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Ternary codes associated with $O(3, 3^r)$ and power moments of Kloosterman sums with trace nonzero square arguments

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Abstract. In this paper, we construct two ternary linear codes $C(SO(3,q))$ and $C(O(3,q))$, respectively associated with the orthogonal groups $SO(3,q)$ and $O(3,q)$. Here q is a power of three. Then we obtain two recursive formulas for the power moments of Kloosterman sums with "trace nonzero square arguments" in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by utilizing the explicit expressions of Gauss sums for the orthogonal groups.

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Key words: power moment, Kloosterman sum, trace nonzero square argument, orthogonal group, Pless power moment identity, weight distribution, Gauss sum

1. Introduction

Let ψ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements (p a prime). Then the Kloosterman sum $K(\psi; a)$([12]) is defined by

$$K(\psi; a) = \sum_{\alpha \in \mathbb{F}_q^*} \psi(\alpha + a\alpha^{-1}) \ (a \in \mathbb{F}_q^*).$$

The Kloosterman sum was introduced in 1926 ([11]) to give an estimate for the Fourier coefficients of modular forms.

For each nonnegative integer h, by $MK(\psi)^h$ we will denote the h-th moment of the Kloosterman sum $K(\psi; a)$. Namely, it is given by

$$MK(\psi)^h = \sum_{\alpha \in \mathbb{F}_q^*} K(\psi; a)^h.$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK(\lambda)^h$ will be simply denoted by MK^h.

Explicit computations on power moments of Kloosterman sums were initiated in the paper [16] of Salié in 1931, where it is shown that for any odd prime q,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1} \ (h \geq 1).$$

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Here $M_0 = 0$, and, for $h \in \mathbb{Z}_{>0}$,

$$M_h = |\{(\alpha_1, \ldots, \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1}\}|.$$

For $q = p$ an odd prime, Salié obtained MK^1, MK^2, MK^3, MK^4 in [16] by determining M_1, M_2, M_3. On the other hand, MK^5 can be expressed in terms of the p-th eigenvalue for a weight 3 newform on $\Gamma_0(15)$ (cf. [13], [15]). MK^6 can be expressed in terms of the p-th eigenvalue for a weight 4 newform on $\Gamma_0(6)$ (cf.[4]). Also, based on numerical evidence, in [1] Evans was led to propose a conjecture which expresses MK^7 in terms of Hecke eigenvalues for a weight 3 newform on $\Gamma_0(525)$ with quartic nebentypus of conductor 105.

From now on, let us assume that $q = 3^r$. Recently, Moisio was able to find explicit expressions of MK^h, for $h \leq 10$ (cf. [14]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the ternary Melas code of length $q-1$, which were known by the work of Geer, Schoof and Vlugt in [3].

In order to describe our results, we introduce three incomplete power moments of Kloosterman sums. For every nonnegative integer h, ψ as before and with $tr : \mathbb{F}_q \to \mathbb{F}_3$ the trace function, we define

$$T_0SK(\psi)^h = \sum_{a \in \mathbb{F}_q^*,\ tra=0} K(\psi; a^2)^h,\ T_{12}SK(\psi)^h = \sum_{a \in \mathbb{F}_q^*,\ tra \neq 0} K(\psi; a^2)^h, \qquad (1)$$

which will be respectively called the h-th moment of Kloosterman sums with "trace zero square arguments" and those with "trace nonzero square arguments." Then, clearly we have

$$2SK(\psi)^h = T_0SK(\psi)^h + T_{12}SK(\psi)^h, \qquad (2)$$

where

$$SK(\psi)^h = \sum_{a \in \mathbb{F}_q^*,\ a\ square} K(\psi; a)^h, \qquad (3)$$

which is called the h-th moment of Kloosterman sums with "square arguments." If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $SK(\lambda)^h$, $T_0SK(\lambda)^h$, and $T_{12}SK(\lambda)^h$ will be respectively denoted by SK^h, T_0SK^h and $T_{12}SK^h$, for brevity.

In [8], for both n, q powers of two, a binary linear code $C(SL(n,q))$ associated with the finite special linear group $SL(n,q)$ was constructed in order to produce a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in that code. On the other hand, in [9], two infinite families of ternary linear codes associated with double cosets in the symplectic group $Sp(2n,q)$ were constructed in order to generate infinite families of recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in those codes.

In this paper, we will show the main Theorem 1 giving recursive formulas for the power moments of Kloosterman sums with "trace nonzero square arguments." To do that, we construct ternary linear codes $C(SO(3,q))$ and $C(O(3,q))$, respectively

associated with the orthogonal groups $SO(3,q)$ and $O(3,q)$, and express those power moments in terms of the frequencies of weights in the codes. Then, thanks to our previous results on the explicit expressions of "Gauss sums" for the orthogonal group $O(2n+1,q)$ [6], we can express the weight of each codeword in the duals of the codes in terms of Kloosterman sums. Then our formulas will follow immediately from the Pless power moment identity.

Henceforth, we agree that, for nonnegative integers a, b, c,

$$\binom{c}{a,b} = \frac{c!}{a!\ b!\ (c-a-b)!},\ if\ a+b \le c, \tag{4}$$

and

$$\binom{c}{a,b} = 0,\ if\ a+b > c. \tag{5}$$

We observe that, in addition to orthogonal groups $SO(3,q)$ and $O(3,q)$, the symplectic group $Sp(2,q)$ is used in the following theorem.

Theorem 1. *Let $q = 3^r$. Then we have the following.*

(a) For $h = 1, 2, 3, \ldots$,

$$\begin{aligned}
&((-1)^{h+1} + 2^{-h})T_{12}SK^h \\
&\quad = -\sum_{j=1}^{h-1}((-1)^{j+1} + 2^{-j})\binom{h}{j}(q^2-1)^{h-j}T_{12}SK^j \\
&\qquad + q^{1-h}\sum_{j=0}^{min\{N_1,h\}}(-1)^j(C_{1,j} - \hat{C}_j)\sum_{t=j}^{h} t!S(h,t)3^{h-t}2^{t-h-j}\binom{N_1-j}{N_1-t},
\end{aligned} \tag{6}$$

where $N_1 = |SO(3,q)| = q(q^2-1)$, and $\{C_{1,j}\}_{j=0}^{N_1}$ and $\{\hat{C}_j\}_{j=0}^{N_1}$ are the weight distributions of $C(SO(3,q))$ and $C(Sp(2,q))$ respectively given by: for $j = 0, \ldots, N_1$,

$$\begin{aligned}
C_{1,j} = &\sum \binom{q^2}{\nu_0, \mu_0}\binom{q^2}{\nu_2, \mu_2} \\
&\times \prod_{\beta^2-2\beta\neq 0\ square}\binom{q^2+q}{\nu_\beta, \mu_\beta}\prod_{\beta^2-2\beta\ nonsquare}\binom{q^2-q}{\nu_\beta, \mu_\beta},
\end{aligned} \tag{7}$$

$$\begin{aligned}
\hat{C}_j = &\sum \binom{q^2}{\nu_1, \mu_1}\binom{q^2}{\nu_{-1}, \mu_{-1}} \\
&\times \prod_{\beta^2-1\neq 0\ square}\binom{q^2+q}{\nu_\beta, \mu_\beta}\prod_{\beta^2-1\ nonsquare}\binom{q^2-q}{\nu_\beta, \mu_\beta}.
\end{aligned} \tag{8}$$

Here the first sum in (6) is 0 if $h = 1$ and the unspecified sums in (7) and (8) run over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ and $\{\mu_\beta\}_{\beta\in\mathbb{F}_q}$ satisfying

$$\sum_{\beta\in\mathbb{F}_q}\nu_\beta + \sum_{\beta\in\mathbb{F}_q}\mu_\beta = j,\ and\ \sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = \sum_{\beta\in\mathbb{F}_q}\mu_\beta\beta.$$

In addition, $S(h,t)$ is the Stirling number of the second kind defined by

$$S(h,t) = \frac{1}{t!}\sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j}j^h. \tag{9}$$

(*b*) *For $h = 1, 2, 3, \ldots,$*

$$\begin{aligned}
&((-1)^{h+1} + 2^{-h})T_{12}SK^h \\
&\quad = -\sum_{j=1}^{h-1}((-1)^{j+1} + s^{-j})\binom{h}{j}(q^2-1)^{h-j}T_{12}SK^j \\
&\qquad + q^{1-h}\sum_{j=0}^{min\{N_2,h\}}(-1)^j C_{2,j}\sum_{t=j}^{h} t!S(h,t)3^{h-t}2^{t-2h-j}\binom{N_2-j}{N_2-t} \\
&\qquad - q^{1-h}\sum_{j=0}^{min\{N_1,h\}}(-1)^j \hat{C}_j\sum_{t=j}^{h} t!S(h,t)3^{h-t}2^{t-h-j}\binom{N_1-j}{N_1-t},
\end{aligned} \tag{10}$$

where $N_2 = |O(3,q)| = 2q(q^2-1)$, and $\{C_{2,j}\}_{j=0}^{N_1}$ is the weight distribution of $C(O(3,q))$ given by: for $j = 0, \ldots, N_2$,

$$\begin{aligned}
C_{2,j} &= \sum\prod_{\beta\in\mathbb{F}_q}\binom{n_2(\beta)}{\nu_\beta, \mu_\beta}\ (j = 0, \ldots, N_2), \\
&\text{with } n_2(\beta) = 2q^2 - 2q + q\delta(1,q;\beta-1) + q\delta(1,q;\beta+1).
\end{aligned} \tag{11}$$

Here the first sum in (10) is 0 if $h = 1$, the unspecified sum in (11) runs over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ and $\{\mu_\beta\}_{\beta\in\mathbb{F}_q}$ satisfying

$$\sum_{\beta\in\mathbb{F}_q}\nu_\beta + \sum_{\beta\in\mathbb{F}_q}\mu_\beta = j, \quad \text{and} \quad \sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = \sum_{\beta\in\mathbb{F}_q}\mu_\beta\beta,$$

$S(h,t)$ indicates the Stirling number of the second as in (9), $\hat{C}_j$'s are as in (8), and

$$\begin{aligned}
\delta(1,q;\beta) &= |\{x\in\mathbb{F}_q | x^2 - \beta x + 1 = 0\}| \\
&= \begin{cases} 2, & \text{if } \beta^2-1 \neq 0 \text{ is a square,} \\ 1, & \text{if } \beta^2 - 1 = 0, \\ 0, & \text{if } \beta^2 - 1 \text{ is a nonsquare.} \end{cases}
\end{aligned} \tag{12}$$

The above result follows from Theorems 5.3 and 5.4 and the argument after Theorem 5.4.

2. $O(2n+1,q)$

Here we will go over some elementary facts about the orthogonal groups $O(2n+1,q)$ and $SO(2n+1,q)$, and the symplectic group $Sp(2n,q)$. For more details about the

results of this section, one is referred to the paper [6]. Especially, one can find an elementary proof, at the level of linear algebra, of the Bruhat decomposition in Theorem 3.1 of [7]. Throughout this paper, the following notations will be used:

$$\begin{aligned} q &= 3^r \ (r \in \mathbb{Z}_{>0}), \\ \mathbb{F}_q &= \textit{ the finite field with } q \textit{ elements}, \\ TrA &= \textit{ the trace of } A \textit{ for a square matrix } A, \\ {}^tB &= \textit{ the transpose of } B \textit{ for any matrix } B. \end{aligned}$$

The orthogonal group $O(2n+1,q)$ is defined as:

$$O(2n+1,q) = \{w \in GL(2n+1,q) | {}^twJw = J\},$$

where

$$J = \begin{bmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It consists of the matrices

$$\begin{bmatrix} A & B & e \\ C & D & f \\ g & h & i \end{bmatrix} (A,B,C,D\ n\times n, e,f\ n\times 1, g,h\ 1\times n, i\ 1\times 1)$$

in $GL(2n+1,q)$ satisfying the relations:

$$\begin{aligned} &{}^tAC + {}^tCA + {}^tgg = 0,\ {}^tBD + {}^tDB + {}^thh = 0, \\ &{}^tAD + {}^tCB + {}^tgh = 1_n,\ {}^tef + {}^tfe + i^2 = 1, \\ &{}^tAf + {}^tCe + {}^tgi = 0,\ {}^tBf + {}^tDe + {}^thi = 0. \end{aligned}$$

Let $P(2n+1,q)$ be the maximal parabolic subgroup of $O(2n+1,q)$ given by

$$\begin{aligned} P &= P(2n+1,q) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^th \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{c} A \in GL(n,q),\ i = \pm 1 \\ B + {}^tB + {}^thh = 0 \end{array} \right\}, \end{aligned}$$

and let $Q = Q(2n+1,q)$ be the subgroup of $P(2n+1,q)$ of index 2 defined by

$$\begin{aligned} Q &= Q(2n+1,q) \\ &= \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1_n & B & -{}^th \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{c} A \in GL(n,q) \\ B + {}^tB + {}^thh = 0 \end{array} \right\}. \end{aligned}$$

Then we see that

$$P(2n+1,q) = Q(2n+1,q) \amalg \rho Q(2n+1,q),$$

with

$$\rho = \begin{bmatrix} 1_n & 0 & 0 \\ 0 & 1_n & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let σ_r denote the following matrix in $O(2n+1,q)$

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (0 \le r \le n).$$

Then the Bruhat decomposition of $O(2n+1,q)$ with respect to $P = P(2n+1,q)$ is given by

$$O(2n+1,q) = \coprod_{r=0}^{n} P\sigma_r P = \coprod_{r=0}^{n} P\sigma_r Q,$$

which can further be modified as

$$\begin{aligned} O(2n+1,q) &= \coprod_{r=0}^{n} P\sigma_r(B_r \setminus Q) \\ &= \coprod_{r=0}^{n} Q\sigma_r(B_r \setminus Q) \amalg \coprod_{r=0}^{n} \rho Q\sigma_r(B_r \setminus Q), \end{aligned} \tag{13}$$

with

$$B_r = B_r(q) = \{w \in Q(2n+1,q) | \sigma_r w \sigma_r^{-1} \in P(2n+1,q)\}.$$

The special orthogonal group $SO(2n+1,q)$ is defined as

$$SO(2n+1,q) = \{w \in O(2n+1,q) | \det w = 1\}.$$

Then we see from (13) that

$$SO(2n+1,q) = \coprod_{0 \le r \le n,\ r\ even} Q\sigma_r(B_r \setminus Q) \amalg \coprod_{0 \le r \le n,\ r\ odd} \rho Q\sigma_r(B_r \setminus Q). \tag{14}$$

The sympletic group $Sp(2n,q)$ is defined as:

$$Sp(2n,q) = \{w \in GL(2n,q) | {}^t w \hat{J} w = \hat{J}\},$$

with

$$\hat{J} = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

As is well-known or mentioned in [5] and [6],

$$|O(2n+1,q)| = 2q^{n^2}\prod_{j=1}^{n}(q^{2j}-1), \tag{15}$$

$$|SO(2n+1,q)| = |Sp(2n,q)| = q^{n^2}\prod_{j=1}^{n}(q^{2j}-1). \tag{16}$$

For integers n, r with $0 \le r \le n$, the q-binomial coefficients are defined as:

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1}(q^{n-j}-1)/(q^{r-j}-1).$$

It is shown in [6] that

$$|B_r(q) \setminus Q(2n+1)| = q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q. \tag{17}$$

3. Gauss sums for $O(2n+1,q)$

Here we will recall our previous results about the Gauss sums for the finite classical groups $GL(t,q)$, $SO(2n+1,q)$, and $O(2n+1,q)$.

The following notations will be employed throughout this paper.

$$\begin{aligned} tr(x) &= x + x^3 + \cdots + x^{3^{r-1}} \text{ the trace function } \mathbb{F}_q \to \mathbb{F}_3, \\ \lambda_0(x) &= e^{2\pi i x/3} \text{ the canonical additive character of } \mathbb{F}_3, \\ \lambda(x) &= e^{2\pi i tr(x)/3} \text{ the canonical additive character of } \mathbb{F}_q. \end{aligned}$$

Then any nontrivial additive character ψ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$. Also, since $\lambda(a)$ for any $a \in \mathbb{F}_q$ is a 3rd root of 1, we have

$$\lambda(-a) = \lambda(2a) = \lambda(a)^2 = \lambda(a)^{-1} = \overline{\lambda(a)}. \tag{18}$$

For any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi;a)$ for $GL(t,q)$ is defined as

$$K_{GL(t,q)}(\psi;a) = \sum_{w \in GL(t,q)} \psi(Trw + aTrw^{-1}).$$

Observe that, for $t=1$, $K_{GL(1,q)}(\psi;a)$ denotes the Kloosterman sum $K(\psi;a)$.
In [5], it is shown that $K_{GL(t,q)}(\psi;a)$ satisfies the following recursive relation: for integers $t \ge 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi;a) = q^{t-1}K_{GL(t-1,q)}(\psi;a)K(\psi;a) + q^{2t-2}(q^{t-1}-1)K_{GL(t-2,q)}(\psi;a),$$

where we understand that $K_{GL(0,q)}(\psi;a) = 1$.

Proposition 1 (see [6]). *Let ψ be a nontrivial additive character of $\mathbb{F}_q$. For each positive integer r, let Ω_r be the set of all $r \times r$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then we have*

$$a_r(\psi) = \sum_{B \in \Omega_r} \sum_{h \in \mathbb{F}_q^{r \times 1}} \psi({}^t hBh) = \begin{cases} q^{r(r+2)/4} \prod_{j=1}^{r/2} (q^{2j-1} - 1), & \text{for } r \text{ even}, \\ 0, & \text{for } r \text{ odd}. \end{cases} \tag{19}$$

From [5] and [6], the Gauss sums for $SO(2n+1,q)$ and $O(2n+1,q)$ are equal to $\psi(1)$ times that for $Sp(2n,q)$ and $\psi(1)+\psi(-1)$ times that for $Sp(2n,q)$, respectively. Indeed, using the decomposition in (14), for any nontrivial additive character ψ of $\mathbb{F}_q$, it is shown that

$$\begin{aligned} \sum_{w \in SO(2n+1,q)} \psi(Trw) &= \sum_{\substack{0 \le r \le n \\ r \ even}} |B_r \setminus Q| \sum_{w \in Q} \psi(Trw\sigma_r) \\ &\quad + \sum_{\substack{0 \le r \le n \\ r \ odd}} |B_r \setminus Q| \sum_{w \in Q} \psi(Tr\rho w \sigma_r) \\ &= q^{\binom{n+1}{2}} \{\psi(1) \sum_{\substack{0 \le r \le n \\ r \ even}} |B_r \setminus Q| q^{r(n-r-1)} a_r(\psi) K_{GL(n-r,q)}(\psi;1) \\ &\quad + \psi(-1) \sum_{\substack{0 \le r \le n \\ r \ odd}} |B_r \setminus Q| q^{r(n-r-1)} a_r(\psi) K_{GL(n-r,q)}(\psi;1)\} \\ &= \psi(1) q^{\binom{n+1}{2}} \sum_{\substack{0 \le r \le n \\ r \ even}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q \\ &\quad \times \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,q)}(\psi;1) \ (cf. (17),\ (19)) \\ &(= \psi(1) \sum_{w \in Sp(2n,q)} \psi(Trw)) \ (cf.\ [5]). \end{aligned}$$

Similarly, from the decomposition in (13) it is shown in [6] that

$$\begin{aligned} \sum_{w \in O(2n+1,q)} \psi(Trw) &= (\psi(1) + \psi(-1)) q^{\binom{n+1}{2}} \sum_{\substack{0 \le r \le n \\ r \ even}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q \\ &\quad \times \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,q)}(\psi;1) \\ &(= (\psi(1) + \psi(-1)) \sum_{w \in Sp(2n,q)} \psi(Trw)). \end{aligned}$$

For our purposes, we only need the $n = 1$ case of expressions of Gauss sums for $SO(2n+1,q)$ and $O(2n+1,q)$ in the above. So we state them separately as a theorem. Also, for the ease of notations, we introduce

$$G_1(q) = SO(3,q),\ G_2(q) = O(3,q).$$

Theorem 2. *Let ψ be any nontrivial additive character of $\mathbb{F}_q$. Then we have*

$$\sum_{w\in G_1(q)} \psi(Trw) = \psi(1)qK(\psi;1),$$
$$\sum_{w\in G_2(q)} \psi(Trw) = (\psi(1)+\psi(-1))qK(\psi;1).$$

Corollary 1. *Let λ be the canonical additive character of $\mathbb{F}_q$, and let $a\in\mathbb{F}_q^*$. Then we have*

$$\sum_{w\in G_1(q)} \lambda(aTrw) = \lambda(a)qK(\lambda;a^2), \tag{20}$$

$$\begin{aligned}\sum_{w\in G_2(q)} \lambda(aTrw) &= (\lambda(a)+\lambda(-a))qK(\lambda;a^2)\\ &= 2(Re\lambda(a))qK(\lambda;a^2)\ (cf.\ (18)).\end{aligned} \tag{21}$$

Proof. This follows from Theorem 3.2 by replacing ψ by $\lambda(a\ \cdot)$ and by a simple change of variables. □

Proposition 2 (see [7, 5.3-5])**.** *Let λ be the canonical additive character of $\mathbb{F}_q$, $m\in\mathbb{Z}_{\geq 0}$, $\beta\in\mathbb{F}_q$. Then*

$$\sum_{a\in\mathbb{F}_q^*} \lambda(-a\beta)K(\lambda;a^2)^m = q\delta(m,q;\beta)-(q-1)^m, \tag{22}$$

where, for $m\geq 1$,

$$\delta(m,q;\beta) = |\{(\alpha_1,\dots,\alpha_m)\in(\mathbb{F}_q^*)^m | \alpha_1+\alpha_1^{-1}+\cdots,\alpha_m+\alpha_m^{-1}=\beta\}|, \tag{23}$$

and

$$\delta(0,q;\beta) = \begin{cases} 1, & \beta{=}0,\\ 0, & otherwise.\end{cases}$$

Remark 1. *Here one notes that*

$$\begin{aligned}\delta(1,q;\beta) &= |\{x\in\mathbb{F}_q | x^2-\beta x+1=0\}|\\ &= \begin{cases} 2, & if\ \beta^2-1\neq 0\ is\ a\ square,\\ 1, & if\ \beta^2-1{=}0,\\ 0, & if\ \beta^2-1\ is\ a\ nonsquare.\end{cases}\end{aligned} \tag{24}$$

Let $G(q)$ be $G_1(q)$ or $G_2(q)$. Then we put, for each $\beta\in\mathbb{F}_q$,

$$N_{G(q)}(\beta) = |\{w\in G(q) | Tr(w)=\beta\}|.$$

Then it is easy to see that

$$qN_{G(q)}(\beta) = |G(q)| + \sum_{a\in\mathbb{F}_q^*} \lambda(-a\beta) \sum_{w\in G(q)} \lambda(aTrw). \tag{25}$$

For brevity, we write

$$n_1(\beta) = N_{G_1(q)}(\beta),\ n_2(\beta) = N_{G_2(q)}(\beta). \tag{26}$$

Proposition 3. *With the notations in (23), (24), and (26), we have:*

$$n_1(\beta) = q^2 - q + q\delta(1, q; \beta - 1), \tag{27}$$
$$n_2(\beta) = 2q^2 - 2q + q\delta(1, q; \beta - 1) + q\delta(1, q; \beta + 1). \tag{28}$$

Proof. The equation in (27) follows from (16), (20), and (22), while that in (28) is obtained by combining (15), (21), and (22). □

Corollary 2. *With $G_1(q) = SO(3, q)$, $G_2(q) = O(3, q)$, $Tr : G_1(q) \to \mathbb{F}_q$, and $Tr : G_2(q) \to \mathbb{F}_q$ are surjective.*

Proof. This is immediate from the above Proposition 3. □

4. Construction of codes

With $G_1(q) = SO(3, q)$, $G_2(q) = O(3, q)$, we let

$$N_1 = |G_1(q)| = q(q^2 - 1),\ N_2 = |G_2(q)| = 2q(q^2 - 1). \tag{29}$$

Here we will construct ternary linear codes $C(G_1(q))$ of length N_1 and $C(G_2(q))$ of length N_2, respectively associated with the orthogonal groups $G_1(q)$ and $G_2(q)$. By abuse of notations, let $g_1, g_2, \ldots, g_{N_i}$ be a fixed ordering of the elements in the group $G_i(q)$, for $i = 1, 2$. Also, we put

$$v_i = (Trg_1, Trg_2, \ldots, Trg_{N_i}) \in \mathbb{F}_q^{N_i},\ for\ i = 1, 2.$$

Then the ternary linear code is defined as

$$C(G_i(q)) = \{u \in \mathbb{F}_3^{N_i} | u \cdot v_i = 0\},\ for\ i = 1, 2, \tag{30}$$

where the dot denotes the usual inner product in $\mathbb{F}_q^{N_i}$.

The following theorem of Delsarte is well-known.

Theorem 3 (see [2])**.** *Let B be a linear code over $\mathbb{F}_q$. Then*

$$(B|_{\mathbb{F}_3})^\perp = tr(B^\perp).$$

In view of this theorem, the dual $C(G_i(q))^\perp$ is given by

$$C(G_i(q))^\perp = \{c_i(a) = (tr(aTrg_1), \ldots, tr(aTrg_{N_i})) | a \in \mathbb{F}_q\},\ for\ i = 1, 2. \tag{31}$$

Proposition 4. *For every $q = 3^r$, the map $\mathbb{F}_q \to C(G_i(q))^\perp (a \mapsto c_i(a))$ is an $\mathbb{F}_3$-linear isomorphism, for $i = 1, 2$.*

Proof. The maps are clearly $\mathbb{F}_3$-linear and surjective. Let a be in the kernel of either of the map. Then, in view of Corollary 2, $tr(a\beta) = 0$, for all $\beta \in \mathbb{F}_q$. Since the trace function $\mathbb{F}_q \to \mathbb{F}_3$ is surjective, $a = 0$. □

5. Power moments of Kloosterman sums with trace nonzero square arguments

In this section, we find, via Pless power moment identity, recursive formulas for the power moments of Kloosterman sums with trace nonzero square arguments in terms of the frequencies of weights in $C(SO(3,q))$ and $C(O(3,q))$.

Theorem 4 (Pless power moment identity, see [4, p.257 (P-1)])**.** *Let B be a q-ary $[n,k]$ code, and let B_i(resp. $B_i^{\perp}$) denote the number of codewords of weight i in B(resp. in $B^{\perp}$). Then, for $h = 0,1,2,\ldots,$*

$$\sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{min\{n,h\}} (-1)^j B_j^{\perp} \sum_{t=j}^{h} t! S(h,t) q^{k-t} (q-1)^{t-j} \binom{n-j}{n-t}, \tag{32}$$

where $S(h,t)$ is the Stirling number of the second kind defined in (9).

Lemma 1. *Let $c_i(a) = (tr(aTrg_1),\ldots,tr(aTrg_{N_i})) \in C(G_i(q))^{\perp}$, for $a \in \mathbb{F}_q^*$, and $i = 1,2$. Then the Hamming weight $w(c_i(a))$ can be expressed as follows:*

$$w(c_i(a)) = \frac{2qi}{3}(q^2 - 1 - (Re\lambda(a))K(\lambda; a^2)), \ \ for\ i = 1,2. \tag{33}$$

Proof. For $i = 1,2$,

$$\begin{aligned} w(c_i(a)) &= \sum_{j=1}^{N_i}(1 - \frac{1}{3}\sum_{\alpha\in\mathbb{F}_3} \lambda_0(\alpha tr(aTrg_j))) \\ &= N_i - \frac{1}{3}\sum_{\alpha\in\mathbb{F}_3}\sum_{w\in G_i(q)} \lambda(\alpha aTrw) \\ &= \frac{2}{3}N_i - \frac{1}{3}\sum_{\alpha\in\mathbb{F}_3^*}\sum_{w\in G_i(q)} \lambda(\alpha aTrw). \end{aligned}$$

Our results now follow from (18), (20), (21) and (29). □

Theorem 5. *Let $q = 3^r$ be as before, and let $\{C_{i,j}\}_{j=0}^{N_i}$ be the weight distribution of $C(G_i(q))$, for $i = 1,2$. Then*

(a)

$$\begin{aligned} C_{1,j} &= \sum \prod_{\beta\in\mathbb{F}_q} \binom{n_1(\beta)}{\nu_\beta, \mu_\beta} \ (j = 0,\ldots,N_1), \\ &= \sum \binom{q^2}{\nu_0,\mu_0}\binom{q^2}{\nu_2,\mu_2} \prod_{\beta^2-2\beta\neq 0\ square} \binom{q^2+q}{\nu_\beta,\mu_\beta} \\ &\quad \times \prod_{\beta^2-2\beta\ nonsquare} \binom{q^2-q}{\nu_\beta,\mu_\beta}, \end{aligned} \tag{34}$$

with

$$n_1(\beta) = q^2 - q + q\delta(1, q; \beta - 1),$$

and

(*b*)

$$C_{2,j} = \sum \prod_{\beta \in \mathbb{F}_q} \binom{n_2(\beta)}{\nu_\beta, \mu_\beta} (j = 0, \ldots, N_2), \tag{35}$$

with

$$n_2(\beta) = 2q^2 - 2q + q\delta(1, q; \beta - 1) + q\delta(1, q; \beta + 1).$$

Here in both (34) and (35) the unspecified sums run over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta \in \mathbb{F}_q}$ *and* $\{\mu_\beta\}_{\beta \in \mathbb{F}_q}$ *satisfying*

$$\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \quad \text{and} \quad \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta,$$

and, for every $\beta \in \mathbb{F}_q$, $\delta(1, q; \beta)$ *is as in (24).*

Proof. Fix $i (i = 1, 2)$, and let $u = (u_1, \ldots, u_{N_i}) \in \mathbb{F}_3^{N_i}$, with ν_β 1's and μ_β 2's in the coordinate places where $Tr(g_j) = \beta$, for each $\beta \in \mathbb{F}_q$. Then we see from the definition of the code $C(G_i(q))$(cf. (30)) that u is a codeword with weight j if and only if $\sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j$ and $\sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta$(an identity in $\mathbb{F}_q$). Note that there are $\prod_{\beta \in \mathbb{F}_q} \binom{n_i(\beta)}{\nu_\beta, \mu_\beta}$(cf. (4), (5)) many such codewords with weight j. Now, we get the formulas in (34)-(35), by using the explicit values of $n_i(\beta)$ in (27), (28) (cf. (23), (24)). □

The recursive formula in the following theorem follows from the study of ternary linear codes associated with the symplectic group $Sp(2, q) = SL(2, q)$. It is slightly modified from its original version, which makes it more usable in below.

Theorem 6 (see [10]). *For* $h = 1, 2, 3, \ldots$,

$$\begin{aligned} &2(\frac{2q}{3})^h \sum_{j=0}^{h} (-1)^j \binom{h}{j} (q^2 - 1)^{h-j} SK^j \\ &\quad = q \sum_{j=0}^{min\{N_1, h\}} (-1)^j \hat{C}_j \sum_{t=j}^{h} t! S(h, t) 3^{-t} 2^{t-j} \binom{N_1 - j}{N_1 - t}, \end{aligned} \tag{36}$$

where $N_1 = q(q^2 - 1) = |Sp(2, q)| = |SO(3, q)|$, $S(h, t)$ *indicates the Stirling number of the second kind as in (9), and* $\{\hat{C}_j\}_{j=0}^{N_1}$ *denotes the weight distribution of the ternary linear code* $C(Sp(2, q))$, *given by*

$$\begin{aligned} \hat{C}_j &= \sum_{\beta \in \mathbb{F}_q} \prod_{\beta} \binom{q\delta(1, q; \beta) + q^2 - q}{\nu_\beta, \mu_\beta} \\ &= \sum \binom{q^2}{\nu_1, \mu_1} \binom{q^2}{\nu_{-1}, \mu_{-1}} \prod_{\beta^2 - 1 \neq 0 \ square} \binom{q^2 + q}{\nu_\beta, \mu_\beta} \prod_{\beta^2 - 1 \ nonsquare} \binom{q^2 - q}{\nu_\beta, \mu_\beta} \end{aligned}$$

$(j = 0, \ldots, N_1)$.
Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ *and* $\{\mu_\beta\}_{\beta\in\mathbb{F}_q}$ *satisfying* $\sum_{\beta\in\mathbb{F}_q}\nu_\beta + \sum_{\beta\in\mathbb{F}_q}\mu_\beta = j$ *and* $\sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = \sum_{\beta\in\mathbb{F}_q}\mu_\beta\beta$.

We now begin the proof of Theorem 1.1 (cf. (6)-(8), (10)-(12)) about recursive formulas by applying the Pless power moment identity (32) to $C(G_i(q))^\perp$. We do this for $i = 1, 2$ at the same time.

The left-hand side of that identity in (32) is equal to

$$\sum_{a\in\mathbb{F}_q^*} w(c_i(a))^h, \tag{37}$$

with $w(c_i(a))$ given by (33).

In below, "the sum over $tra = 0$ (resp. $tra \neq 0$)" will mean "the sum over all $a \in \mathbb{F}_q^*$ with $tra = 0$(resp. $tra \neq 0$)."

(37) is given by

$$(\frac{2qi}{3})^h \sum_{a\in\mathbb{F}_q^*}(q^2 - 1 - (Re\lambda(a))K(\lambda; a^2))^h = (\frac{2qi}{3})^h \sum_{tra=0}(q^2 - 1 - K(\lambda; a^2))^h$$
$$+(\frac{2qi}{3})^h \sum_{tra\neq 0}(q^2 - 1 + \frac{1}{2}K(\lambda; a^2))^h$$

(noting that $Re\lambda(a) = 1$, if $tra = 0$; $Re\lambda(a) = -\frac{1}{2}$, if $tra \neq 0$, i.e., $tra = 1, 2$)

$$= (\frac{2qi}{3})^h \sum_{tra=0}\sum_{j=0}^{h}(-1)^j\binom{h}{j}(q^2-1)^{h-j}K(\lambda; a^2)^j$$
$$+(\frac{2qi}{3})^h \sum_{tra\neq 0}\sum_{j=0}^{h}\binom{h}{j}(q^2-1)^{h-j}2^{-j}K(\lambda; a^2)^j$$
$$= (\frac{2qi}{3})^h \sum_{j=0}^{h}(-1)^j\binom{h}{j}(q^2-1)^{h-j}(2SK^j - T_{12}SK^j)$$
$$(\psi = \lambda \text{ case of (1), (2)})$$
$$+(\frac{2qi}{3})^h \sum_{j=0}^{h}\binom{h}{j}(q^2-1)^{h-j}2^{-j}T_{12}SK^j$$
$$= i^h 2(\frac{2q}{3})^h \sum_{j=0}^{h}(-1)^j\binom{h}{j}(q^2-1)^{h-j}SK^j$$
$$+(\frac{2qi}{3})^h \sum_{j=0}^{h}((-1)^{j+1} + 2^{-j})\binom{h}{j}(q^2-1)^{h-j}T_{12}SK^j$$

$$
\begin{aligned}
&= i^h q \sum_{j=0}^{min\{N_1,h\}} (-1)^j \hat{C}_j \sum_{t=j}^{h} t!S(h,t)3^{-t}2^{t-j}\binom{N_1-j}{N_1-t} \text{ (from (36))} \\
&+ (\frac{2qi}{3})^h \sum_{j=0}^{h}((-1)^{j+1}+2^{-j})\binom{h}{j}(q^2-1)^{h-j}T_{12}SK^j.
\end{aligned} \tag{38}
$$

On the other hand, the right-hand side of (32) is

$$
q \sum_{j=0}^{min\{N_1,h\}} (-1)^j C_{i,j} \sum_{t=j}^{h} t!S(h,t)3^{-t}2^{t-j}\binom{N_i-j}{N_i-t}. \tag{39}
$$

Here one has to note that $dim_{\mathbb{F}_2}C(SO(3,q)) = dim_{\mathbb{F}_2}C(O(3,q)) = r$(cf. Prop. 4) and to separate the term corresponding to $l = h$ of the second sum in (38). Our main results in Theorem 1 now follow by equating (38) and (39).

Corollary 3. *Let $q = 3^r$, and let, for the canonical addition character λ of $\mathbb{F}_q$, $SK = SK(\lambda)$, $T_0SK = T_0SK(\lambda)$, $T_{12}SK = T_{12}SK(\lambda)$ (cf. (1), (3)). Then we have the following.*

(a) $SK = \frac{1}{2}\{(-1)^r q + 1\}$,

(b) $T_0SK = \frac{1}{3}(-1)^r q + 1$,

(c) $T_{12}SK = \frac{2}{3}(-1)^r q$.

Proof. From either (6) or (10), we get (c). (a) follows from our previous result ([10], (4)) or can be derived directly as follows.

$$
\begin{aligned}
SK &= \frac{1}{2}\sum_{a\in\mathbb{F}_q^*} K(\lambda;a^2) \\
&= \frac{1}{2}\sum_{a\in\mathbb{F}_q^*}\sum_{\alpha\in\mathbb{F}_q^*} \lambda(\alpha + a^2\alpha^{-1}) \\
&= \frac{1}{2}\sum_{a\in\mathbb{F}_q^*}\sum_{\alpha\in\mathbb{F}_q^*} \lambda(a(\alpha + \alpha^{-1})) \\
&= \frac{1}{2}\sum_{\alpha\in\mathbb{F}_q^*}\sum_{a\in\mathbb{F}_q} \lambda(a(\alpha + \alpha^{-1})) - \frac{1}{2}(q-1) \\
&= \begin{cases} \frac{1}{2}2q - \frac{1}{2}(q-1), & \text{if } r \text{ even}, \\ -\frac{1}{2}(q-1), & \text{if } r \text{ odd}. \end{cases}
\end{aligned} \tag{40}
$$

In (40), we note that $\alpha + \alpha^{-1} = 0$ has a solution in $\mathbb{F}_q^*$ if and only if -1 is a square in $\mathbb{F}_q$ if and only if r is even, in which case there are two distinct solutions. Finally, (b) follows from relation (2) with $h = 1$ and $\psi = \lambda$. □

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