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# On nilpotent elements in a nearring of polynomials

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**Abstract.** For a ring $R$, $R[x]$ is a left nearring under addition and substitution, and we denote it by $(R[x], +, \circ)$. In this note, we show that if $nil(R)$ is a locally nilpotent ideal of $R$, then $nil(R[x], +, \circ) = nil(R)_0[x]$, where $nil(R)$ is the set of nilpotent elements of $R$ and $nil(R)_0[x]$ is the 0-symmetric left nearring of polynomials with coefficients in $nil(R)$. As a corollary, if $R$ is a 2-primal ring, then $nil(R[x], +, \circ) = nil(R)_0[x]$.

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## 1. Introduction

Throughout this paper, all rings are associative and unitary and all nearrings are left nearrings; subrings of a ring need not have the same unit, and *subrng* will denote a subring without unit. For a ring or nearring $N$, $nil(N)$ denotes the set of nilpotent elements of $N$. Also, $P(R)$ denotes the prime radical of a ring $R$. Recall that a ring or a nearring is said to be *reduced* if it has no nonzero nilpotent elements.

Rege and Chhawchharia [20] introduce the notion of an Armendariz ring. A ring $R$ is called *Armendariz* if whenever $f(x)g(x) = 0$ where $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x]$, then $a_ib_j = 0$ for each $i, j$. The name of the ring was given to E. Armendariz who proved in [3] that reduced rings satisfied this condition. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring $R$ and the annihilators of the polynomial ring $R[x]$. Let us recall two known facts: A ring $R$ is called *Baer* by Kaplansky [11] if the right annihilator of every nonempty subset of $R$ is generated by an idempotent. An example of Chon shows that the matrix ring $M_2(\mathbb{Z})$ is Baer but $M_2(\mathbb{Z})[x]$ is not. A well-known example of Kerr [12] shows that there exists a right Goldie ring $R$ such that $R[x]$ is not right Goldie. But, for an Armendariz ring $R$, $R$ is Baer if and only if $R[x]$ is Baer (Armendariz [3]; Kim and Lee [14]), and $R$ is right Goldie if and only if $R[x]$ is right Goldie (Hirano [8]). The reason behind these is a natural bijection between the set of annihilators of $R$ and the set of annihilators of $R[x]$ (see Hirano [8]). We refer to [1, 2, 3, 9, 10, 14, 15, 16, 17, 20] for more detail on Armendariz rings.

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Recall from [4] $R$ is said to satisfy the IFP (*insertion of factors property*) if $r_R(a) = \{b \in R | ab = 0\}$ is an ideal for all $a \in R$. Reduced rings satisfy the IFP. Shin [21] proved that $R$ is a division ring if and only if $R$ is a von Neumann regular prime ring and satisfies the IFP. Smoktunowicz [22] showed that there exists a nil ring $R$ such that $R[x]$ is not nil. But if $R$ satisfies the IFP, then Liu and Zhao [16] proved that $nil(R)$ is an ideal of $R$ and $nil(R[x]) = nil(R)[x]$. Also Antoine [2] proved that if $R$ is an Armendariz ring, then $nil(R)$ is a subrng of $R$ and $nil(R[x]) = nil(R)[x]$. Properties, examples and counterexamples of rings which satisfy the IFP are given in [4, 7, 8, 10, 18, 21, 23].

A ring $R$ is called *2-primal* if the prime radical of $R$ coincides with the set of all nilpotent elements of $R$ (see [6] for details). The class of 2-primal rings is also closed under subrings by [6, Proposition 2.2]. If $R$ satisfies the IFP, then $R$ is 2-primal.

Let $R$ be a ring. Since $R[x]$ is an abelian nearring under addition and substitution, it is natural to investigate the nearring of polynomials $(R[x], +, \circ)$. The binary operation of substitution, denoted by $\circ$, of one polynomial into another is both natural and important in the theory of polynomials. We adopt the convention that for polynomials $(x)g$ and $(x)f = \sum_{i=0}^{m} f_i x^i \in R[x]$, $(x)g \circ (x)f = \sum_{i=0}^{m} f_i((x)g)^i$. For example, $(a_0 + a_1x) \circ x^2 = (a_0 + a_1x)^2 = a_0^2 + (a_0a_1 + a_1a_0)x + a_1^2x^2$. However, the operation "$\circ$", left distributes but does not right distribute over addition. Thus $(R[x], +, \circ)$ forms a left nearring but not a ring. Unless specifically indicated otherwise, $R[x]$ denotes the left nearring of polynomials $(R[x], +, \circ)$ with coefficients from $R$ and $R_0[x] = \{f \in R[x] | f$ has zero constant term$\}$ is the 0-symmetric left nearring of polynomials with coefficients in $R$.

We say that a set $S \subseteq R$ is *locally nilpotent* if for any subset $\{s_1, s_2, \cdots, s_n\} \subseteq S$, there exists an integer $t$, such that any product of $t$ elements from $\{s_1, s_2, \cdots, s_n\}$ is zero.

Antoine [2, Corollaries 3.3 and 5.2] proved that if $R$ is an Armendariz ring, then $nil(R)$ is a subrng of $R$ and $nil(R)[x] = nil(R[x])$. Hence $nil(R)$ is a locally nilpotent subrng of $R$, when $R$ is an Armendariz ring.

The following examples show that there exist non Armendariz rings such that the set of its nilpotent elements is a locally nilpotent ideal.

**Example 1.** *Let $\mathbb{Z}$ be the ring of integers and let*

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} | a - b \equiv c \equiv 0 (mod\ 2) \right\}.$$

*Then by [14, Example 13], $R$ is not Armendariz. Since*

$$nil(R) = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} | c \equiv 0 (mod\ 2) \right\},$$

*hence $nil(R)$ is a locally nilpotent ideal of $R$.*

**Example 2.** *Let $T$ be a reduced ring and*

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in T \right\}.$$

*Let*

$$S = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} | A, B \in R \right\}.$$

*Then by [14, Example 5], $S$ is not Armendariz. Since*

$$nil(S) = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} | A \in nil(R) \text{ and } B \in R \right\},$$

*hence $nil(S)$ is a locally nilpotent ideal of $S$.*

If $R$ satisfies the IFP, then $R$ is *abelian* (i.e., each idempotent of $R$ is central). The following example shows that there exists a ring $R$ such that it does not satisfy the IFP, but $nil(R)$ is a locally nilpotent ideal of $R$.

**Example 3.** *Let $F$ be a division ring and consider the 2-by-2 upper triangular ring*

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}.$$

*Then $R$ does not satisfy the IFP, since $R$ is not abelian. But*

$$nil(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} | b \in F \right\}$$

*is a locally nilpotent ideal of $R$.*

## 2. Nilpotent elements in a nearring of polynomials

**Definition 1.** *Let $R$ be a ring. We say $R$ has property $(*)$, whenever $(x)f = a_0+a_1x+\cdots+a_mx^m$, $(x)g = b_0+b_1x+\cdots+b_nx^n$ are elements of nearring $(R[x], +, \circ)$ and $f \circ g \in nil(R)[x]$, then $a_ib_j \in nil(R)$ for $i = 1, \cdots, m$, $j = 0, 1, \cdots, n$.*

By [5, Lemma 3.4], every reduced ring has property $(*)$.

**Proposition 1.** *Let $I$ be a nil ideal of a ring $R$. Then $R/I$ has property $(*)$ if and only if $R$ has property $(*)$.*

**Proof**. We denote $\overline{R} = R/I$. Since $I$ is nil, then $nil(\overline{R}) = \overline{nil(R)}$. Let $(x)f = \sum_{i=0}^m a_ix^i$ and $(x)g = \sum_{j=0}^n b_jx^j$ be elements of nearring $R[x]$. Then $f \circ g \in nil(R)[x]$, if and only if $(\sum_{i=0}^m \overline{a}_ix^i) \circ (\sum_{j=0}^n \overline{b}_jx^j) \in nil(\overline{R})[x]$. Also, $a_ib_j \in nil(R)$ if and only if $\overline{a_i}\overline{b_j} \in nil(\overline{R})$, for $i = 1, \cdots, m$ and $j = 0, 1, \cdots, n$. $\square$

**Proposition 2.** *Let $R$ be a ring and $e$ a central idempotent element of $R$. Then the following statements are equivalent:*

1. *$R$ has property $(*)$.*
2. *$eR$ and $(1-e)R$ have property $(*)$.*

**Proof**. (1) ⇒ (2) It is clear, since $eR$ and $(1-e)R$ are subrings of $R$.

(2) ⇒ (1) Let $(x)f = \sum_{i=0}^m a_ix^i$ and $(x)g = \sum_{j=0}^n b_jx^j$ be elements of nearring $R[x]$ such that $f \circ g \in nil(R)[x]$. Let $(x)f_1 = \sum_{i=0}^m ea_ix^i$, $(x)f_2 = \sum_{i=0}^m (1-e)a_ix^i$, $(x)g_1 = \sum_{j=0}^n eb_jx^j$ and $(x)g_2 = \sum_{j=0}^n (1-e)b_jx^j$. Then $f_1 \circ g_1 = (\sum_{i=0}^m ea_ix^i) \circ (\sum_{j=0}^n eb_jx^j) = e.(f \circ g) \in nil(eR)[x]$ and $f_2 \circ g_2 = (\sum_{i=0}^m (1-e)a_ix^i) \circ (\sum_{j=0}^n (1-e)b_jx^j) = (1-e).(f \circ g) \in nil((1-e)R)[x]$, since $f \circ g \in nil(R)[x]$ and $e, (1-e)$ are central idempotent elements of $R$. Hence $ea_ib_j$ and $(1-e)a_ib_j$ are nilpotent, for each $i = 1, \cdots, m$ and $j = 0, 1, \cdots, n$, since $eR$ and $(1-e)R$ have property $(*)$. Thus there exists $t \geq 2$ such that $(ea_ib_j)^t = ((1-e)a_ib_j)^t = 0$ for each $i = 1, \cdots, m$ and $j = 0, 1, \cdots, n$. Hence $(a_ib_j)^t = 0$ for each $i = 1, \cdots, m$ and $j = 0, 1, \cdots, n$. Therefore $R$ has property $(*)$. □

**Proposition 3.** *Let $R$ be a finite subdirect sum of rings which have property $(*)$. Then $R$ has property $(*)$.*

**Proof**. Let $I_k$ $(k = 1, 2, \cdots, \ell)$ be ideals of $R$ such that $R/I_k$ has property $(*)$ and $\cap_{k=1}^{\ell} I_k = 0$. Suppose that $(x)f = \sum_{i=0}^m a_ix^i$ and $(x)g = \sum_{j=0}^n b_jx^j$ be elements of nearring $R[x]$ such that $f \circ g \in nil(R)[x]$. Then there exists $p_{ij} \geq 1$, such that $(\overline{a}_i\overline{b}_j)^{p_{ij}} = 0$ in $R/I_k$. Thus $(a_ib_j)^{p_{ij}} \in I_k$. Set $p = max\{p_{ij} | i, j \geq 1\}$. Then $(a_ib_j)^{p_{ij}} \in I_k$, for any $k$, which implies that $(a_ib_j)^p = 0$. Therefore $R$ has property $(*)$. □

For a ring $R$, we denote the $n$-by-$n$ upper triangular and full matrix ring over $R$ by $T_n(R)$ and $M_n(R)$, respectively.

**Proposition 4.** *A ring $R$ has property $(*)$ if and only if, for any $n$, $T_n(R)$ has property $(*)$.*

**Proof**. If $T_n(R)$ has property $(*)$, then so $R$ has property $(*)$ as a subring of $T_n(R)$.

Conversely, let $(x)f = \sum_{i=0}^p A_ix^i$ and $(x)g = \sum_{j=0}^q B_jx^j$ be elements of nearring $T_n(R)[x]$ such that $f \circ g \in nil(T_n(R))[x]$. Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}$$

and

$$B_j = \begin{pmatrix} b_{11}^i & b_{12}^i & \cdots & b_{1n}^i \\ 0 & b_{22}^i & \cdots & b_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^i \end{pmatrix}.$$

Then from $f \circ g \in nil(T_n(R))[x]$ it follows that $(\sum_{i=0}^p a_{ss}^ix^i) \circ (\sum_{j=0}^q b_{ss}^jx^j) \in nil(R)[x]$ for $s = 1, \cdots, n$. Since $R$ has property $(*)$, $a_{ss}^ib_{ss}^j \in nil(R)$, for each $i = 1, \cdots, p$, $j = 0, 1, \cdots, q$ and $s = 1, \cdots, n$. Then $A_iB_j \in nil(T_n(R))$ for each $i = 1, \cdots, p$, $j = 0, 1, \cdots, q$. Therefore $T_n(R)$ has property $(*)$. □

Let $R$ be a ring. Then

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} | a, a_{ij} \in R \right\}$$

is a subring of $T_n(R)$, for each $n \geq 2$. By a similar argument as used in the proof of Proposition 4, we can show that $R$ has property $(*)$ if and only if, for any $n$, $R_n$ has property $(*)$.

The same idea can be used to prove the following.

**Proposition 5.** *Let $R, S$ be rings and ${}_RM_S$ an $(R, S)$-bimodule. Then $T = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ has property $(*)$ if and only if $R$ and $S$ have property $(*)$.*

**Theorem 1.** *If $nil(R)$ is an ideal of $R$, then $R$ has property $(*)$.*

**Proof**. Since $R/nil(R)$ is a reduced ring, hence by [5, Lemma 3.4], $R/nil(R)$ has property $(*)$. Hence by Proposition 1, $R$ has property $(*)$. □

**Lemma 1** (see [16]). *If $R$ satisfies the IFP, then*

1. *$nil(R)$ is a locally nilpotent ideal of $R$;*
2. *$nil(R[x]) = nil(R)[x]$.*

**Proposition 6.** *If $R$ satisfies the IFP, then $R$ has property $(*)$.*

**Proof**. It follows from Lemma 1 and Theorem 1. □

The following example shows that the condition " $nil(R)$ be an ideal of $R$ " in Theorem 1 is not superfluous.

**Example 4.** *Let $F$ be a field and $S = M_2(F)$. Then $nil(S)$ is not ideal of $R$. Let*

$$(x)f = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} x$$

*and*

$$(x)g = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^2$$

*be polynomials in $S[x]$. Then $f \circ g = 0 \in nil(S)[x]$, but*

$$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin nil(S).$$

**Lemma 2.** *Let $nil(R)$ be an ideal of $R$, and $a_1, a_2, \cdots, a_n, a, b \in R$.*

1. *If $ab \in nil(R)$, then $arb \in nil(R)$ for each $r \in R$.*

*2. If $ab^n \in nil(R)$ for some $n \geq 1$, then $ab \in nil(R)$.*

*3. If $b_1b_2\cdots b_m \in nil(R)$, where $b_i \in \{a_1, a_2, \cdots, a_n\}$, then $a_1a_2\cdots a_n \in nil(R)$.*

**Proof**. The details are left to the reader. □

For any $(x)f \in R[x]$, we denote by $C_f$ the set of all coefficients of $f$. Let $C^*_f = C_f - \{a_0\}$, where $a_0$ is the constant term of $f$.

**Proposition 7.** *Let $(x)f_1, (x)f_2, \cdots, (x)f_n$ be elements of nearring $R[x]$, such that $f_1 \circ f_2 \circ \cdots \circ f_n \in nil(R)[x]$. If $nil(R)$ is an ideal of $R$, then $C^*_{f_1}C^*_{f_2}\cdots C^*_{f_n} \subseteq nil(R)$.*

**Proof**. We use induction on $n$. The case $n = 2$ follows from Theorem 1.

Suppose $n > 2$. Consider $g = f_2 \circ f_3 \circ \cdots \circ f_n$. Then $f_1 \circ g \in nil(R)[x]$ and hence by Theorem 1, $a_1a_g \in nil(R)$ where $a_g \in C_g$ and $a_1 \in C^*_{f_1}$. Therefore for all $a_1 \in C^*_{f_1}$,

$$\begin{aligned} g \circ a_1x &= (f_2 \circ f_3 \circ \cdots \circ f_n) \circ a_1x = f_2 \circ f_3 \circ \cdots \circ f_{n-1} \circ (f_n \circ a_1x) \\ &= f_2 \circ f_3 \circ \cdots \circ f_{n-1} \circ (a_1f_n) \in nil(R)[x] \end{aligned}$$

and by induction, since the coefficients of $a_1f_n$ are $a_1a_n$, where $a_n$ is a coefficient of $f_n$, we obtain $a_2a_3\cdots a_{n-1}a_1a_n \in nil(R)$. Hence $C^*_{f_1}C^*_{f_2}\cdots C^*_{f_n} \subseteq nil(R)$, by Lemma 2. □

**Theorem 2.** *Let $(x)f = a_0 + a_1x + \cdots + a_mx^m$ be a nilpotent element of nearring $R[x]$ and $nil(R)$ an ideal of $R$. Then $a_i \in nil(R)$ for $i = 0, 1, \cdots, m$.*

**Proof**. Let $(x)f = a_0 + a_1x + \cdots + a_mx^m \in nil(R[x])$. Then there exists $k \geq 2$ such that

$$f^k = \underbrace{f \circ f \circ f \cdots \circ f}_{k} = 0 \in nil(R)[x].$$

By Proposition 7, $a_i \in nil(R)$ for each $i = 1, \cdots, m$. We claim that $a_0 \in nil(R)$. The constant term of $f^k$ is $a_0 + \beta$, where $\beta$ is a sum of elements $a_{i_1}a_{i_2}\cdots a_{i_t}$ such that $t \geq 2$ and $\{a_{i_1}, a_{i_2}, \cdots, a_{i_t}\} \cap \{a_1, a_2, \cdots, a_m\} \neq \phi$. Then $\beta \in nil(R)$, and since $a_0 + \beta \in nil(R)$, we have $a_0 \in nil(R)$. Therefore $a_i \in nil(R)$, for $i = 0, 1, \cdots, m$. □

**Theorem 3.** *If $nil(R)$ is a locally nilpotent ideal of a ring $R$, then $nil(R[x]) = nil(R)_0[x]$.*

**Proof**. Let $(x)f = a_0 + a_1x + \cdots + a_mx^m$ be a nilpotent element of nearring $R[x]$. By Theorem 2, $a_i \in nil(R)$ for $i = 0, 1, \cdots, m$. Thus $\{a_0, a_1, \cdots, a_m\} \subseteq nil(R)$, and since $nil(R)$ is a locally nilpotent subset of $R$, there exists $t \geq 2$ such that $\{a_0, a_1, \cdots, a_n\}^t = 0$. Since $f \in nil(R[x])$, hence

$$f^k = \underbrace{f \circ f \circ f \cdots \circ f}_{k} = 0,$$

for some $k \geq t$. For each $j \geq 1$, the coefficient of $x^j$ in the polynomial $f^k$ is a sum of elements $a_{i_1}a_{i_2}\cdots a_{i_\ell}$, where $a_{i_r} \in \{a_0, a_1, \cdots, a_m\}$, and $\ell \geq k$. Also the constant term of the polynomial $f^k$ is $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 + \alpha$, where $\alpha$ is a sum of elements $a_{i_1}a_{i_2}\cdots a_{i_\ell}$, where $a_{i_r} \in \{a_0, a_1, \cdots, a_m\}$ and $\ell \geq k$. Since $\{a_0, a_1, \cdots, a_m\}^t = 0$, hence $\alpha = 0$, and since $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 + \alpha = 0$, we have $a_0 + a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 = 0$. Multiplying this equation by $a_1$ from the left yields $a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-1}a_0 = 0$, and since $a_1^{k-1}a_0 = 0$ we have $a_1a_0 + a_1^2a_0 + \cdots + a_1^{k-2}a_0 = 0$. Hence $a_0 = 0$ and $nil(R[x]) \subseteq nil(R)_0[x]$.

Now let $(x)f = a_1x + \cdots + a_mx^m \in nil(R)_0[x]$. Since $nil(R)$ is a locally nilpotent subset of $R$, there exists $t \geq 2$ such that $\{a_1, \cdots, a_m\}^t = 0$. Since for each $j \geq 2$, the coefficient of $x^j$ in the polynomial

$$f^t = \underbrace{f \circ f \circ f \cdots \circ f}_{t}$$

is a sum of elements $a_{i_1}a_{i_2}\cdots a_{i_\ell}$, where $a_{i_r} \in \{a_1, \cdots, a_m\}$ and $\ell \geq t$, hence $f^t = 0$. Therefore $nil(R)_0[x] \subseteq nil(R[x])$ and hence $nil(R[x]) = nil(R)_0[x]$. □

By [13, Proposition 10.31], the sum of all locally nilpotent ideals in a ring $R$ (denoted by $L\text{-}rad\,R$) is locally nilpotent, and $P(R) \subseteq L\text{-}rad\,R \subseteq nil(R)$. Then $P(R) = L\text{-}rad\,R = nil(R)$, if $R$ is a 2-primal ring. Thus we have the following result:

**Corollary 1.** *If $R$ is a 2-primal ring, then $nil(R[x]) = nil(R)_0[x]$.*

**Corollary 2.** *If $R$ satisfies the IFP, then $nil(R[x]) = nil(R)_0[x]$.*

**Corollary 3.** *If $R$ is an Armendariz ring and $nil(R)$ an ideal of $R$, then $nil(R[x]) = nil(R)_0[x]$.*

**Proof**. Since $R$ is an Armendariz ring, hence by [2, Corollary 5.2] $nil(R)[x] = nil(R[x])$. Thus by [1, Proposition 1], $nil(R)$ is a locally nilpotent subset of $R$. Now the result follows from Theorem 3. □

**Proposition 8.** *If $nil(R)$ is a locally nilpotent ideal of $R$, then $nil(R[x])$ is a right ideal of $(R[x], +, \circ)$.*

**Proof**. Let

$$(x)f = f_0 + f_1x + \cdots + f_mx^m,$$
$$(x)h = h_0 + h_1x + \cdots + h_mx^m \in R[x]$$

and

$$(x)g = g_1x + \cdots + g_mx^m \in nil(R[x]).$$

Then

$$(f+g)\circ h - f\circ h = h_1[(f+g) - f] + h_2[(f+g)^2 - f^2] + \cdots + h_m[(f+g)^m - f^m].$$

Since for each $i \geq 2$, $[(f+g)^i - f^i] \in nil(R)_0[x]$, hence $(f+g)\circ h - f\circ h \in nil(R)_0[x]$. Thus $nil(R[x])$ is a right ideal of $(R[x], +, \circ)$. □

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