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On a decomposition of partitioned J-unitary matrices*

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Abstract. We propose a new decomposition of hyperbolic block-unitary matrices into a product of a hyperbolic block-rotation and a block-diagonal hyperbolic unitary matrix. A similar result is known in the real space equipped with the Euclidean scalar product, but we generalize it to the complex spaces equipped with hyperbolic scalar products.
We shall also present an example how such a decomposition might be used to calculate other decompositions with block-operations.

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1. Introduction

Among the most important tools in linear algebra are unitary ($U^*U = \mathrm{I}$) or, in the real case, orthogonal ($U^\tau U = \mathrm{I}$) matrices. Their importance lies in the fact that they preserve the scalar product and their inverse is calculated trivially.

Two classes of elementary unitary matrices are widely used: rotations and reflectors. Rotations are usually used in some two-dimensional subspace for the annihilation of a single element. Reflectors, on the other hand, are used for the annihilation of the whole matrix column. For details, see [6, Section 5.1].

It is easy to show that every 2×2 unitary matrix can be decomposed into the product of one rotation and one unitary diagonal matrix.

Zakrajšek and Vidav have shown in [14] that every block-orthogonal matrix of an arbitrary order can be decomposed into the product of one block-rotation and one block-diagonal orthogonal matrix, with the complex case being a trivial generalization of that result. We shall call such a decomposition the ZV decomposition.

In [13], Veselić has proposed more general block-rotations, with regard to an arbitrary scalar product $[x, y]_J := y^* Jx$ induced by any symmetric orthogonal block-diagonal matrix J.

Some well researched elementary classes of matrices, resembling those from the Euclidean scalar products, also exist with regard to the hyperbolic ones. For a given hyperbolic J (i.e., $J = \mathrm{diag}(\pm 1)$), a matrix U is J-unitary if and only if $U^*JU = J$,

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which is equivalent to $UJU^* = J$ and $U^{[*]} = U^{-1}$, where $U^{[*]} := JU^*J$ is J-adjoint of U. If a matrix A is such that $A = A^{[*]}$, it is called J-Hermitian. More on the indefinite scalar products and matrices with the special structure with regard to such products can be found in [5].

We shall also use J-positive and J-nonnegative matrices, which are the hyperbolic counterparts of positive definite and positive semidefinite matrices, respectively. We say that a matrix A is J-positive (J-nonnegative) if A is J-Hermitian and JA is positive (semi-)definite. More on these matrices can be found in [4], where the semidefinite J-polar decomposition, a direct generalization of the traditional polar decomposition, was proposed.

In this paper, we shall focus on using these block-rotations proposed by Veselić to generalize the ZV decomposition to the hyperbolic scalar products, i.e., for $J = \operatorname{diag}(\pm 1)$. A special case of the hyperbolic ZV decomposition, for $J = \operatorname{diag}(\mathrm{I}_p, -\mathrm{I}_q)$, with p and q corresponding to the partitions of U, was proposed in [1, Theorem 2.4].

As we shall see, the original result of Zakrajšek and Vidav can be proven via the CS decomposition. But, since the hyperbolic CS decomposition is known only for a very limited case $J = \operatorname{diag}(\mathrm{I}_p, -I_q)$, we shall mimic the actual proof done by Zakrajšek and Vidav which is based on an SVD. For that reason, we shall need SVD's hyperbolic counterpart, the two-sided hyperbolic singular value decomposition (2HSVD) from [12] which decomposes a given matrix A into $A = U\Sigma V^{[*]}$, where U and V are J-unitary and Σ is real diagonal. Unlike the traditional SVD, not all matrices have the 2HSVD and the diagonal elements of Σ usually cannot be ordered in some specific (i.e., descending) order.

Another decomposition we shall need is an aforementioned semidefinite J-polar decomposition which states that if somewhat complex conditions are met, a matrix A can be decomposed as $A = WX$, where W is J-unitary and X is J-nonnegative. As shown in [12], every matrix that has the 2HSVD also has the semidefinite J-polar decomposition, so those "somewhat complex conditions" will not be a concern.

We shall also use the principal matrix roots, which Veselić defines in [13, Formula (6)] as

$$\sqrt[p]{X} := \sum_{k=0}^{\infty} \binom{1/p}{k} (X - \mathrm{I})^k, \tag{1}$$

for all matrices X with no negative eigenvalues, such that zero is their at most non-defective eigenvalue. In Sections 2 and 3, we shall need the following simple result which follows trivially from (1):

Lemma 1. *If $\sqrt{\mathrm{I} \pm XY}$ exists, then $\sqrt{\mathrm{I} \pm YX}$ exists as well and*

$$Y\sqrt{\mathrm{I} \pm XY} = \sqrt{\mathrm{I} \pm YX}Y.$$

The following trivial, yet not very obvious fact shall be used in the proof of Lemma 2: the principal square root of X^2 does not always exists. For example, if $X = \mathcal{J}_n(0)$ is a Jordan block of order $n > 2$ corresponding to the eigenvalue 0, then X^2 doesn't have a principal square root, as 0 is its defective eigenvalue.

More on matrix roots can be found in [8, Chapter 6]. Principal root is particularly nice as it preserves the group of J-unitary matrices (sometimes referred to as an automorphism group $\mathbb{G}$). For more on this subject, see [9].

In Section 2, we state Zakrajšek and Vidav's theorem and prove it via the CS decomposition. In Section 3, we present our main result: the hyperbolic ZV decomposition, along with some of its consequences (among which are the ZV and the hyperbolic CS decomposition). In Section 4, we consider how such a decomposition might be used to improve algorithms for the computation of other decompositions by making them work on blocks instead of single elements.

2. The Euclidean case

In this section, we provide Zakrajšek and Vidav's theorem (naturally generalized to complex matrices), along with its simple proof based on the CS decomposition. The CS decomposition is a standard tool when dealing with unitary matrices and is discussed in many books and papers, like [11, Section 5.1] and [6, Sections 2.6 and 12.4].

Theorem 1 (Zakrajšek-Vidav)**.** *Every block-unitary matrix*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

can be decomposed into $U = R\Delta$ *such that* $\Delta = \operatorname{diag}(\Delta_{11}, \Delta_{22})$ *is block-diagonal unitary and*

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where $R_{11} = \sqrt{\mathrm{I} - R_{12}R_{12}^*}$, $R_{21} = -R_{12}^*$, $R_{22} = \sqrt{\mathrm{I} - R_{12}^*R_{12}}$*.*

Proof. Let us assume that $U_{11} \in \mathbb{C}^{p\times p}$, $U_{22} \in \mathbb{C}^{q\times q}$, $p + q = n$, $p \leq q$. Then there exists a CS decomposition of U:

$$U = \left[\begin{array}{c|c} V_{11} & \\ \hline & V_{22} \end{array}\right] \left[\begin{array}{c|cc} \Gamma & \Sigma & 0 \\ \hline -\Sigma & \Gamma & 0 \\ 0 & 0 & \mathrm{I}_{n-2p} \end{array}\right] \left[\begin{array}{c|c} W_{11}^* & \\ \hline & W_{22}^* \end{array}\right], \quad \Gamma^2 + \Sigma^2 = \mathrm{I}_p,$$

where $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_p) \geq 0$ and $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_p) \geq 0$. Using Lemma 1, we easily see that

$$\begin{aligned} V_{11}\Gamma &= V_{11}\sqrt{\mathrm{I} - \Sigma^2} = V_{11}\sqrt{\mathrm{I} - V_{11}^*V_{11}\Sigma^2V_{11}^*V_{11}} = \sqrt{\mathrm{I} - V_{11}\Sigma^2V_{11}^*}V_{11}, \\ V_{22}\hat{\Gamma} &= V_{22}\sqrt{\mathrm{I} - \hat{\Sigma}^*\hat{\Sigma}} = V_{22}\sqrt{\mathrm{I} - V_{22}^*V_{22}\hat{\Sigma}^*\hat{\Sigma}V_{22}^*V_{22}} = \sqrt{\mathrm{I} - V_{22}\hat{\Sigma}^*\hat{\Sigma}V_{22}^*}V_{22}, \end{aligned}$$

where $\hat{\Gamma} = \operatorname{diag}(\Gamma, \mathrm{I}_{n-2p})$ and $\hat{\Sigma} = [\Sigma\ 0]$ denote the bottom-right and the top-right blocks of the CS factor of U, respectively. Finally, let $R_{12} := V_{11}\hat{\Sigma}V_{22}^*$. Then:

$$R_{11} = \sqrt{I - V_{11}\Sigma^2V_{11}^*}, \quad R_{21} = -V_{22}\hat{\Sigma}V_{11}^*, \quad R_{22} = \sqrt{I - V_{22}\hat{\Sigma}^*\hat{\Sigma}V_{22}^*}.$$

Furthermore, let $\Delta = VW^*$, where $V = \operatorname{diag}(V_{11}, V_{22})$ and $W = \operatorname{diag}(W_{11}, W_{22})$. It is now easy to see that $U = R\Delta$.

The case when $p > q$ is proved in a similar manner, using the CS decomposition

$$U = \left[\begin{array}{c|c} V_{11} & \\ \hline & V_{22} \end{array}\right] \left[\begin{array}{cc|c} \Gamma & 0 & \Sigma \\ 0 & \mathrm{I}_{n-2q} & 0 \\ \hline -\Sigma & 0 & \Gamma \end{array}\right] \left[\begin{array}{c|c} W_{11}^* & \\ \hline & W_{22}^* \end{array}\right], \quad \Gamma^2 + \Sigma^2 = \mathrm{I}_q,$$

where $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_q) \geq 0$ and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_q) \geq 0$. □

The original proof of Theorem 1 by Zakrašjek and Vidav uses the SVD instead of the CS decomposition and is much longer, but it will prove quite useful for the hyperbolic case in which we shall mimic it.

3. The hyperbolic case

In this section, we show how certain J-unitary matrices can be decomposed into the product of a block J-rotation and a block-diagonal J-unitary matrix. We assume that given hyperbolic J and J-unitary U are partitioned into 2×2 blocks:

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}, \quad J = J_1 \oplus J_2, \quad J_1, J_2 = \mathrm{diag}(\pm 1), \tag{2}$$

where U_{11} and J_1 are square matrices of the same order, as well as U_{22} and J_2.

Hyperbolic decompositions are often done for a special case $J = \mathrm{diag}(I_k, -I_l)$. Good examples of this are the hyperbolic CS decomposition [7] and the early version of the hyperbolic QR [3]. Even our main result was investigated for this special case and proven via the hyperbolic CS decomposition in [1, Theorem 2.4].

However, our aim is to make an accessory to other decompositions, especially a more general hyperbolic QR from [10]. Such applications usually allow a simple $J = \mathrm{diag}(I_k, -I_l)$ to be used (along with appropriate permutations), but the sizes k and l need not be the same as partitioning of U, which makes these previous results unfitting.

For example, a general hyperbolic QR usually annihilates the first one or two rows in each pass, which means that, in order to use the existing results, the appropriate k would have to be 1 or 2, thus imposing a strong limit on the inertia of J and so greatly limiting the application. We could, of course, partition $J = \mathrm{diag}(\mathrm{I}_2, \mathrm{I}_{k-2}, -I_l)$, or similarly, but this would lead to almost the same hardness of the problem. As we shall see, the only gain of such partitioning of J would be a somewhat simpler Lemma 4, which is only a matter of a small technical convenience.

For these reasons, throughout this section we shall use the most general form of J, as introduced in (2).

A class of elementary J-unitary matrices, a block-generalization of plane rotations, was proposed by Veselić in [13] for real matrices. His results also apply to complex matrices of the following form:

$$R = \begin{bmatrix} \sqrt{\mathrm{I} + BC} & B \\ C & \sqrt{\mathrm{I} + CB} \end{bmatrix}, \quad C = -J_2 B^* J_1.$$

Now we state the main result of this paper:

Theorem 2. *Let J and U be given by (2). If U is a J-unitary matrix such that both $U_{11} \in \mathbb{C}^{n_1 \times n_1}$ and $U_{22} \in \mathbb{C}^{n_2 \times n_2}$ have the 2HSVD with regard to J_1 and J_2, respectively, it can be written in the form*

$$U = R\Delta, \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix}, \tag{3}$$

where Δ_1 and Δ_2 are J_1- and J_2-unitary (respectively) and

$$R_{11} = \sqrt{\mathrm{I} + R_{12}R_{21}}, \quad R_{21} = -J_2 R_{12}^* J_1, \quad R_{22} = \sqrt{\mathrm{I} + R_{21}R_{12}}. \tag{4}$$

For better clarity, we prove Theorem 2 through a series of lemmas, somewhat mimicking the proof of Zakrajšek and Vidav, although the switch to the hyperbolic case introduces quite a few concerns which are non-existent in the case of the Euclidean scalar product. In the end, the ZV decomposition will be a simple corollary of Theorem 2. The outline of the proof is as follows:

1. Lemma 2 gives all of the properties except the relation $R_{21} = -J_2 R_{12}^* J_1$. The rest of the proof aims at proving that particular relation.
2. Corollary 1 introduces the missing relation providing that an additional condition is met (U_{11} or U_{22} nonsingular).
3. Lemma 3 shows that the missing relation holds if U_{11} and U_{22} are diagonal, with the zeroes in U_{11} grouped in the bottom right corner.
4. Lemma 4 makes up for the grouping of zeroes needed by the previous lemma (but not provided by the 2HSVD we are using).
5. Proof of Theorem 2 on page 276 summarizes these results.

Before we start with the proof itself, note that because J is diagonal, unitary and Hermitian, so are J_1 and J_2 and the following holds:

$$J_1 = J_1^{-1}, \quad J_1^2 = \mathrm{I}, \quad J_2 = J_2^{-1}, \quad J_2^2 = \mathrm{I}.$$

These properties will be used often throughout the remainder of this paper.

The first step in proving Theorem 2 will be to show some basic relations between the blocks of R in (3).

Lemma 2. *Let J and U be given by (2). If U is a J-unitary matrix such that both $U_{11} \in \mathbb{C}^{n_1 \times n_1}$ and $U_{22} \in \mathbb{C}^{n_2 \times n_2}$ have the 2HSVD with regard to J_1 and J_2, respectively, it can be written in the form (3), where Δ_1 and Δ_2 are J_1- and J_2-unitary polar factors of blocks U_{11} and U_{22}, respectively, and*

$$\begin{aligned} R_{11} &= \sqrt{\mathrm{I} - R_{12} J_2 R_{12}^* J_1} = \sqrt{\mathrm{I} - J_1 R_{21}^* J_2 R_{21}}, \\ 0 &= R_{11}(J_1 R_{21}^* + R_{12} J_2) = (J_1 R_{21}^* + R_{12} J_2) R_{22}^*, \\ R_{22} &= \sqrt{\mathrm{I} - J_2 R_{12}^* J_1 R_{12}} = \sqrt{\mathrm{I} - R_{21} J_1 R_{21}^* J_2}. \end{aligned}$$

Proof. Since we have assumed that U_{11} has the J_1-2HSVD and U_{22} has the J_2-2HSVD, then according to [12, Theorem 5.4], U_{11} and U_{22} have the J_1- and the J_2-polar decomposition, respectively. This means that there exist a J_1-nonnegative matrix A, a J_2-nonnegative matrix D (note that both are Hermitian with regard to J_1 and J_2, respectively), a J_1-unitary matrix P and a J_2-unitary matrix Q such that

$$U_{11} = AP, \quad U_{22} = DQ. \tag{5}$$

If we define

$$B := U_{12}Q^{[*]} = U_{12}J_2Q^*J_2, \quad C := U_{21}P^{[*]} = U_{21}J_1P^*J_1,$$

then the following applies:

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} AP & BQ \\ CP & DQ \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & \\ & Q \end{bmatrix}.$$

From (5) and J-unitarity of U (i.e., $U^*JU = J$ and $UJU^* = J$) we have:

$$\begin{aligned}
A^*J_1A + C^*J_2C &= J_1, \\
A^*J_1B + C^*J_2D &= 0, \\
B^*J_1B + D^*J_2D &= J_2, \\
AJ_1A^* + BJ_2B^* &= J_1, && (6)\\
AJ_1C^* + BJ_2D^* &= 0, && (7)\\
CJ_1C^* + DJ_2D^* &= J_1,
\end{aligned}$$

From (6), we see that $J_1 - BJ_2B^* = AJ_1A^*$ and since A is J_1-nonnegative,

$$\mathrm{I} - BJ_2B^*J_1 = AJ_1A^*J_1 = AA^{[*]} = A^2 = (AJ_1)(J_1A).$$

The obvious question now is the existence of $\sqrt{\mathrm{I} - BJ_2B^*J_1}$ and whether it is equal to A or not. This is not as trivial as it may seem because, as explained in Section 1, X^2 need not have a square root for some matrices X.

Note that $A^2 = (AJ_1)(J_1A)$ is the product of two positive semidefinite matrices. Unfortunately, A^2 need not be positive semidefinite as it may be non-Hermitian. But, by [2, Fact 8.13.9], it is $\mathbb{R}$-diagonalizable with nonnegative eigenvalues and, since it is diagonalizable, zero is at most its non-defective eigenvalue. As said in the introduction, this means that A^2 has a square root, so $\sqrt{\mathrm{I} - BJ_2B^*J_1}$ exists.

Unfortunately, $\sqrt{\mathrm{I} - BJ_2B^*J_1}$ is rarely equal to A, unlike the traditional case, handled by Zakrajšek and Vidav, where A was positive definite and always equal to $\sqrt{\mathrm{I} - BJ_2B^*J_1}$.

To resolve this problem, we need to note that the existence of the 2HSVD of the matrix U_{11} and J-unitarity of P (from (5)) imply the existence of the 2HSVD for A as well. If $U_{11} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^{[*]}$ is the 2HSVD of U_{11}, then

$$A = U_{11}P^{[*]} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^{[*]}P^{[*]} = \widetilde{U}\Sigma(P\widetilde{V})^{[*]},$$

where $\widetilde{U}$, $\widetilde{V}$ and P are J_1-unitary, which shows that A has the 2HSVD. Since A is J_1-nonnegative, by [12, Theorem 5.2] A has the 2HSVD $A = V\Sigma J_1 V^{[*]}$, where Σ is nonnegative diagonal and V is J_1-unitary. In other words:

$$V\Sigma^2 V^{[*]} = (V\Sigma J_1 V^{[*]})(VJ_1\Sigma V^{[*]}) = A^2 = \mathrm{I} - BJ_2B^*J_1.$$

Since $V^{[*]} = V^{-1}$, we see that $V\Sigma V^{[*]} = \sqrt{A^2} = \sqrt{\mathrm{I} - BJ_2B^*J_1}$. Let us examine how A relates to $\sqrt{\mathrm{I} - BJ_2B^*J_1}$:

$$A = V\Sigma J_1 V^{[*]} = V\Sigma V^{[*]} VJ_1V^{[*]} = \sqrt{A^2}VJ_1V^{[*]} = \sqrt{\mathrm{I} - BJ_2B^*J_1}VJ_1V^{[*]}, \quad (8)$$

i.e.,

$$\sqrt{\mathrm{I} - BJ_2B^*J_1} = A(VJ_1V^{[*]})^{[*]} = AVJ_1V^{[*]}.$$

We can now define

$$R_{11} := A(VJ_1V^{[*]}), \quad \Delta_1 := (VJ_1V^{[*]})P. \quad (9)$$

Since V is J_1-unitary, $U_{11} = AP = R_{11}\Delta_1$. Analogously,

$$R_{22} := D(WJ_2W^{[*]}), \quad \Delta_2 := (WJ_2W^{[*]})Q, \quad R_{22}\Delta_2 = DQ = U_{22}, \quad (10)$$

where W is the J_2-unitary matrix from the 2HSVD of D (with the same argumentation as with A). The relations between B and R_{12}, as well as between C and R_{21}, should now be obvious:

$$R_{12} := BWJ_2W^{[*]}, \qquad R_{21} := CVJ_1V^{[*]}. \quad (11)$$

Since $WJ_2W^{[*]}$ and $VJ_1V^{[*]}$ are involutory, it follows from (11) that

$$B = R_{12}WJ_2W^{[*]}, \quad C = R_{21}VJ_1V^{[*]}. \quad (12)$$

From (12) it follows that

$$\begin{aligned} A &= \sqrt{\mathrm{I} - BJ_2B^*J_1}(VJ_1V^{[*]}) = \sqrt{\mathrm{I} - R_{12}J_2R_{12}^*J_1}(VJ_1V^{[*]}) \\ R_{11} &= AVJ_1V^{[*]} = \sqrt{\mathrm{I} - R_{12}J_2R_{12}^*J_1}. \end{aligned} \quad (13)$$

Analogously, we get:

$$\begin{aligned} R_{11} &= \sqrt{\mathrm{I} - J_1R_{21}^*J_2R_{21}}, \\ R_{22} &= \sqrt{\mathrm{I} - R_{21}J_1R_{21}^*J_2} = \sqrt{\mathrm{I} - J_2R_{12}^*J_1R_{12}}. \end{aligned} \quad (14)$$

Let us now find the relation between R_{12} and R_{21}. By substituting A and C from (8) and (12) and using involutority of $VJ_1V^{[*]}$, we see that

$$AJ_1C^* = R_{11}J_1R_{21}^*. \quad (15)$$

Reasoning in the same way, by using (10), (12) and involutority of $WJ_2W^{[*]}$, we get

$$BJ_2D^* = R_{12}J_2R_{22}^*. \quad (16)$$

Finally, using (7), (13)–(16) and Lemma 1 we get

$$\begin{aligned} 0 &= AJ_1C^* + BJ_2D^* = R_{11}J_1R_{21}^* + R_{12}J_2\sqrt{\mathrm{I} - R_{12}^*J_1R_{12}J_2} \\ &= R_{11}J_1R_{21}^* + \sqrt{\mathrm{I} - R_{12}J_2R_{12}^*J_1}R_{12}J_2 = R_{11}(J_1R_{21}^* + R_{12}J_2). \end{aligned} \tag{17}$$

Analogously:

$$0 = (J_1R_{21}^* + R_{12}J_2)R_{22}^*, \tag{18}$$

which completes the proof. □

The remainder of this section will be dedicated to solving

$$0 = R_{11}(J_1R_{21}^* + R_{12}J_2) \quad \text{or} \quad 0 = (J_1R_{21}^* + R_{12}J_2)R_{22}^*,$$

which is a much more difficult problem than it may seem. This part is quite troublesome in the positive definite case (i.e., in the Zakrajšek and Vidav's proof) as well, but it introduces even more problems in the hyperbolic case.

Theorem 2 trivially holds if either U_{11} or U_{22} is nonsingular:

Corollary 1. *Let J and U be given by (2). If U is a J-unitary matrix such that both $U_{11} \in \mathbb{C}^{n_1\times n_1}$ and $U_{22} \in \mathbb{C}^{n_2\times n_2}$ have the 2HSVD with regard to J_1 and J_2, respectively, and either U_{11} or U_{22} is nonsingular, U can be written in the form (3), such that (4) holds.*

Proof. From (5), (9) and (10), it is obvious that R_{11} and/or R_{22} are nonsingular if and only if U_{11} and/or U_{22} are non-singular, respectively.

If R_{11} is non-singular, then from it (17) follows that $J_1R_{21}^* + R_{12}J_2 = 0$. Since J_1 is unitary, $R_{21} = -J_2R_{12}^*J_1$. An analogous result follows from (18) if R_{22} is nonsingular. □

Now, we need to generalize the obtained results to matrices U with singular submatrices U_{11} and U_{22}. We shall do this by using the 2HSVD, which will lead us to the case of the real diagonal matrices on the block-diagonals of U (presented in the next two lemmas).

Unlike the Euclidian case of the SVD, the 2HSVD does not provide a "sorted" diagonal (which we assume to have in the following lemma). This is discussed in [12].

Fortunately, this problem can be corrected with a simple permutation, as described later, in Lemma 4. So, first we shall assume that the upper-left diagonal block is partly "sorted", i.e., it has all the diagonal zeroes grouped in the bottom right block (middle block in the matrix U).

Lemma 3. *If $J = \mathrm{diag}(\pm 1)$ and U is J-unitary,*

$$U = \begin{bmatrix} U_{11} & 0 & U_{13} \\ 0 & 0 & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix}, \quad J = J_1 \oplus J_2 \oplus J_3,$$

such that U_{11} is nonsingular real diagonal, U_{33} is (possibly, but not necessarily, singular) real diagonal and orders of the blocks of J match those of U, then U can be written in the form (3), such that (4) holds.

Proof. Obviously, diagonal matrices U_{11} and U_{33} always have trivial 2HSVD and $\hat{U}_{11} = \hat{U}_{11}\hat{\Delta}_{11}$, where

$$\hat{U}_{11} := \begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\Delta}_{11} := \begin{bmatrix} \mathrm{I} & 0 \\ 0 & X \end{bmatrix},$$

and X is an arbitrary matrix of the same order as U_{22} and J_2. We need $\widehat{\Delta}_{11}$ to be $\widehat{J}_1$-unitary, where $\widehat{J}_1 := J_1 \oplus J_2$. For $\widehat{\Delta}_{11}$ to be $\widehat{J}_1$-unitary, the following must hold: $\widehat{\Delta}_{11}^* \widehat{J}_1 \widehat{\Delta}_{11} = \widehat{J}_1$, which yields that X has to be J_2-unitary.

We can divide U into four blocks:

$$U = \begin{bmatrix} \widehat{U}_{11} & \widehat{U}_{12} \\ \widehat{U}_{21} & \widehat{U}_{22} \end{bmatrix}, \tag{19}$$

where

$$\widehat{U}_{12} = \begin{bmatrix} U_{13} \\ U_{23} \end{bmatrix}, \quad \widehat{U}_{21} = \begin{bmatrix} U_{31} & U_{32} \end{bmatrix}, \quad \widehat{U}_{22} = U_{33}. \tag{20}$$

We shall also define $\hat{J}_2 := J_3$. All this, along with Lemma 2, yields the following:

$$\begin{aligned} U &= R\Delta, \qquad (21) \\ \widehat{\Delta}_{22} &= \mathrm{I}, \\ R_{11} &= \widehat{U}_{11} = \sqrt{\mathrm{I} - R_{12}J_2R_{12}^*J_1} = \sqrt{\mathrm{I} - J_1R_{21}^*J_2R_{21}}, \\ 0 &= R_{11}(J_1R_{21}^* + R_{12}J_2) = (J_1R_{21}^* + R_{12}J_2)R_{22}^*, \\ R_{22} &= \widehat{U}_{22} = U_{33} = \sqrt{\mathrm{I} - J_2R_{12}^*J_1R_{12}} = \sqrt{\mathrm{I} - R_{21}J_1R_{21}^*J_2}, \end{aligned}$$

where R_{ij} are blocks of R and $\widehat{\Delta}_{kk}$ are diagonal $\hat{J}_k$-unitary blocks of Δ, as described in Lemma 2.

We want to prove that we can choose X such that $R_{21} = -\widehat{J}_2R_{12}^*\widehat{J}_1$. Let

$$R_{12} = \begin{bmatrix} R_{12}^{(1)} \\ R_{12}^{(2)} \end{bmatrix}, \quad R_{21} = \begin{bmatrix} R_{21}^{(1)} & R_{21}^{(2)} \end{bmatrix},$$

where the dimensions of $R_{12}^{(1)}$, $R_{12}^{(2)}$, $R_{21}^{(1)}$ and $R_{21}^{(2)}$ are such that

$$\begin{bmatrix} R_{11} & 0 & R_{12}^{(1)} \\ 0 & 0 & R_{12}^{(2)} \\ R_{21}^{(1)} & R_{21}^{(2)} & R_{33} \end{bmatrix}$$

is a block matrix with the block sizes matching those in (19) and (20). From (21), it easily follows that

$$\begin{bmatrix} U_{11} & 0 & U_{13} \\ 0 & 0 & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} R_{11} & 0 & R_{12}^{(1)} \\ 0 & 0 & R_{12}^{(2)} \\ R_{21}^{(1)} & R_{21}^{(2)}X & R_{33} \end{bmatrix},$$

i.e.,

$$U_{13} = R_{12}^{(1)}, \quad U_{23} = R_{12}^{(2)}, \quad U_{31} = R_{21}^{(1)}, \quad U_{32} = R_{21}^{(2)} X. \tag{22}$$

Since X is an arbitrary J_2-unitary matrix, we see that

$$R_{21}^{(2)} J_2 = R_{21}^{(2)} X J_2 X^* = U_{32} J_2 X^*$$

or, in a more appropriate form,

$$J_2 (R_{21}^{(2)})^* = X J_2 U_{32}^*. \tag{23}$$

From $A(\widehat{J}_1 C^* + B\widehat{J}_2) = 0$ in Lemma 2 it follows that $U_{11}(J_1(R_{21}^{(1)})^* + R_{12}^{(1)} J_3) = 0$. Since U_{11} is nonsingular, we see that $J_1(R_{21}^{(1)})^* + R_{12}^{(1)} J_3 = 0$, i.e.,

$$J_1 U_{31}^* + U_{13} J_3 = 0. \tag{24}$$

Note that from (24) we easily get the following relations which shall be used later in this proof:

$$U_{31} J_1 U_{31}^* = -J_3 U_{13}^* U_{31}^*, \tag{25}$$

$$-U_{13}^* J_1 = J_3 U_{31}. \tag{26}$$

We still need to show that

$$J_2 (R_{21}^{(2)})^* + R_{12}^{(2)} J_3 = 0. \tag{27}$$

From J-unitarity of U, i.e., $U^* J U = U J U^* = J$, follows that

$$0 = U_{32}^* J_3 U_{31}, \tag{28}$$

$$J_2 = U_{32}^* J_3 U_{32}, \tag{29}$$

$$0 = U_{32}^* J_3 U_{33}, \tag{30}$$

$$J_3 = U_{13}^* J_1 U_{13} + U_{23}^* J_2 U_{23} + U_{33}^* J_3 U_{33}, \tag{31}$$

$$0 = U_{23} J_3 U_{13}^*, \tag{32}$$

$$0 = U_{23} J_3 U_{33}^*, \tag{33}$$

$$J_3 = U_{31} J_1 U_{31}^* + U_{32} J_2 U_{32}^* + U_{33} J_3 U_{33}^*, \tag{34}$$

$$J_2 = U_{23} J_3 U_{23}^*. \tag{35}$$

We now construct a J_2-unitary X that satisfies (27), by using (22) and (23). Since

$$0 = J_2 (R_{21}^{(2)})^* + R_{12}^{(2)} J_3 = X J_2 U_{32}^* + U_{23} J_3 = X J_2 (U_{32}^* + X^* J_2 U_{23} J_3),$$

and $X J_2$ is non-singular, we want $X^* J_2 U_{23} J_3 = -U_{32}^*$, which, using (35), leads to

$$-U_{32}^* U_{23}^* = X^* J_2 U_{23} J_3 U_{23}^* = X^*.$$

Therefore, we define:

$$X := -U_{23} U_{32}. \tag{36}$$

Note that this was just a construction of a good candidate X to satisfy (27). Since U_{23} can be singular (even non-square!), we have to check that the obtained X satisfies (27) and that it is J_2-unitary. Using (34), let us first show that X defined by (36), satisfies (27):

$$\begin{aligned} J_2(R_{21}^{(2)})^* + R_{12}^{(2)} J_3 &= X J_2 U_{32}^* + U_{23} J_3 = -U_{23} U_{32} J_2 U_{32}^* + U_{23} J_3 \\ &= U_{23}(U_{31} J_1 U_{31}^* + U_{33} J_3 U_{33}^*). \end{aligned} \tag{37}$$

Since we have assumed that U_{33} and J_3 are real diagonal, they are Hermitian and their product commutes, so

$$U_{33}^* J_3 = U_{33} J_3 = J_3 U_{33}. \tag{38}$$

Finally, from (37), using (25), (32), (33) and (38), we get:

$$\begin{aligned} J_2(R_{21}^{(2)})^* + R_{12}^{(2)} J_3 &= U_{23}(-J_3 U_{13}^* U_{31}^* + J_3 U_{33} U_{33}^*) \\ &= -U_{23} J_3 U_{13}^* U_{31}^* + U_{23} J_3 U_{33}^* U_{33} = -0 \cdot U_{31}^* + 0 \cdot U_{33} = 0. \end{aligned}$$

This proves that (27) holds for X defined by (36). The only thing that remains to be shown is that X is J_2-unitary. To show this, we shall use (26), (28)–(31) and (38):

$$\begin{aligned} X^* J_2 X &= U_{32}^* U_{23}^* J_2 U_{23} U_{32} = U_{32}^*(J_3 - U_{13}^* J_1 U_{13} - U_{33}^* J_3 U_{33}) U_{32} \\ &= U_{32}^* J_3 U_{32} - U_{32}^* U_{13}^* J_1 U_{13} U_{32} - U_{32}^* U_{33}^* J_3 U_{33} U_{32} \\ &= J_2 + U_{32}^* J_3 U_{31} U_{13} U_{32} - U_{32}^* J_3 U_{33} U_{33} U_{32} \\ &= J_2 + 0 \cdot U_{13} U_{32} + 0 \cdot U_{33} U_{32} = J_2. \end{aligned}$$

□

As we have announced, the previous lemma is not enough to complete the proof of Theorem 2, because the 2HSVD does not group diagonal zeroes in Σ. This is only a minor inconvenience, as can be seen from the following lemma.

Lemma 4. *If* $J = \operatorname{diag}(\pm 1)$ *and* U *is* J*-unitary,*

$$U = \begin{bmatrix} U_{11} & 0 & \cdots & 0 & U_{1,k+1} \\ 0 & U_{22} & \cdots & 0 & U_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & U_{kk} & U_{k,k+1} \\ U_{k+1,1} & U_{k+1,2} & \cdots & U_{k+1,k} & U_{k+1,k+1} \end{bmatrix},$$

such that each U_{ii} $(1 \le i \le k)$ *is either a nonsingular real diagonal or a zero matrix and* $U_{k+1,k+1}$ *is real diagonal, then* U *can be written in the form (3), such that (4) holds.*

Proof. Obviously, there exists a permutation $\widehat{S}$ that groups zeroes and non-zeroes in $U[1:k,1:k]$ (diagonal submatrix defined by blocks U_{11} through U_{kk}). In other words,

$$\widehat{U} := S^{-1}US, \quad \text{where } S := \operatorname{diag}(\widehat{S}, \mathrm{I}),$$

has the form as in Lemma 3 which we can apply to the matrix $\widehat{U}$ and the indefinite product defined by $\widehat{J} = S^*JS$. In other words, $\hat{U} = \hat{R}\hat{\Delta}$,

$$\hat{U} := \begin{bmatrix} \widehat{U}_{11} & \widehat{U}_{12} \\ \widehat{U}_{21} & \widehat{U}_{22} \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} \widehat{R}_{11} & \widehat{R}_{12} \\ \widehat{R}_{21} & \widehat{R}_{22} \end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix} \widehat{\Delta}_1 & 0 \\ 0 & \widehat{\Delta}_2 \end{bmatrix},$$

where $\widehat{R}_{11}$, $\widehat{R}_{21}$ and $\widehat{R}_{22}$ are as in (4), and $\widehat{\Delta}_1$ is $\widehat{S}^*J_1\widehat{S}$-unitary, while $\widehat{\Delta}_2$ is J_2-unitary. Since $U = S\widehat{U}S^*$, we have:

$$U = S\hat{R}\hat{\Delta}S^* = \begin{bmatrix} \widehat{S}\widehat{R}_{11}\widehat{S}^* & \widehat{S}\widehat{R}_{12} \\ \widehat{R}_{21}\widehat{S}^* & \widehat{R}_{22} \end{bmatrix} \cdot \begin{bmatrix} \widehat{S}\widehat{\Delta}_1\widehat{S}^* & 0 \\ 0 & \widehat{\Delta}_2 \end{bmatrix}.$$

Note that

$$(\widehat{S}\widehat{\Delta}_1\widehat{S}^*)^*J_1(\widehat{S}\widehat{\Delta}_1\widehat{S}^*) = \widehat{S}\widehat{\Delta}_1^*(\widehat{S}^*J_1\widehat{S})\widehat{\Delta}_1\widehat{S}^* = \widehat{S}(\widehat{S}^*J_1\widehat{S})\widehat{S}^* = J_1,$$

which means that $\widehat{S}\widehat{\Delta}_1\widehat{S}^*$ is J_1-unitary. Also, if we define $R_{12} := \widehat{S}\widehat{R}_{12}$ and R_{11}, R_{21} and R_{22} accordingly (as in (4)), then:

$$\begin{aligned} R_{21} &= -J_2R_{12}^*J_1 = -J_2\widehat{R}_{12}^*\widehat{S}^*J_1\widehat{S}\widehat{S}^* = -\widehat{J}_2\widehat{R}_{12}^*\widehat{J}_1\widehat{S}^* = \widehat{R}_{21}\widehat{S}^*, \\ R_{11} &= \sqrt{\mathrm{I} + R_{12}R_{21}} = \sqrt{\mathrm{I} + \widehat{S}\widehat{R}_{12}\widehat{R}_{21}\widehat{S}^*} = \widehat{S}\sqrt{\mathrm{I} + \widehat{R}_{12}\widehat{R}_{21}}\widehat{S}^* = \widehat{S}\widehat{R}_{11}\widehat{S}^*, \\ R_{22} &= \sqrt{\mathrm{I} + R_{21}R_{12}} = \sqrt{\mathrm{I} + \widehat{R}_{21}\widehat{S}^*\widehat{S}\widehat{R}_{12}} = \sqrt{\mathrm{I} + \widehat{R}_{21}\widehat{R}_{12}} = \widehat{R}_{22}. \end{aligned}$$

In other words, we have

$$U = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \cdot \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix},$$

which completes the proof of this Lemma. □

Using the results from Lemma 2, Corollary 1, Lemma 3 and Lemma 4, we can now prove Theorem 2, using the 2HSVD:

Proof of Theorem 2. If either U_{11} or U_{22} is nonsingular, then the statement of Theorem 2 follows directly from Lemma 2 and Corollary 1.

Let us assume that both U_{11} and U_{22} are singular. By the assumption of the Theorem, we can decompose both of them using the 2HSVD:

$$U_{11} = V_1\Sigma_1W_1^{-1}, \quad U_{22} = V_2\Sigma_2W_2^{-1},$$

where V_1 and W_1 are J_1-unitary, V_2 and W_2 are J_2-unitary and Σ_1 and Σ_2 are real diagonal. By applying the 2HSVD and Lemma 4, we obtain the following:

$$\begin{aligned} U &= \begin{bmatrix} V_1\Sigma_1 W_1 & U_{12} \\ U_{21} & V_2\Sigma_2 W_2 \end{bmatrix} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \cdot \begin{bmatrix} \Sigma_1 & V_1^{-1}U_{12}W_2^{-1} \\ V_2^{-1}U_{21}W_1^{-1} & \Sigma_2 \end{bmatrix} \cdot \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \\ &= \begin{bmatrix} V_1\sqrt{\mathrm{I} + \widehat{R}_{12}J_2\widehat{R}_{12}^* J_1} & V_1\widehat{R}_{12} \\ -V_2 J_2 \widehat{R}_{12}^* J_1 & V_2\sqrt{\mathrm{I} + J_2\widehat{R}_{12}^* J_1 \widehat{R}_{12}} \end{bmatrix} \cdot \begin{bmatrix} \widehat{\Delta}_1 W_1 & 0 \\ 0 & \widehat{\Delta}_2 W_2 \end{bmatrix}. \end{aligned} \tag{39}$$

Note that the right matrix in (39) can be expanded in the following way:

$$\begin{bmatrix} \widehat{\Delta}_1 W_1 & 0 \\ 0 & \widehat{\Delta}_2 W_2 \end{bmatrix} = \begin{bmatrix} J_1 V_1^* J_1 & 0 \\ 0 & J_2 V_2^* J_2 \end{bmatrix} \cdot \begin{bmatrix} V_1\widehat{\Delta}_1 W_1 & 0 \\ 0 & V_2\widehat{\Delta}_2 W_2 \end{bmatrix}. \tag{40}$$

Let $R_{12} := V_1\widehat{R}_{12}J_2V_2^*J_2$ and $R_{21} := -V_2J_2\widehat{R}_{12}^*V_1^*J_1$. It is easy to see that the following holds:

$$R_{21} = -V_2J_2\widehat{R}_{12}^*V_1^*J_1 = -J_2(J_2V_2J_2\widehat{R}_{12}^*V_1^*)J_1 = -J_2R_{12}^*J_1.$$

We define: $\Delta_1 := V_1\widehat{\Delta}_1W_1$ and $\Delta_2 := V_2\widehat{\Delta}_2W_2$. Obviously, Δ_1 is J_1- and Δ_2 is J_2-unitary (since V_1, $\widehat{\Delta}_1$ and W_1 are J_1- and V_2, $\widehat{\Delta}_2$ and W_2 are J_2-unitary).

Using Lemma 1, (39), (40) and described substitutions, we obtain:

$$U = \begin{bmatrix} \sqrt{\mathrm{I} - R_{12}J_2R_{12}^*J_1} & R_{12} \\ -J_2R_{12}^*J_1 & \sqrt{\mathrm{I} - J_2R_{12}^*J_1R_{12}} \end{bmatrix} \cdot \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix},$$

which completes the proof. □

The ZV decomposition of the traditional unitary matrices is a simple corollary of Theorem 2.

Corollary 2 (Zakrajšek-Vidav)**.** *Every unitary matrix U partitioned as in (2) can be written in the form (3) such that*

$$R_{11} = \sqrt{\mathrm{I} - R_{12}R_{12}^*}, \quad R_{21} = -R_{12}^*, \quad R_{22} = \sqrt{\mathrm{I} - R_{12}^*R_{12}}.$$

Proof. We use $J = \mathrm{I} \oplus \mathrm{I}$ in Theorem 2. Note that the 2HSVD with regard to the scalar product induced by I is actually a traditional SVD and therefore it always exists. □

The following simple corollary shall be used in the next section.

Corollary 3. *Let $J = \mathrm{I} \oplus (-\mathrm{I})$ and J-unitary U be given by (2). Then U can be written in the form (3) such that*

$$R_{11} = \sqrt{\mathrm{I} + R_{12}R_{12}^*}, \quad R_{21} = R_{12}^*, \quad R_{22} = \sqrt{\mathrm{I} + R_{12}^*R_{12}}. \tag{41}$$

Proof. Again, it is enough to note that the 2HSVD with regard to the scalar product induced by I or −I is actually a traditional SVD and therefore it always exists, so we can apply Theorem 2. □

There is a fairly simple proof of the hyperbolic CS decomposition [7, Theorem 3.2] which uses the previous corollary. We shall state the theorem a bit differently from the original (using Σ instead of $-S$), but this is only a matter of a simple multiplication from the left and the right with the matrix $\operatorname{diag}(\mathrm{I}, -\mathrm{I})$ which is J-unitary for every hyperbolic J.

Theorem 3 (Hyperbolic CS decomposition)**.** *Let $J_1 = \mathrm{I}_p$, $J_2 = -\mathrm{I}_q$, $p+q = n$, and J-unitary U be given by (2), such that the order of U_{11} is p and the order of U_{22} is q.*

If $p \le q$, U can be written in the form

$$U = VRW^* = \left[\begin{array}{c|c} V_1 & \\ \hline & V_2 \end{array}\right] \left[\begin{array}{c|cc} \Gamma & \Sigma & 0 \\ \hline \Sigma & \Gamma & 0 \\ 0 & 0 & \mathrm{I}_{n-2p} \end{array}\right] \left[\begin{array}{c|c} W_1^* & \\ \hline & W_2^* \end{array}\right], \tag{42}$$

where V_1 and W_1 are unitary matrices of order p, V_2 and W_2 are unitary matrices of order q and Γ, Σ are diagonal matrices of order p such that $\Gamma^2 - \Sigma^2 = \mathrm{I}_p$.

If $p \ge q$, U can be written in the form

$$U = VRW^* = \left[\begin{array}{c|c} V_1 & \\ \hline & V_2 \end{array}\right] \left[\begin{array}{cc|c} \Gamma & 0 & \Sigma \\ 0 & \mathrm{I}_{n-2q} & 0 \\ \hline \Sigma & 0 & \Gamma \end{array}\right] \left[\begin{array}{c|c} W_1^* & \\ \hline & W_2^* \end{array}\right],$$

where V_1 and W_1 are unitary matrices of order p, V_2 and W_2 are unitary matrices of order q and Γ, Σ are diagonal matrices of order q such that $\Gamma^2 - \Sigma^2 = \mathrm{I}_q$.

Proof. Let us consider the case where $p \le q$, as the other one is very similar. By Corollary 3, there exists a hyperbolic ZV decomposition $U = R\Delta$ as in (3) such that (41) holds.

Note that J_1-unitary and J_2-unitary matrices are traditional unitary matrices of orders p and q, respectively. That means that R_{12} has a traditional SVD $R_{12} = V_1[\Sigma\ 0]V_2^*$, where Σ is a non-negative real diagonal matrix of order p. We now see that:

$$\begin{aligned} R_{11} &= \sqrt{\mathrm{I} + R_{12}R_{12}^*} = V_1\sqrt{\mathrm{I} + \Sigma^2}V_1^*, \\ R_{21} &= R_{12}^* = V_2 \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V_1^*, \\ R_{22} &= \sqrt{\mathrm{I} + R_{12}^*R_{12}} = V_2\sqrt{\mathrm{I} + \operatorname{diag}(\Sigma, 0)^2}V_2^*. \end{aligned}$$

If we denote $V = \operatorname{diag}(V_1, V_2)$, $W = \Delta^* V$ and $\Gamma = \sqrt{\mathrm{I}_p - \Sigma}$, we get the form of U as in (42). □

Note that the converse to the previous theorem is also true, i.e., it would be easy to prove a hyperbolic ZV decomposition from the hyperbolic CS decomposition, following steps similar to the proof of Theorem 1, but only for the case $J = \pm\operatorname{diag}(\mathrm{I}_p, -\mathrm{I}_q)$, for which the hyperbolic CS decomposition is known.

Let us now see an example of the described decomposition.

Example 1. *Let* $J_1 = J_2 = \operatorname{diag}(1, -1)$*,* $J = \operatorname{diag}(J_1, J_2)$ *and*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \left[\begin{array}{rr|rr} 0.924692 & -0.0324821 & 0.383786 & -0.0359435 \\ -0.130595 & 1.03105 & 0.222018 & 0.0573754 \\ \hline -0.419525 & 0.249001 & 0.968408 & 0.227629 \\ -0.118328 & -0.00127776 & 0.189249 & 1.0246 \end{array}\right].$$

A simple multiplication shows that $U^{[*]}U = \mathrm{I}$*, so* U *is* J*-unitary. Furthermore, the spectrum of* $U_{11}^{[*]_{J_1}}U_{11}$ *and* $U_{22}^{[*]_{J_2}}U_{22}$ *is*

$$\sigma(U_{11}^{[*]_{J_1}}U_{11}) = \sigma(U_{22}^{[*]_{J_2}}U_{22}) = \{0.91, 0.99\},$$

so $U_{11}^{[*]_{J_1}}U_{11}$ *and* $U_{22}^{[*]_{J_2}}U_{22}$ *are* $\mathbb{R}_0^+$*-diagonalizable, so* U_{11} *and* U_{22} *have the 2HSVD with regard to* J_1 *and* J_2*, respectively.*

Since these matrices are of order 2*, finding their 2HSVDs can be done directly, with no iterations. We now get:*

$$\begin{aligned} U_{11} &= V_{11}\Sigma_{11}W_{11}^{[*]_{J_1}} \\ &= \begin{bmatrix} 1.30384 & 0.83666 \\ 0.83666 & 1.30384 \end{bmatrix} \begin{bmatrix} 0.953939 & \\ & 0.994987 \end{bmatrix} \begin{bmatrix} 1.3784 & 0.948683 \\ 0.948683 & 1.3784 \end{bmatrix}^{[*]_{J_1}}, \\ U_{22} &= V_{22}\Sigma_{22}W_{22}^{[*]_{J_2}} \\ &= \begin{bmatrix} 1.14018 & 0.547723 \\ 0.547723 & 1.14018 \end{bmatrix} \begin{bmatrix} 0.953939 & \\ & 0.994987 \end{bmatrix} \begin{bmatrix} 1.04881 & 0.316228 \\ 0.316228 & 1.04881 \end{bmatrix}^{[*]_{J_2}}. \end{aligned}$$

Finally, we get the desired decomposition:

$$\begin{aligned} \Delta_1 &= V_{11}W_{11}^{[*]_{J_1}} = \begin{bmatrix} 1.00349 & -0.0836754 \\ -0.0836754 & 1.00349 \end{bmatrix}, \\ \Delta_2 &= V_{22}W_{22}^{[*]_{J_2}} = \begin{bmatrix} 1.02262 & 0.213901 \\ 0.213901 & 1.02262 \end{bmatrix}, \\ R_{12} &= U_{12}\Delta_2^{[*]_{J_1}} = \begin{bmatrix} 0.400156 & -0.118849 \\ 0.214767 & 0.0111834 \end{bmatrix}, \end{aligned}$$

$$R_{21} = -J_2 R_{12}^* J_1 = \begin{bmatrix} -0.400156 & 0.214767 \\ -0.118849 & -0.0111834 \end{bmatrix},$$

$$R_{11} = \sqrt{\mathrm{I} + R_{12}R_{21}} = \begin{bmatrix} 0.925205 & 0.0447783 \\ -0.0447783 & 1.02372 \end{bmatrix},$$

$$R_{22} = \sqrt{\mathrm{I} + R_{21}R_{12}} = \begin{bmatrix} 0.941625 & 0.0256346 \\ -0.0256346 & 1.0073 \end{bmatrix}.$$

As we can see, the absolute error is within the machine precision:

$$R_{11}^2 - (\mathrm{I} + R_{12}R_{21}) = \begin{bmatrix} -1.11022 \cdot 10^{-16} & -8.32667 \cdot 10^{-17} \\ -3.46945 \cdot 10^{-16} & -4.44089 \cdot 10^{-16} \end{bmatrix},$$

$$R_{22}^2 - (\mathrm{I} + R_{21}R_{12}) = \begin{bmatrix} -2.22045 \cdot 10^{-16} & -5.55112 \cdot 10^{-17} \\ 5.55112 \cdot 10^{-17} & -2.22045 \cdot 10^{-16} \end{bmatrix},$$

$$U - R\Delta = \begin{bmatrix} 2.22045 \cdot 10^{-16} & -9.71445 \cdot 10^{-17} & 5.55112 \cdot 10^{-17} & 0 \\ 1.38778 \cdot 10^{-16} & 0 & 5.55112 \cdot 10^{-17} & -6.93889 \cdot 10^{-18} \\ -5.55112 \cdot 10^{-17} & 1.11022 \cdot 10^{-16} & 1.11022 \cdot 10^{-16} & 1.66533 \cdot 10^{-16} \\ 0 & 6.76542 \cdot 10^{-17} & -8.32667 \cdot 10^{-17} & 4.44089 \cdot 10^{-16} \end{bmatrix}.$$

4. An application

The described decomposition can help us use block transformations in other decompositions, which can make their computation faster. In this section we shall present a general idea through the example of calculating an indefinite QR, proposed by Singer in [10]. At the end of the section, we shall provide an example of such calculation.

The indefinite QR states that a matrix $\hat{G}$, such that $\hat{G}^*\hat{J}\hat{G}$ is nonsingular, can be decomposed into

$$\hat{G} = P_1 G P_2^* = P_1 Q G' P_2^*, \quad G' = \begin{bmatrix} G_1' \\ 0 \end{bmatrix}, \quad Q^* J Q = J, \quad J = P_1^* \hat{J} P_1, \tag{43}$$

where P_1 and P_2 are permutation matrices, matrix Q is J-unitary and G_1' is block upper triangular with diagonal blocks of order 1 and 2.

We assume that $\hat{J} = \mathrm{diag}(\pm 1)$ of order m and $\hat{G} \in \mathbb{C}^{m\times k}$ are given, such that $\hat{G}^*\hat{J}\hat{G}$ is nonsingular. Our aim is to find Q for the indefinite QR. Note that the indefinite QR, as described in [10], is calculated by the one- and two-column annihilations, which would lead to the special case $k = 1$ or $k = 2$. However, we consider a general case, for the annihilation of any number of columns k. The choice of k can then depend on the specific application (i.e., on the properties of some specific matrix $\hat{G}$) and on the available methods of calculation.

Permutations P_1 and P_2 are used for the preparation of $\hat{G}$ for the annihilation of the block G_2. For the better part of this section, we shall deal with already permuted G and J.

From Theorem 2, we see that (under certain conditions) there exist a block J-rotation $B(X)$ (denoted R in the theorem), where X is the top right block of $B(X)$ (denoted R_{12} in the theorem), and a block-diagonal Δ such that $Q = B(X)\Delta$. In other words,

$$G' = \Delta^{-1} B(X)^{-1} G. \tag{44}$$

Note that $B(X)^{-1} = B(-X)$. From (4) and (44) it follows that

$$J_2 X^* J_1 G_1 + \sqrt{\mathrm{I} - J_2 X^* J_1 X}\, G_2 = 0. \tag{45}$$

The following two forms of (45) will be more suitable for solving:

$$\begin{aligned} (\mathrm{I} - J_2 X^* J_1 X)^{-1/2} J_2 X^* J_1 &= -G_2 G_1^{-1}, \\ X(\mathrm{I} - J_2 X^* J_1 X)^{-1/2} &= -J_1 (G_2 G_1^{-1})^* J_2. \end{aligned}$$

These two represent the tangent-like substitutions from [13, formula (13)], where $y = X$ and $x = -J_2 X^* J_1$. So, we define

$$T := t := -G_2 G_1^{-1}, \quad u := J_1 T^* J_2. \tag{46}$$

The solution of equation (45) is obtained by using [13, formula (14)]:

$$\begin{aligned} x &= t(\mathrm{I} + ut)^{-1/2} = T(\mathrm{I} + J_1 T^* J_2 T)^{-1/2}, \\ y &= u(\mathrm{I} + tu)^{-1/2} = J_1 T^* J_2 (\mathrm{I} + T J_1 T^* J_2)^{-1/2}. \end{aligned}$$

Substituting x and y into [13, formula (5)], we see that

$$\begin{aligned} B(X) &= \begin{bmatrix} \sqrt{\mathrm{I} - yx} & y \\ -x & \sqrt{\mathrm{I} - xy} \end{bmatrix} = \begin{bmatrix} (\mathrm{I} + ut)^{-1/2} & u(\mathrm{I} + tu)^{-1/2} \\ -t(\mathrm{I} + ut)^{-1/2} & (\mathrm{I} + tu)^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} (\mathrm{I} + J_1 T^* J_2 T)^{-1/2} & J_1 T^* J_2 (\mathrm{I} + T J_1 T^* J_2)^{-1/2} \\ -T(\mathrm{I} + J_1 T^* J_2 T)^{-1/2} & (\mathrm{I} + T J_1 T^* J_2)^{-1/2} \end{bmatrix}. \end{aligned} \tag{47}$$

Calculating the square root can be done via the Schur decomposition. Since $J_1 T^* J_2 T$ and $T J_1 T^* J_2$ have the rank at most k, which is usually very limited (i.e., $k = 2$), the triangular factors of $\mathrm{I} + J_1 T^* J_2 T$ and $\mathrm{I} + T J_1 T^* J_2$ have the form $\left[\begin{smallmatrix} * & * \\ 0 & \mathrm{I} \end{smallmatrix}\right]$ (with $*$ having k rows), so their square roots are calculated in the fast and stable manner. For details see [8, Section 6.2]. The triangular form also simplifies the calculation of the inverse needed for the calculation of the diagonal blocks in $B(X)$.

Note that we can also improve the calculation of the counter-diagonal blocks of $B(X)$. It is easy to see that

$$x = t(\mathrm{I} + ut)^{-1/2} = (\mathrm{I} + tu)^{-1/2} t, \quad y = u(\mathrm{I} + tu)^{-1/2} = (\mathrm{I} + ut)^{-1/2} u.$$

In other words, we can choose the appropriate formulas to calculate those blocks, depending on the dimensions of t and u, i.e., depending on the choice of k.

If permutation P_1 is chosen in a way that $J_1 = \pm\mathrm{I}$ (note: J_2 does not have to be equal to $\mp\mathrm{I}$), after $B(-X)G$ is calculated we can easily apply the traditional QR on its top $k \times k$ block, thus obtaining Δ_1, while Δ_2 is not needed for the described annihilation. In other words, when such choice of P_1 is possible, we can avoid getting an irreducible 2×2 block in G', while still performing our calculations in a block-wise manner.

Note that if U is J-unitary, then U^{-1} is also J-unitary. This means that, instead of (44), we could have looked for $\hat{X}$ and $\hat{\Delta}$ such that $G' = B(\hat{X})\hat{\Delta}G$. This might help in terms of performance and/or stability, but would not make the calculation of $B(\hat{X})$ possible when $B(X)$ does not exist. Obviously, instead of "tangent" $T = -G_2G_1^{-1}$, we would have $\hat{T} = -\hat{\Delta}_2G_2G_1^{-1}\hat{\Delta}_1^{-1}$. To calculate $B(\hat{X})$, we would still need square roots and inverses of $\mathrm{I} + \hat{T}J_1\hat{T}^*J_2$ and $\mathrm{I} + J_1\hat{T}^*J_2\hat{T}$. It is easy to see that

$$\mathrm{I} + \hat{T}J_1\hat{T}^*J_2 = \Delta_2(\mathrm{I} + TJ_1T^*J_2)\Delta_2^{-1}, \quad \mathrm{I} + J_1\hat{T}^*J_2\hat{T} = \Delta_1(\mathrm{I} + TJ_1T^*J_2)\Delta_1^{-1}.$$

These are the similarity relations, which means that $B(X)$ exists if and only if $B(\hat{X})$ exists.

Let us now consider an example of calculating the indefinite QR of a 6×2 real matrix, using the formulas derived in this section.

Example 2. *Let* $J_1 = \mathrm{diag}(1,-1)$, $J_2 = \mathrm{diag}(1,-1,1,-1)$, $J = J_1 \oplus J_2$ *and*

$$G_1 = \begin{bmatrix} -2.08529 & -1.11372 \\ -0.886879 & -1.08692 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -0.12679 & -0.361292 \\ -2.75626 & 2.82151 \\ -2.87042 & 2.83256 \\ -2.12738 & 1.43712 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}.$$

From (46), we see that

$$T = t = G_2G_1^{-1} = \begin{bmatrix} 0.142797 & -0.478716 \\ -4.29941 & 7.00127 \\ -4.4041 & 7.1187 \\ -2.8048 & 4.19613 \end{bmatrix}, \quad u^\tau = \begin{bmatrix} 0.142797 & 0.478716 \\ 4.29941 & 7.00127 \\ -4.4041 & -7.1187 \\ 2.8048 & 4.19613 \end{bmatrix}.$$

Using (47), we get

$$B(X) = \begin{bmatrix} 1.52166 & -0.799964 & -0.165667 & 0.941495 & -1.00685 & 0.911208 \\ 0.799964 & -0.21252 & 0.0124955 & 1.95147 & -2.01026 & 1.35198 \\ 0.165667 & 0.0124955 & 0.99318 & 0.16511 & -0.171781 & 0.126268 \\ 0.941495 & -1.95147 & -0.16511 & -1.27195 & 2.30796 & -1.34695 \\ 1.00685 & -2.01026 & -0.171781 & -2.30796 & 3.34366 & -1.36218 \\ 0.911208 & -1.35198 & -0.126268 & -1.34695 & 1.36218 & 0.244255 \end{bmatrix}.$$

Once again, the absolute error is within the machine precision:

$$B(X)^{-1}G = B(-X)G = \mathrm{diag}(\mathrm{I}_2, -\mathrm{I}_4)B(X)\,\mathrm{diag}(\mathrm{I}_2, -\mathrm{I}_4)G$$
$$= \begin{bmatrix} -0.841248 & -1.99904 \\ 1.00654 & -2.41029 \\ -8.32667 \cdot 10^{-16} & 1.13798 \cdot 10^{-15} \\ 1.64313 \cdot 10^{-14} & -1.26565 \cdot 10^{-14} \\ 1.64313 \cdot 10^{-14} & -1.11022 \cdot 10^{-14} \\ 9.4369 \cdot 10^{-15} & -6.93889 \cdot 10^{-15} \end{bmatrix}.$$

5. Conclusion

In this paper we have presented a hyperbolic generalization of Zakrajšek and Vidav's decomposition of block-unitary matrices. We have shown that this decomposition has a theoretical value as the tool for deriving other decompositions (i.e., the Euclidean ZV and the hyperbolic CS decomposition).

We have also shown a practical application of such decomposition as an accessory to other decompositions in a block-wise manner. This was shown in Example 2, where we presented such application on the indefinite QR.

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