Construction of a kind of abundant semigroups

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Abstract. *In this paper, we discuss properties of medial idempotents on abundant semigroups, study quasi-adequate semigroups with a normal medial idempotent and some of their extreme cases, and give a description of structure of every type of such semigroups, respectively.*

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The abundant semigroup with medial idempotents has received a great deal of attention in past decades. As its important special case, the abundant semigroups with normal medial idempotents were considered in [1] and their constructions were also given there. El-Qallali has successfully uniformed and generalized relative results of [2] and [3]. In this paper, we study further properties of medial idempotents on abundant semigroups, apply the construction theory of such semigroups in paper [3] to quasi-adequate semigroups and some of their extreme cases, and also give more exact constructive description of every type of semigroup. First of all, we give several important properties of medial idempotents on abundant semigroups after making some preparation. Secondly, we describe construction of quasi-adequate semigroups with normal medial idempotents. Thirdly, we investigate several extreme cases, that is, the quasi-adequate semigroups with normal medial idempotents whose sets are left normal, right normal and rectangular band, respectively.

1. Medial idempotents

Here we adopt some definitions and notations of paper [3]. Let S be a semigroup, $\mathcal{L}^*$ and $\Re^*$ are Green's relations on S. $\mathcal{L}^*$ and $\Re^*$ can be defined equivalently as follows:

$$\mathcal{L}^* = \{(a, b) \in S \times S : \forall(x, y \in S^1)ax = ay \Leftrightarrow bx = by\}.$$

$$\Re^* = \{(a, b) \in S \times S : \forall(x, y \in S^1)xa = ya \Leftrightarrow xb = yb\}.$$

Following Fountain's [5], a semigroup whose each $\mathcal{L}^*$-class and each $\Re^*$-class contains idempotents is called an abundant semigroup. If the set of idempotents

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forms a band in S, then it is called a quasi-adequate semigroup. If the set of idempotents forms a semilattice in S, then it is called an adequate semigroup. An adequate semigroup S is called a type A semigroup if $eS \cap aS = ea, Se \cap Sa = Sae$, for any $e^2 = e, a \in S$.

Lemma 1.1(see [5]) *Let S be a semigroup, $e^2 = e, a \in S$. Then $\mathcal{L}^* a(e\Re^* a)$ if and only if $ae = a(ea = a)$ and*

$$\forall (x, y \in S^1) ax = ay(xa = ya) \Longrightarrow ex = ey(xe = ye).$$

Let S be an abundant semigroup with the set E of idempotents and set

$$f \in \mathcal{L}_a^* \cap E,\ e \in \Re_a^* \cap E.$$

Here, $\mathcal{L}_a^*$ is a $\mathcal{L}^*$-class of S and $\Re_a^*$ is a $\Re^*$-class of S.

From *Lemma 1.1* it is easy to see that $a = ea = af = eaf$. If S is an adequate semigroup, then each $\mathcal{L}^*$-class and each $\Re^*$-class of S contains a unique idempotent. In this case, the unique elements of $\mathcal{L}_a^*$ and $\Re_a^*$ are denoted by a^* and a^+ respectively. Evidently, we have that $a = a^+ a = aa^* = a^+ aa^*$.

Lemma 1.2(see [6]) *Let S be an adequate semigroup, then $(\forall a \in S, e \in E)$ $(ea)^+ = ea^+, (ae)^* = a^* e$.*

Definition 1.3. *An idempotent u of an abundant semigroup S is called medial if $x = xux$, for any $x \in \overline{E} =< E >$. A medial idempotent u is called normal if the subband $u\overline{E}u = \{uxu : x \in \overline{E}\}$ is a semilattice.*

Proposition 1.4. *If S possesses medial idempotents, then $\overline{E}$ must be the normal subsemigroup of S. Thus, $RegS$ is the subsemigroup of S.*

EL-Qallali [1] considered a kind of abundant semigroups in which $\overline{E}$ is a regular subsemigroup with a normal medial idempotent u such that uSu is a type A semigroup. It follows immediately from *Proposition 1.4* that it is not necessary to assume that $\overline{E}$ is a regular subsemigroup.

More conveniently, we have the following.

Definition 1.5. *A medial idempotent u of an abundant semigroup S is called strong normal if the subsemigroup uSu is a type A semigroup.*

From the definition and *Proposition 1.4*, we can see that it is abundant semigroups with strong normal medial idempotents that are the semigroups considered by EL-Qallali[1]. In the *Definition 1.5*, we need not suppose that u is normal. In the sequel, however, we will successively prove that strong normal must be normal, but normal medial idempotents cannot be strong normal. So these two concepts are identical only in the regular case. The following proposition, a generalization of a result in [2], is obvious.

Proposition 1.6. *Let S be an abundant semigroup with a medial idempotent u. Then*

$$(\forall x \in S, e \in \Re_x^* \cap E, f \in \mathcal{L}_x^* \cap E) x = eux = xuf = euxuf.$$

As proved by EL-Qallali [1], if S is a quasi-adequate semigroup with a strong normal medial idempotent u, then uS and Su are quasi-adequate subsemigroups.

In fact, we can get the same result if we suppose that S is an abundant semigroup with a medial idempotent instead of the above.

Proposition 1.7. *Let S be an abundant semigroup with a medial idempotent u, then uS, Su and uSu are quasi-adequate subsemigroups. Moreover,*

$$E(uS) = u\overline{E} = uE,\ E(Su) = \overline{E}u = Eu,\ E(uSu) = u\overline{E}u = uEu.$$

Proof. Clearly, $uE \subseteq u\overline{E} \subseteq E(uS)$ and $u\overline{E}$ is a subband of uS. Note that for any $e = ux \in E(uS)$, $e = ux = uux = ue \in uE$. It follows that $E(uS) \in uE$ and therefore $uE = u\overline{E} = E(uS)$.

To see that uS is an abundant semigroup, we assume that x is an element in uS, say, $x = uy$ for some $y \in S$. Since S is an abundant semigroup, there exist $e \in \mathfrak{R}^*_y \cap E$ and $f \in \mathcal{L}^*_y \cap E$. Notice that $\mathfrak{R}^*$ is a left congruence. We have $ue \in uE = E(uS)$ and $ue\mathfrak{R}^*x$. Therefore, each $\mathfrak{R}^*$-class of uS contains idempotents.

In order to see that $uf\mathcal{L}^*x$, we can easily get $xuf = x$ and for any $s, t \in (uS)^1$,

$$\begin{aligned} xs = xt &\Rightarrow uys = uyt \Rightarrow ueuys = ueuyt \Rightarrow eueuye = eueuyt \\ &\Rightarrow euys = euyt \Rightarrow ys = yt \Rightarrow fs = ft \Rightarrow ufs = uft. \end{aligned}$$

Therefore, by *Lemma 1.1*, each $\mathcal{L}^*$-class of uS or $uf\mathcal{L}^*x$ contains idempotents. Hence uS is a quasi-adequate subsemigroup. Other cases can be proved in a similar way. □

Corollary 1.8. *Any strong normal medial idempotent is a normal medial idempotents.*

By using the following example we point out that in general a normal medial idempotent is not a strong normal medial idempotent.

Example 1.9. *Let A be the infinite cyclic semigroup generated by a and B be the infinite cyclic monoid generated by $b, b^0 = e$. Set $S \cup A \cup B \cup \{1\}$ and define a binary operation as follows:*

$$a^m b^n = b^{m+n}, b^n a^m = a^{n+m}, m > 0, n \geq 0.$$

It is easy to see that S is a semigroup with the identity 1 and that $\mathcal{L}^*-$classes of S are $A \cup \{1\}$, B and $\mathfrak{R}^*$-classes of S are $\{1\}$, $A \cup B$, $E(S) = \{1, e\}$ is a semilattice. Therefore, S is an adequate semigroup and 1 is a normal medial idempotent. But $S = 1S1$ is not a type A semigroup, since $Se \cap Sa = B \cap A \neq B - \{e\} = Sae$. So 1 is not a strong normal medial idempotent.

Corollary 1.10. *Let S be a regular semigroup, then medial idempotent u is normal if and only if u is strong normal.*

2. Construction of a kind of abundant semigroups

The main goal of this section is to describe construction of quasi-adequate semigroups with normal medial idempotents. Let $\overline{E}$ be an idempotent-generated regular semigroup with a normal medial idempotent u and $\mathcal{L}$ and $\mathfrak{R}$ be the Green's-relations on $\overline{E}$. Assume that S is an adequate semigroup with the semilattice of idempotents $E^0 = u\overline{E}u$.

Lemma 2.1. *Let $x, y \in S, e, f, g, h \in \overline{E}$ and $e\mathcal{L}x^+, f\Re x^*, g\mathcal{L}y^+, h\Re y^*$. Then $e, g \in \overline{E}u, , f, h \in u\overline{E}$, moreover, $e(xfg)^+\mathcal{L}(xfgy)^+$, $(fgh)^*h\Re(xfgy)^*$.*

Let
$$W = W(\overline{E}, S) = \{(e, x, f) \in \overline{E}u \times S \times u\overline{E} : e\mathcal{L}x^+, f\Re x^*\}.$$

Then the expression $(e, x, f)(g, y, h) = (e(xfy)^+, xfgy, (fgy)^*h)$ defines a binary operation on W. Further, we have the following constructive theorem established in [3], which is a generalization of the union forms of the corresponding results in [2] and [1].

Theorem 2.2. *W is an abundant semigroup with a normal medial idempotent $\overline{u} = (u, u, u)$, and*

$$E(W) = \{(e, x, f) \in W : x \in E^0, fe = x\},$$
$$\overline{E(W)} = \{(e, x, f) \in W : x \in E^0\} \cong \overline{E}, \overline{u}\overline{W}\overline{u} \cong S.$$

Conversely, any abundant semigroup with a normal medial idempotent can be constructed in this way.

Since the result of *Lemma 2.1* is terser than the corresponding result of paper [3], *Theorem 2.2* improves and deepens the construction theory of abundant semigroups with normal medial idempotents in [3]. In order to describe the quasi-adequate semigroups with normal medial idempotents, we need following lammas.

Lemma 2.3. *Let S be an abundant semigroup with a normal medial idempotent u, then*

(1) S is a quasi-adequate semigroup if and only if u is a middle unit element.

(2) S is an adequate semigroup if and only if u is a unit element.

Lemma 2.4. *Suppose that E is a band which contains a normal medial idempotent u. Then E must be a normal band.*

Proof. It follows from *Lemma 2.3* that u is the middle unit element of E, $\forall e, x, y \in E$. Notice that uEu is a semilattice, so we have $exye = euxuuyue = euyuuxue = eyxe$. Therefore, E is a normal band.

Now, assume that $\overline{E} = E$ is a band with a normal medial idempotent u. It follows from *Lemma 2.4* that E is a normal band. □

Further, we have

Lemma 2.5. *Let $x, y \in S, f, g \in E, f\Re x^*, g\mathcal{L}y^+$, then*

(1) $xfgy = xy$,

(2) $(xfg)^+ = (xy)^+, (fgy)^* = (xy)^*$.

Proof.

(1) Noting that $fx^* = x^*, y^+g = y^+, f, g \in E^0$ and E is normal, we have

$$xfgy = xx^*fgx^*y = xy^+x^*gfx^*y = xy^+fgx^*y = xy^+x^*x = xx^*y^+y = xy.$$

(2) It can be easily proved that $xfgy^+\Re xfgy$, so

$$(xfg)^+ = (xfgy^+)^+ = (xfgy)^+ = (xy)^+.$$

$(fgy)^* = (xy)^*$ can be proved similarly.

□

Now, we define

$$Q = Q(E,S) = \{(e,x,f) \in Eu \times S \times uE : e\mathcal{L}x^+, f\Re x^*\}$$

and a binary operation

$$(e,x,f)(g,y,h) = (e(xy)^+, xy, (xy)^*h)$$

on it. Combining *Theorem 2.2* and *Lemma 2.5*, we obtain

Theorem 2.6. *Q is a quasi-adequate semigroup that contains a normal medial idempotent $\overline{u} = (u,u,u)$, and $E(Q) = \{(e,x,f) \in Q : x \in E^0\} \cong E, \overline{u}Q\overline{u} \cong S$. Conversely, any quasi-adequate semigroup with a normal medial idempotent can be constructed in this way.*

3. Extreme casee

We have seen above that a band with a normal medial idempotent must be a normal band. In general, it is neither a left normal band nor a right normal band.

Lemma 3.1. *Suppose that E is a band, $u \in E$, then*

(1) E is left normal, then u is a normal medial idempotent if and only if u is a right unit element.

(2) E is right normal, then u is a normal medial idempotent if and only if u is a left unit element.

Proof.

(1) If u is a normal medial idempotent, then

$$(\forall e \in E)eu = eeu = eue = e.$$

So, u is s a right unit element. If u is a left unit element, then u is obviously a medial idempotent. Because

$$(\forall e, f \in E)ueuufu = uufueu = ufuueu,$$

we can see that uEu is a semilattice, that is, u is a normal medial idempotent.

(2) It can be proved similarly. □

Now suppose that E is a left normal band and u is a right unit element. From $Eu = E, uE = uEu$, we know that for any $x \in S, f \in uE, f\Re x^*$ if and only if $f = x^*$.

Thereupon, we obtain

$$\begin{aligned} Q = Q(E,S) &= \{(e,x,f) \in E \times S \times uE : e\mathcal{L}x^+, f = x*\} \\ &= \{(e,x,x*) \in E \times S \times uE : e\mathcal{L}x^+\}. \end{aligned}$$

Consider the set

$$L = L(E,S) = \{(e,x) \in E \times S : e\mathcal{L}x^+\},$$

and define multiplication

$$(e,x)(g,y)=(e(xy)^+,xy).$$

Thus, we can obtain the following

Theorem 3.2. *$L=L(E,S)$ is a quasi-adequate semigroup, whose set of idempotents is a left normal band with a right unit element. Conversely, any such semigroup can be constructed in this way.*

Assume that E is a right normal band with a left unit element dually. Denote

$$R=R(S,E)=\{(x,f)\in S\times E: f\Re x^*\}.$$

Define multiplication

$$(x,f)(y,h)=(xy,(xy)^*h).$$

We have

Theorem 3.3. *$R=R(S,E)$ is a quasi-adequate semigroup, whose set of idempotents is a right normal band with a left unit element. Conversely, any such semigroup can be constructed in this way.*

Now, suppose that E is a rectangular band and u is an element of E. Then u is a normal medial idempotent. Furthermore, Eu is a left zero semigroup and uE is a right zero semigroup and $uEu=\{u\}$. In this case, S is an adequate semigroup containing the unique idempotent u. It can be easily proved that for any $x\in S, e\in Eu, f\in uE$, we have $x^+=x^*=u, e\mathcal{L}u\Re f$. So we get

$$Q=Q(E,S)=\{(e,x,f)\in Eu\times S\times uE: e\mathcal{L}u\Re f\}=Eu\times S\times uE.$$

The multiplication above is

$$(e,x,f)(g,y,h)=(e(xy)^+,xy,(xy)^*h)=(eu,xy,uf)=(eg,xy,fh).$$

Hence, we get the following conclusion:

Theorem 3.4. *The set of idempotents of an abundant semigroup is a rectangular band if and only if it is isomorphic to the direct product of a left zero semigroup, an adequate semigroup with a unique idempotent and a right zero semigroup.*

Especially, we have

Corollary 3.5. *A regular semigroup whose set of idempotents is a rectangular band must be a rectangular group.*

References

[1] A. El-Qallali, *On the construction of a class of regular semigroups*, Acta Math Hung. **56**(1990), 77-91.

[2] T. S. Blyth, R. McFadden, *On the construction of a class of regular semigroups*, J. Algebra **81**(1983), 1-22.

[3] F. Jing, *Abundant semigroups with a medial idempotent*, Semigroups Forum **51**(1995), 247-261.

[4] J. M. HOWIE, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[5] J. B. FOUNTAIN, *Adequate semigroups*, Proc. Edinburgh Math. Soc. **22**(1979), 113-125.

[6] J. B. FOUNTAIN, *Abundant semigroups*, Proc. London Math. Soc. **44**(1982), 103-129.