *in* K *iteratively defined by:*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_n T y_n + u_n, \\ y_n = (1 - b'_n)x_n + b'_n T x_n + v_n, \quad n \geq 0, \end{cases}$$

where $\{b_n\}, \{b'_n\}$ *are sequences in* $[0, 1]$ *and* $\{u_n\}$, $\{v_n\}$ *are sequences in* K *satisfying* $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$, *is known as an Ishikawa iterative scheme with errors.*

Algorithm 2. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *iteratively defined by:*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - b_n)x_n + b_n T x_n + u_n, \quad n \geq 0, \end{cases}$$

where $\{b_n\}$ *is a sequence in* $[0, 1]$ *and* $\{u_n\}$ *a sequence in* K *satisfying* $\sum_{n=1}^{\infty} \|u_n\| < \infty$, *is known as a Mann iterative scheme with errors.*

In 1998, Xu [14] devised a new iteration scheme to study the unique solution of the nonlinear strongly accretive operator equation $Tx = f$ and the convergence problem of the revised iterative sequences for strongly pseudocontractive mappings without the Lipschitz condition.

Algorithm 3. *For any given* $u_0 \in K$ *the sequence* $\{u_n\}_{n=0}^{\infty}$ *defined by*

$$\begin{aligned} u_{n+1} &= a'_n u_n + b'_n T v_n + c'_n s_n, \\ v_n &= a_n u_n + b_n T u_n + c_n t_n, \qquad n \geq 0, \end{aligned} \tag{XU-I}$$

where $\{s_n\}_{n=0}^{\infty}$ *and* $\{t_n\}_{n=0}^{\infty}$ *are arbitrary bounded sequences in* K *and* $\{a_n\}_{n=0}^{\infty}$*,* $\{b_n\}_{n=0}^{\infty}$*,* $\{c_n\}_{n=0}^{\infty}$*,* $\{a'_n\}_{n=0}^{\infty}$*,* $\{b'_n\}_{n=0}^{\infty}$ *and* $\{c'_n\}_{n=0}^{\infty}$ *are real sequences in* $[0,1]$ *such that* $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ *for all* $n \geq 0$ *is called the Ishikawa iterative sequence with errors in the sense of Xu [14].*

Algorithm 4. *The sequence* $\{x_n\}_{n=0}^{\infty}$ *now defined by*

$$\begin{aligned} &x_0 \in K, \\ &x_{n+1} = a'_n x_n + b'_n T x_n + c'_n w_n, \qquad n \geq 0, \end{aligned} \tag{XU-M}$$

is called the Mann iterative sequence with errors in the sense of Xu [14], where $\{w_n\}_{n=0}^{\infty}$ *is an arbitrary bounded sequence in* K*.*

It is clear that the Mann and Ishikawa iterative sequences [6, 9] are all special cases of the Ishikawa iterative sequences with errors in the sense of Xu [14].

In [11], Rhoades and Soltuz showed that for several classes of mappings Mann [9] and Ishikawa [6] iteration procedures are equivalent.

In [12], Soltuz showed that Mann-Ishikawa iterations and Mann-Ishikawa iterations with errors in the sense of Liu [8] are equivalent models for several classes of operators.

While it is clear that consideration of error terms in iterative schemes is an important part of the theory, it is also clear that the iterative schemes with errors introduced by Liu [8] are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms which say that they tend to zero as n tends to infinity are, therefore, unreasonable.

In this paper, we show that for several classes of mappings Mann (XU-M) and Ishikawa (XU-M) iteration procedures with errors in the sense of Xu [14] are equivalent.

2. Fundamentals

Lemma 1 [see[13]]. *Let* $\{\Phi_n\}_{n\geq 0}$ *be a nonnegative sequence that satisfies the inequality*

$$\Phi_{n+1} \leq (1-\delta_n)\Phi_n + \sigma_n, \qquad n \geq 0, \tag{2.1}$$

where $\delta_n \in [0,1]$ *for each* $n \in N$*,* $\sum\limits_{n\geq 0} \delta_n = \infty$ *and* $\sigma_n = 0(\delta_n)$*. Then* $\Phi_n \to 0$ *as* $n \to \infty$*.*

Lemma 2 [see [7]]. *Let* $x,\ y \in X$*. Then*

$$\|x\| \leq \|x + ry\|, \tag{2.2}$$

for every $r > 0$ *if and only if there is* $f \in J(x)$ *such that* $\operatorname{Re}\langle y,\ f\rangle \geq 0$*.*

As a consequence of *Lemma 2*, it follows from inequality (1.1) that T is strongly pseudocontractive iff

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq k\,\|x-y\|, \tag{2.3}$$

holds for all $x, y \in K$ and for some $j(x-y) \in J(x-y)$, where $k = \frac{t-1}{t} \in (0,1)$. Consequently, it follows easily from *Lemma 2* and inequality (2.3) that T is strongly pseudocontractive iff the following inequality holds:

$$\|x-y\| \leq \|x-y+s[(I-T-kI)x-(I-T-kI)y\|, \tag{2.4}$$

for all $x, y \in K$ and for all $s > 0$.

3. Main results

We are able now to prove the following results.

Theorem 1. *Let X be an arbitrary Banach space, K a nonempty closed convex subset of X and T a Lipschitzian selfmap of K with Lipschitz constant $L \leq 1$. Suppose that T has a fixed point $q \in F(T)$. Let $x_\circ = u_\circ \in K$ and define u_n and x_n by (XU-I) and (XU-M), with $\{s_n\}_{n=0}^\infty$, $\{t_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$ bounded sequences in K and $\{a_n\}_{n=0}^\infty$,$\{b_n\}_{n=0}^\infty$,$\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$ and $\{c'_n\}_{n=0}^\infty$ sequences in $[0,1]$ satisfying*

$$\begin{aligned} a_n + b_n + c_n &= a'_n + b'_n + c'_n = 1, \qquad n \geq 0, \\ \lim_{n\to\infty} c_n &= 0, \\ c'_n &= 0(b'_n), \\ \sum_{n\geq 0} b'_n &= \infty. \end{aligned}$$

Then the following are equivalent :

(1-a) the Mann iteration (XU-M) converges strongly to q,

(1-b) the Ishikawa iteration (XU-I) converges strongly to q.

Proof. That (1-b) implies (1-a) is obvious by setting $b_n = 0 = c_n$ in (XU-I). We prove that (1-a) implies (1-b). From $c'_n = 0(b'_n)$, we have $c'_n = \epsilon_n b'_n$, where $\epsilon_n \to 0$ as $n \to \infty$. From (XU-I) and (XU-M),

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|a'_n(x_n - u_n) + b'_n(Tx_n - Tv_n) + c'_n(w_n - s_n)\| \\ &\leq a'_n \|x_n - u_n\| + b'_n L \|x_n - v_n\| + c'_n(\|w_n - x_n\| + \|s_n - x_n\|). \end{aligned}$$

$$\begin{aligned} \|x_n - v_n\| &= \|a_n(x_n - u_n) + b_n(x_n - Tu_n) + c_n(x_n - t_n)\| \\ &\leq a_n \|x_n - u_n\| + b_n(\|x_n - Tx_n\| + \|Tx_n - Tu_n\|) + c_n \|t_n - x_n\| \\ &\leq (a_n + b_n L) \|x_n - u_n\| + b_n \|x_n - Tx_n\| + c_n \|t_n - x_n\| . \end{aligned}$$

Thus

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| \leq\ & [a'_n + b'_n L(a_n + b_n L)] \|x_n - u_n\| + b_n b'_n \|x_n - Tx_n\| \\ & + b'_n c_n L \|t_n - x_n\| + c'_n(\|w_n - x_n\| + \|s_n - x_n\|). \end{aligned}$$

Note that

$$\begin{aligned} a'_n + b'_n L(a_n + b_n L) &\leq 1 - c'_n - b'_n c_n L \\ &= 1 - b'_n(\epsilon_n + c_n L) \\ &\leq 1 - L b'_n. \end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \le (1-Lb'_n)\|x_n-u_n\| + b'_n[\|x_n - Tx_n\| \\ +c_nL\|t_n-x_n\| + \epsilon_n(\|w_n-x_n\|+\|s_n-x_n\|)].\end{aligned}$$

Since $x_n \to q$ and T is Lipschitzian, T is continuous. Therefore, $\lim_{n\to\infty} Tx_n = Tq = q$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. With

$$\begin{aligned}\Phi_n &= \|x_n-u_n\|, \\ \delta_n &= Lb'_n, \text{ and} \\ \sigma_n &= b'_n[\|x_n-Tx_n\| + +c_nL\|t_n-x_n\| + \epsilon_n(\|w_n-x_n\|+\|s_n-x_n\|)],\end{aligned}$$

for each $n \in N$, inequality (2.1) of *Lemma 1* is satisfied. Therefore

$$\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{3.1}$$

Since (1-a) is true, using (3.1),

$$\|u_n-q\| \le \|x_n-q\| + \|x_n-u_n\|,$$

which implies that $\lim_{n\to\infty}\|u_n-q\| = 0$. □

Theorem 2. *Let X be an arbitrary Banach space, K a nonempty closed convex subset of X and T a Lipschitzian strongly pseudocontractive selfmap of K. Suppose that T has a fixed point $q \in F(T)$. Let $x_\circ = u_\circ \in K$ and define u_n and x_n by (XU-I) and (XU-M), with $\{s_n\}_{n=0}^\infty$, $\{t_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$ bounded sequences in K and $\{a_n\}_{n=0}^\infty$,$\{b_n\}_{n=0}^\infty$,$\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$ and $\{c'_n\}_{n=0}^\infty$ are sequences in $[0,1]$ satisfying*

$$\begin{aligned}a_n+b_n+c_n &= a'_n+b'_n+c'_n = 1, \qquad n\ge 0, \\ \lim_{n\to\infty} c_n &= 0, \\ c'_n &= 0(b'_n), \\ \sum_{n\ge 0} b'_n &= \infty, \\ \lim_{n\to\infty} b_n &= 0 = \lim_{n\to\infty} b'_n.\end{aligned}$$

Then the following are equivalent:

(2-a) the Mann iteration (XU-M) converges strongly to q,

(2-b) the Ishikawa iteration (XU-I) converges strongly to q.

Proof. The existence of a fixed point q comes from [4, Corollary 1] which holds in an arbitrary Banach space. That (2-b) implies (2-a) is obvious by setting $b_n = 0 = c_n$ in (XU-I). Without loss of generality, we may assume that the Lipschitz constant L of T is greater than or equal to 1. If $L \in [0,1]$, then the result follows from *Theorem 1.* To prove that (2-a) implies (2-b), it is necessary to express $\|x_{n+1}-u_{n+1}\|$ in terms of (2.2). From $c'_n = 0(b'_n)$, we have $c'_n = \epsilon_n b'_n$, where

$\epsilon_n \to 0$ as $n \to \infty$. It is very clear that (XU-I) and (XU-M) are equivalent to the following

$$\begin{aligned} u_{n+1} &= (1-b'_n)u_n + b'_n T v_n + c'_n(s_n - u_n), \\ v_n &= (1-b_n)u_n + b_n T u_n + c_n(t_n - u_n), \quad n \geq 0, \end{aligned} \tag{3.2}$$

$$x_{n+1} = (1-b'_n)x_n + b'_n T x_n + c'_n(w_n - x_n), \quad n \geq 0. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} u_n = {} & (1+b'_n)u_{n+1} + b'_n(I - T - kI)u_{n+1} - (1-k)b'_n u_n \\ & + (2-k)b'^2_n(u_n - Tv_n) + b'_n(Tu_{n+1} - Tv_n) \\ & - [1 + (2-k)b'_n]c'_n(s_n - u_n), \end{aligned} \tag{3.4}$$

$$\begin{aligned} x_n = {} & (1+b'_n)x_{n+1} + b'_n(I - T - kI)x_{n+1} - (1-k)b'_n x_n \\ & + (2-k)b'^2_n(x_n - Tx_n) + b'_n(Tx_{n+1} - Tx_n) \\ & - [1 + (2-k)b'_n]c'_n(w_n - x_n), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|x_n - u_n\| = {} & \|(1+b'_n)(u_{n+1} - x_{n+1}) + b'_n[(I - T - kI)u_{n+1} \\ & -(I - T - kI)x_{n+1}] - (1-k)b'_n(u_n - x_n) \\ & +(2-k)b'^2_n(u_n - Tv_n - x_n + Tx_n) \\ & +b'_n(Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n) \\ & -[1 + (2-k)b'_n]c'_n(s_n - u_n - w_n + x_n)\|. \end{aligned}$$

Using the triangular inequality and (2.4),

$$\begin{aligned} \|x_n - u_n\| \geq {} & (1+b'_n)\|x_{n+1} - u_{n+1}\| - (1-k)b'_n\|x_n - u_n\| \\ & -(2-k)b'^2_n\|u_n - Tv_n - x_n + Tx_n\| \\ & -b'_n\|Tu_{n+1} - Tv_n - Tx_{n+1} + Tx_n\| \\ & -[1 + (2-k)b'_n]c'_n\|s_n - u_n - w_n + x_n\|. \end{aligned}$$

Solving the above inequality for $\|x_{n+1} - u_{n+1}\|$ gives

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| \leq {} & \frac{[1 + (1-k)b'_n]}{1 + b'_n}\|x_n - u_n\| \\ & + (2-k)b'^2_n\|u_n - Tv_n\| + (2-k)b'^2_n\|x_n - Tx_n\| \\ & + b'_n\|Tu_{n+1} - Tv_n\| + b'_n\|Tx_{n+1} - Tx_n\| \\ & + 3c'_n(\|s_n - x_n\| + \|x_n - u_n\| + \|w_n - x_n\|). \end{aligned} \tag{3.6}$$

Observe that

$$\|u_n - Tv_n\| \leq \|x_n - u_n\| + \|x_n - Tx_n\| + \|Tx_n - Tv_n\|. \tag{3.7}$$

$$\begin{aligned}\|Tx_n - Tv_n\| &\leq L\|x_n - v_n\| \\ &= L\|(1-b_n)(x_n - u_n) + b_n(x_n - Tu_n) + c_n(t_n - u_n)\| \\ &\leq L[(1-b_n)\|x_n - u_n\| + b_n\|x_n - Tx_n\| + b_nL\|x_n - u_n\| \\ &\quad + c_n\|t_n - x_n\| + c_n\|x_n - u_n\|] \\ &= L[(1-b_n+b_nL+c_n)\|x_n - u_n\| + b_n\|x_n - Tx_n\| \\ &\quad + c_n\|t_n - x_n\|].\end{aligned}$$

Note that, for $L \geq 1$, $1 - b_n + b_nL \leq L$. Therefore

$$\|Tx_n - Tv_n\| \leq L[(L+c_n)\|x_n - u_n\| + b_n\|x_n - Tx_n\| + c_n\|t_n - x_n\|]. \tag{3.8}$$

Substituting (3.8) in (3.7) gives

$$\begin{aligned}\|u_n - Tv_n\| \leq{} & [1 + L(1+L)]\|x_n - u_n\| + (1+L)\|x_n - Tx_n\| \\ & + Lc_n\|t_n - x_n\|.\end{aligned} \tag{3.9}$$

$$\begin{aligned}\|Tu_{n+1} - Tv_n\| &\leq L\|u_{n+1} - v_n\| \\ &= L\|(1-b_n')(u_n - v_n) + b_n'(Tv_n - v_n) + c_n'(s_n - u_n)\| \\ &\leq L[(1-b_n')\|u_n - v_n\| + b_n'\|Tv_n - v_n\| + c_n'(\|s_n - x_n\| \\ &\quad + \|x_n - u_n\|)].\end{aligned} \tag{3.10}$$

$$\begin{aligned}\|u_n - v_n\| &= \|b_n(u_n - Tu_n) - c_n(t_n - u_n)\| \\ &\leq b_n\|u_n - Tu_n\| + c_n\|t_n - u_n\| \\ &\leq [b_n(1+L) + c_n]\|x_n - u_n\| + b_n\|x_n - Tx_n\| \\ &\quad + c_n\|t_n - x_n\|.\end{aligned} \tag{3.11}$$

$$\begin{aligned}\|Tv_n - v_n\| &\leq \|Tv_n - Tx_n\| + \|x_n - Tx_n\| + \|x_n - v_n\| \\ &\leq (1+L)^2\|x_n - u_n\| + [(1+L)b_n + 1]\|x_n - Tx_n\| \\ &\quad + (1+L)c_n\|t_n - x_n\|.\end{aligned} \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we obtain

$$\begin{aligned}\|Tu_{n+1} - Tv_n\| \leq{} & [L(1-b_n')[(1+L)b_n + c_n] + L(1+L)^2b_n' \\ & + Lc_n']\|x_n - u_n\| \\ & + [Lb_n(1-b_n') + L[(1+L)b_n + 1]b_n']\|x_n - Tx_n\| \\ & + L(1+L)c_n\|t_n - x_n\| + Lc_n'\|s_n - x_n\|.\end{aligned} \tag{3.13}$$

Substituting (3.9) and (3.13) into (3.6) and using the fact $(1+b_n')^{-1} \leq 1 - b_n' + b_n'^2$, yields

$$\begin{aligned}\|x_{n+1} - u_{n+1}\| \leq{} & \alpha_n\|x_n - u_n\| + \beta_n\|x_n - Tx_n\| + Lb_n'\|x_{n+1} - x_n\| \\ & + b_n'[L[(2-k) + (1+L)]c_n\|t_n - x_n\| \\ & + (3+L)\epsilon_n\|s_n - x_n\| + 3\epsilon_n\|w_n - x_n\|],\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}\alpha_n &= [1+(1-k)b'_n](1-b'_n+b'^2_n)\\&\quad+(2-k)[1+L(1+L)]b'^2_n+b'_n[L(1-b'_n)[(1+L)b_n+c_n]\\&\quad+L(1+L)^2b'_n+Lc'_n]+3c'_n,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\beta_n &= (2-k)(1+L)b'^2_n+(2-k)b'^2_n\\&\quad+b'_n[Lb_n(1-b'_n)+L[(1+L)b_n+1]b'_n].\end{aligned} \tag{3.16}$$

Note that

$$\begin{aligned}[1+(1-k)b'_n](1-b'_n+b'^2_n) &= 1-kb'_n+kb'^2_n+(1-k)b'^3_n\\&\leq 1-kb'_n+kb'^2_n+(1-k)b'^2_n\\&= 1-kb'_n+b'^2_n.\end{aligned}$$

Therefore,

$$\alpha_n \leq 1-kb'_n+b'_n[Mb'_n+L(1+L)b_n+Lc_n+(3+L)\epsilon_n],$$

where

$$M = 1+(2-k)[1+L(1+L)]+L(1+L)^2.$$

So

$$\alpha_n \leq 1-kb'_n+Mb'_n(b'_n+b_n+c_n+\epsilon_n).$$

Since c'_n, b'_n, c_n and b_n satisfying conditions $c'_n = 0(b'_n)$ and $\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c_n$, there exists an integer N such that

$$M(b'_n+b_n+c_n+\epsilon_n) \leq k(1-k) \quad \text{for all } n \geq N.$$

Thus

$$\begin{aligned}\alpha_n &\leq 1-kb'_n+k(1-k)b'_n\\&= 1-k^2b'_n.\end{aligned} \tag{3.17}$$

Also

$$\begin{aligned}\beta_n &= b'_n[(2-k)(2+L)b'_n+Lb'_n(1+Lb_n)+Lb_n]\\&\leq [2(2+L)+L(1+L)+L]b'_n\\&= (2+L)^2b'_n.\end{aligned} \tag{3.18}$$

Thus

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq (1-k^2b'_n)\|x_n-u_n\|\\&\quad+(2+L)^2b'_n\|x_n-Tx_n\|\\&\quad+Lb'_n\|x_{n+1}-x_n\|\\&\quad+b'_n[L[(2-k)+(1+L)]c_n\|t_n-x_n\|\\&\quad+(3+L)\epsilon_n\|s_n-x_n\|+3\epsilon_n\|w_n-x_n\|].\end{aligned} \tag{3.19}$$

Since T is Lipschitzian, it is continuous. Therefore $x_n \to q$ implies that $\lim_{n\to\infty} Tx_n = Tq = q$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Also $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. With

$$\begin{aligned}\Phi_n &= \|x_n - u_n\|,\\ \delta_n &= k^2 b'_n, \text{ and}\\ \sigma_n &= b'_n[(2+L)^2 \|x_n - Tx_n\| + L\|x_{n+1} - x_n\| + L[(2-k) + (1+L)]c_n \|t_n - x_n\| \\ &\quad +(3+L)\epsilon_n \|s_n - x_n\| + 3\epsilon_n \|w_n - x_n\|],\end{aligned}$$

for each $n \in N$, inequality (2.1) of *Lemma 1* is satisfied. Therefore

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.20}$$

Since (2-a) is true, using (3.20),

$$\|u_n - q\| \le \|x_n - q\| + \|x_n - u_n\|,$$

which implies that $\lim_{n\to\infty} \|u_n - q\| = 0$. □

Remark 1. *All our results hold for multivalued operators provided that they admit appropriate single-valued selections.*

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