

On transient queue-size distribution in the batch arrival system with the N -policy and setup times*

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Received August 30, 2011; accepted February 23, 2012

Abstract. In the paper the $M^X/G/1$ queueing system with the N -policy and setup times is considered. An explicit formula for the Laplace transform of the transient queue-size distribution is derived using the approach consisting of few steps. Firstly, a “special” modification of the original system is investigated and, using the formula of total probability, the analysis is reduced to the case of the corresponding system without limitation in the service. Next, a renewal process generated by successive busy cycles is used to obtain the general result. Sample numerical computations illustrating theoretical results are attached as well.

AMS subject classifications: 60K25, 90B22

Key words: batch arrival queueing system, N -policy, queue-size distribution, setup time, transient state

1. Introduction

Applications of queueing systems with different-type restrictions in the service process are evident. They can model real-life situations occurring in the operation of telecommunication and computer networks, like temporary suspension of data transmission due to server’s switchings off or breakdowns. Such systems can also be used in modelling of random interruptions in the manufacturing process or delays in transport caused e.g. by road works or fixed ferries timetable.

In the paper we deal with an infinite-buffer system with the N -policy and setup times “working” in the exhaustive service regime. In such a system a new busy period is initialized only if at least N customers are accumulated in the system, and the first service is preceded by a random setup time. Introducing the threshold of N customers present, beyond which the server is only activated, can be helpful in reducing the costs of system’s operation. One can change the critical value N in dependence on the arrival rate. A setup time that begins each new busy period can be interpreted as a time “for server” needed to obtain its full readiness for service.

An overview of results related to systems with Poisson arrivals and different-type service restrictions (like single and multiple vacations, the N -policy etc.) can

*Presented at the Seventh Conference on Applied Mathematics and Scientific Computing, Trogir, Croatia, June 2011.

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be found in [21]. The analysis of the stationary queue-size distribution in the system with Poisson arrivals, setup times and the N -policy is done in [5]. In [8], for the system of the $M/G/1$ type, the optimal value of the threshold N is found to minimize operating costs in the stationary state of the system. Three different arrival rates are used in dependence on server's status. A system of the $M^X/G/1$ type with batch arrivals is investigated in [20]. Two thresholds are considered: the lower (under which the server is deactivated) and the upper (ordinary) one. Formulae for the steady-state queue-size distribution and means of busy and idle time durations are derived. Further results for such a system can be found in [2], where the cost structure is investigated and close-down periods occur. In [7] a discrete-time system is considered in that, except for setup times and the N -policy, the Bernoulli feedback is implemented such that the service of a customer can be "successful" or not. The finite-buffer system with the N -policy and setup times is considered in [22] where the queue-size distribution and the mean waiting time are obtained in the stationary case. Different variants of vacation policies in the steady state of the system are analyzed in [9, 10, 11, 12, 13, 14].

It is easy to note that majority of the results relate to the stationary "regime" of the system. However, considering different real-life situations in which queueing models can be applied, one can see that, due to overloading of the system or permanently changing arrival and/or service rates, the stationary state does not occur in practice. Besides, from the mathematical point of view, the convergence rate of transient characteristics to stationary ones can be so slow that the steady state does not describe the real system behavior. These arguments are the motivation for the non-stationary (at fixed time t) analysis of the system.

One can find transient analysis of different characteristics in queueing systems with service limitations in [15, 16, 17, 18, 19]. In particular, in [15] the joint transform of the first busy period, the first idle time and the number of customers served during the first busy period is found for the general-type system with batch arrivals and single exponential vacations. Another approach to the same characteristics is presented in [16]. In [17] the departure counting process is studied for the system of the $M^X/G/1$ type with single vacations. The queue-length distribution for the system with multiple vacations, the N -policy and setup times is analyzed in [18]. In [19] the transient departure counting process in the batch arrival queueing system with single vacations and setup times is investigated.

In the article, we consider the transient queue-size distribution in the $M^X/G/1$ system with batch arrivals, the N -policy and setup times. Our main aim is to find an explicit formula for the LT (Laplace transform) of the number of customers present in the system at fixed moment t . The main idea of the method we propose is to reduce the analysis to the case of the corresponding system of the $M^X/G/1$ type without restrictions in the service process. The approach consists of a number of stages and can be described as follows.

Firstly, we consider the original system working in the non-standard "special" regime in that the initial moment does not coincide with the first batch arrival but with the beginning of the idle time of the system. In such a way, a "special" system starts working empty and waits for customers, and next, after reaching the level N , begins a busy period. Using the formula of total probability we reduce the analysis

to the case of the corresponding ordinary system without limitations in service. We define a busy cycle in a non-standard way: it consists of an idle time followed by a busy period. Finally, applying independence of successive cycles, we use the renewal-theory approach to obtain the general result.

Thus, the paper is organized as follows. In the next Section 2 we give a precise description of queueing models (the original, “special” and ordinary one) and introduce necessary notations. In Section 3 we state results for the ordinary system without limitations in the service process. Section 4 is devoted to the analysis of the queue-size distribution in the “special” system on its first busy cycle. In Section 5 we obtain the main result for the original system. Section 6 contains sample numerical computations which illustrate theoretical results, in particular the influence of “input” parameters of the system (the arrival and service rates, and setup times duration).

2. Description of queueing models

In the original (main) system of the $M^X/G/1$ type batches of customers arrive according to a Poisson process with rate λ and the size of the arriving batch equals k with probability p_k , $\sum_{k=1}^{\infty} p_k = 1$. Customers are served individually with a general-type d.f. (distribution function) $F(\cdot)$ of the service time. The system starts working in the standard way, i.e. at time $t = 0$ the first group of customers enters into the empty system and the service process begins immediately. Each successive busy period (except for the initial one) starts only when the system accumulates at least N customers (the N -policy). The first service in each busy period is preceded by a random setup time that is generally distributed with a d.f. $T(\cdot)$. So, the evolution of the original system can be observed during successive busy cycles C_i defined as follows:

$$C_i = \delta_i + \tau_i, \quad i = 0, 1, \dots, \tag{1}$$

where δ_i denotes the i th idle time of the system and τ_i stands for the i th busy period. Besides, it is assumed that $\delta_0 = 0$. In fact, we will often identify a particular period in the evolution of the system (busy period, idle time or busy cycle) with its duration. Note that we define busy cycles in a non-standard way: each of them begins with an idle time that is followed by a busy period.

We assume that all interarrival times, service times, arriving batch sizes and durations of successive setup times are totally independent random variables.

In addition to the original system, we will also consider the corresponding ordinary system of the $M^X/G/1$ type (with the arrival and service processes described by the same d.fs as in the original one) without limitations in the service process.

Moreover, we introduce a “special” modification of the original system with the N -policy and setup times. A “special” system begins its evolution with the busy cycle C_1 , thus the idle time δ_1 begins at the initial moment of the operation of the system. All probabilities relating to the “special” system will be denoted by $\mathbf{P}_S\{\cdot\}$.

We end this section with introducing some necessary notation and facts. Define

$$f(s) = \int_0^{\infty} e^{-sx} dF(x), \quad \text{Re}(s) > 0, \quad p(\theta) = \sum_{k=1}^{\infty} \theta^k p_k, \quad |\theta| \leq 1. \tag{2}$$

In [3], one can find the following theorem:

Theorem 1. *For any $\mu > 0$ the following factorization identity of Wiener-Hopf type holds true:*

$$1 - \frac{\lambda p(f(\mu - s))}{\lambda + s} = f_+(s, \mu)f_-(s, \mu), \quad 0 \leq \text{Re}(s) \leq \mu. \tag{3}$$

In the formula above the functions $f_{\pm}(s, \mu)$ are regular and non-zero in half-planes $\text{Re}(s) > 0, \text{Re}(s) < \mu$, respectively. The condition $f_{\pm}(\infty, \lambda) = 1$ ensures the uniqueness of factorization (3).

The inverses of functions $f_{\pm}(s, \mu)$ can be written as follows (see [3]):

$$\frac{1}{f_{\pm}(s, \mu)} = 1 \pm \int_0^{\pm\infty} e^{-sx} dP_{\pm}(x, \mu), \quad \pm \text{Re}(s) \geq 0, \tag{4}$$

where functions $P_{\pm}(x, \mu)$ have bounded variations for any positive μ .

Additionally, let us define

$$P_-^{(0)}(x, \mu) = -I\{x < 0\} + P_-(x, \mu), \tag{5}$$

$$P_+^{(0)}(x, \mu) = I\{x > 0\} + P_+(x, \mu), \tag{6}$$

where the notation $I\{\mathbb{A}\}$ stands for the indicator of a random event \mathbb{A} .

In some formulae in the article we use the positive projector I_+ defined on the LT of a function $h(\cdot)$ in the following way:

$$I_+ \left[\int_{-\infty}^{\infty} e^{-sx} h(x) dx \right] = \int_0^{\infty} e^{-sx} h(x) dx, \tag{7}$$

if only $\int_0^{\infty} e^{-\text{Re}(s)x} |h(x)| dx < \infty$.

Finally, denote by p_j^{n*} the j th term of the n -fold convolution of the sequence (p_j) with itself and, similarly, by $F^{n*}(\cdot)$ - the n -fold convolution of the d.f. $F(\cdot)$ with itself.

3. Queue-size distribution in the ordinary system

Let us take into consideration the ordinary system of the $M^X/G/1$ type corresponding to the original one. We will investigate such a system on condition that the size (number of customers) of the first group that enters into the empty system at time $t = 0$ is fixed. Probabilities and means on this condition we denote by $\mathbf{P}_n^O\{\cdot\}$, respectively and $\mathbf{E}_n^O\{\cdot\}$, where the superscript O indicates the ordinary system and n stands for the size of the first arriving group.

Let us put

$$Q_n^O(m, \mu) = \int_0^{\infty} e^{-\mu t} \mathbf{P}_n^O\{X(t) = m, t \in \tau_1\} dt, \quad \mu > 0, \tag{8}$$

where $X(t)$ stands for the number of customers present in the system at time t and τ_1 denotes the first busy period of the system. In [18], one can find the following theorem:

Theorem 2. For any $m \geq 1, n \geq 1$ and $\mu > 0$ the following representation is true:

$$Q_n^O(m, \mu) = \frac{1}{\mu(\lambda + \mu)} \sum_{k=0}^{\infty} p_{m-n}^{k*} \left(\frac{\lambda}{\lambda + \mu}\right)^k + D_1(m, n, \mu) - \frac{f_+(\mu, \mu)}{\mu} \sum_{i=0}^m \left(\frac{\lambda}{\lambda + \mu}\right)^i (p_m^{i*} - p_m^{(i+1)*}) D_2(n, \mu) + D_3(m, n, \mu), \quad (9)$$

where

$$D_1(m, n, \mu) = \sum_{i=0}^m \int_{-\infty}^0 \left[\int_{-\infty}^{+0} \int_0^{\infty} e^{-\mu t} \Theta(t - x - y, m - i, \mu) dH(t, i) dP_-^{(0)}(x, \mu) \right] \times d_y \left[\int_{-\min(0, y)}^{\infty} e^{-\mu v} \int_{-0}^{y+v} (1 - e^{-\lambda(y+v-u)}) dP_+^{(0)}(u, \mu) dF^{n*}(v) \right], \quad (10)$$

$$D_2(n, \mu) = \int_{-\infty}^0 e^{\mu t} \left[\int_{-0}^{-t} e^{-\mu x} dP_+^{(0)}(x, \mu) \right] \times d_t \left(\int_{-\min(0, t)}^{\infty} (1 - e^{-\lambda(t+y)}) e^{-\mu y} dF^{n*}(y) \right), \quad (11)$$

$$D_3(m, n, \mu) = \sum_{i=0}^n \int_0^{\infty} e^{-\mu t} \Theta(t, m - i, \mu) dF^{(n-i)*}(t) \quad (12)$$

and besides

$$H(t, i) = \sum_{j=i+1}^{\infty} p_j F^{(j-i)*}(t), \quad (13)$$

$$\Theta(t, k, \mu) = - \sum_{i=0}^k (p_{k-1}^{i*} - p_k^{i*}) \int_0^{\infty} e^{-(\lambda+\mu)y} \frac{(\lambda(t+y))^i}{i!} dy. \quad (14)$$

As we will see in the next section, the queue-size distribution in the “special” system with the N -policy and setup times will be, in fact, largely determined by the same characteristic in the ordinary system with fixed number of customers present just after the initial moment.

4. Analysis of the “special” system on the first busy cycle

In this section, we deal with the queue-size distribution in the “special” system on its first busy cycle C_1 . The “special” system is a modification of the original one in that $C_0 = 0$. For a fixed moment t let us define the following random events:

- A_1 – the threshold N is reached before t and the first setup time also ends before t (thus, the time t is “inside” the first busy period τ_1);
- A_2 – the threshold N is reached before t but the setup time ends after t ;
- A_3 – the threshold N is reached after time t .

It is easy to note that A_1, A_2 and A_3 are separable in pairs and $\sum_{i=1}^3 \mathbf{P}(A_i) = 1$, so we have

$$\mathbf{P}_S\{X(t) = m, t \in C_1\} = \sum_{i=1}^3 \mathbf{P}_S\{(X(t) = m, t \in C_1) \cap A_i\}. \tag{15}$$

For particular summands on the right side of (15) the following representations are true (to comment these formulae we state them in non-simplified forms):

$$\begin{aligned} \mathbf{P}_S\{(X(t) = m, t \in C_1) \cap A_1\} &= I\{m \geq 1\} \left(\sum_{j=N}^{\infty} p_j \int_0^t \lambda e^{-\lambda y} dy \sum_{r=0}^{\infty} \sum_{l=r}^{\infty} p_l^{r*} \right. \\ &\times \int_0^{t-y} \frac{(\lambda u)^r}{r!} e^{-\lambda u} \mathbf{P}_{j+l}^O\{X(t-y-u) = m, t-y-u \in \tau_1\} dT(u) \\ &+ \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} p_k^{i*} \int_0^t \frac{\lambda^i}{(i-1)!} x^{i-1} e^{-\lambda x} dx \sum_{j=N-k}^{\infty} p_j \int_0^{t-x} \lambda e^{-\lambda y} dy \\ &\times \sum_{r=0}^{\infty} \sum_{l=r}^{\infty} p_l^{r*} \int_0^{t-x-y} \frac{(\lambda u)^r}{r!} e^{-\lambda u} \\ &\left. \times \mathbf{P}_{k+j+l}^O\{X(t-x-y-u) = m, t-x-y-u \in \tau_1\} dT(u) \right), \tag{16} \end{aligned}$$

$$\begin{aligned} \mathbf{P}_S\{(X(t) = m, t \in C_1) \cap A_2\} &= I\{m \geq N\} \left(\sum_{j=N}^m p_j \sum_{r=0}^{m-j} p_{m-j}^{r*} \int_0^t \lambda e^{-\lambda y} (1 - T(t-y)) \right. \\ &\times \frac{(\lambda(t-y))^r}{r!} e^{-\lambda(t-y)} dy \\ &+ \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} p_k^{i*} \int_0^t \frac{\lambda^i}{(i-1)!} x^{i-1} e^{-\lambda x} dx \sum_{j=N-k}^{m-k} p_j \int_0^{t-x} \lambda e^{-\lambda y} \\ &\left. \times (1 - T(t-x-y)) \sum_{r=0}^{m-k-j} p_{m-k-j}^{r*} \frac{(\lambda(t-x-y))^r}{r!} e^{-\lambda(t-x-y)} dy \right), \tag{17} \end{aligned}$$

$$\mathbf{P}_S\{(X(t) = m, t \in C_1) \cap A_3\} = I\{m \leq N - 1\} \sum_{k=0}^m p_m^{k*} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \tag{18}$$

where we take on the agreement $p_0^{0*} = 1$.

Let us briefly comment identities (16)–(18). The first summand on the right side of (16) presents the situation in which the level N of customers present in the system is reached just at the arrival epoch (denoted by y) of the first group, at which $j \geq N$ customers enters into the empty system. During the setup time (that ends before t), next l customers arrive and, beginning with the completion epoch of the setup time, the evolution of the original system coincides with the evolution of the corresponding ordinary system that starts working with $j+l$ customers present. The second summand on the right side of (16) gives the representation for the situation

in which the threshold N is reached for the first time at the $(i + 1)$ th arrival epoch, where $i \geq 1$. This event occurs at time $x + y$ where x denotes a directly preceding arrival epoch. At time $x + y$ the number of customers present in the system equals $k + j$ (k customer enters before the “critical” moment $x + y$, at which (the level N is reached) for the first time, and j customers arrive in the group entering at the moment $x + y$). During the setup time next l customers occur, thus at the end of the setup time the system starts the service process with $k + j + l$ customers present.

The first summand on the right side of (17) describes the case in which the threshold N is obtained just at the first arrival epoch (denoted by y). If at time y the number of $j \geq N$ customers arrive, then - since the setup ends after t - the number of customers entering in the period $(y, t]$ equals exactly $m - j$. The second summand of (17) can be explained similarly, with the difference that the level N is reached at least at the second arrival epoch.

The formula (18) is obvious: until the time t exactly $m \leq N$ customers arrive, where successive batches occur according to a Poisson process with intensity λ .

Denote

$$Q_{S,i}(m, \mu) = \int_0^\infty e^{-\mu t} \mathbf{P}_S\{X(t) = m, t \in C_1 \cap A_i\} dt, \quad \mu > 0, i = 1, 2, 3 \quad (19)$$

and introduce the following functionals:

$$\alpha_k(\mu) = \int_0^\infty e^{-(\mu+\lambda)t} \frac{(\lambda t)^k}{k!} dt, \quad (20)$$

$$\beta_k(\mu) = \int_0^\infty e^{-(\mu+\lambda)t} \frac{(\lambda t)^k}{k!} dT(t), \quad (21)$$

$$\gamma_k(\mu) = \int_0^\infty e^{-(\mu+\lambda)t} \frac{(\lambda t)^k}{k!} (1 - T(t)) dt, \quad (22)$$

$$\delta_k(\mu) = \frac{\lambda^k}{(k - 1)!} \int_0^\infty t^{k-1} e^{-(\mu+\lambda)t} dt. \quad (23)$$

Note that the following identities hold:

$$\int_0^\infty e^{-\mu t} dt \int_0^t \lambda e^{-\lambda y} dy \int_0^{t-y} \frac{(\lambda u)^r}{r!} e^{-\lambda u} \times \mathbf{P}_j^O\{X(t - y - u) = m, t - y - u \in \tau_1\} dT(u) = \beta_r(\mu) \delta_1(\mu) Q_j^O(m, \mu), \quad (24)$$

$$\int_0^\infty e^{-\mu t} dt \int_0^t \frac{\lambda^i}{(i - 1)!} x^{i-1} e^{-\lambda x} dx \int_0^{t-x} \lambda e^{-\lambda y} dy \int_0^{t-x-y} \frac{(\lambda u)^r}{r!} e^{-\lambda u} \times \mathbf{P}_j^O\{X(t - x - y - u) = m, t - x - y - u \in \tau_1\} dT(u) = \beta_r(\mu) \delta_1(\mu) \delta_i(\mu) Q_j^O(m, \mu) = \beta_r(\mu) \delta_{i+1}(\mu) Q_j^O(m, \mu), \quad (25)$$

where the identity $\delta_1(\mu) \delta_i(\mu) = \delta_{i+1}(\mu)$ is a consequence of the fact that $\delta_i(\mu)$ is the LT of the p.d.f. (probability density function) of the i -Erlang distribution with parameter λ .

Similarly, we have

$$\int_0^\infty e^{-\mu t} dt \int_0^t \lambda e^{-\lambda y} (1 - T(t - y)) \frac{(\lambda(t - y))^r}{r!} e^{-\lambda(t - y)} dy = \gamma_r(\mu) \delta_1(\mu) \tag{26}$$

and

$$\begin{aligned} & \int_0^\infty e^{-\mu t} dt \int_0^t \frac{\lambda^i}{(i - 1)!} x^{i-1} e^{-\lambda x} dx \\ & \times \int_0^{t-x} \lambda e^{-\lambda y} (1 - T(t - x - y)) \frac{(\lambda(t - x - y))^r}{r!} e^{-\lambda(t - x - y)} dy \\ & = \gamma_r(\mu) \delta_1(\mu) \delta_i(\mu) = \gamma_r(\mu) \delta_{i+1}(\mu). \end{aligned} \tag{27}$$

Now the formulae (16)–(18) and (24)–(27) lead to the following representations:

$$\begin{aligned} Q_{S,1}(m, \mu) &= I\{m \geq 1\} \\ & \times \sum_{i=0}^{N-1} \delta_{i+1}(\mu) \sum_{k=i}^{N-1} p_k^{i*} \sum_{j=N-k}^\infty p_j \sum_{r=0}^\infty \beta_r(\mu) \sum_{l=r}^\infty p_l^{r*} Q_{k+j+l}^O(m, \mu), \end{aligned} \tag{28}$$

$$Q_{S,2}(m, \mu) = I\{m \geq N\} \sum_{i=0}^{N-1} \delta_{i+1}(\mu) \sum_{k=i}^{N-1} p_k^{i*} \sum_{j=N-k}^{m-k} p_j \sum_{r=0}^{m-k-j} p_{m-k-j}^{r*} \gamma_r(\mu) \tag{29}$$

and

$$Q_{S,3}(m, \mu) = I\{m \leq N - 1\} \sum_{k=0}^m p_m^{k*} \alpha_k(\mu). \tag{30}$$

Denoting

$$Q_S(m, \mu) = \int_0^\infty e^{-\mu t} \mathbf{P}_S\{X(t) = m, t \in C_1\} dt, \quad \mu > 0, \tag{31}$$

since

$$Q_S(m, \mu) = \sum_{i=1}^3 Q_{S,i}(m, \mu), \tag{32}$$

we obtain from (28)–(30) the following theorem:

Theorem 3. *The LT of the queue-size distribution in the “special” system on its first busy cycle C_1 is given by the following formula:*

$$\begin{aligned} Q_S(m, \mu) &= \int_0^\infty e^{-\mu t} \mathbf{P}_S\{X(t) = m, t \in C_1\} dt \\ &= I\{m \geq 1\} \sum_{i=0}^{N-1} \delta_{i+1}(\mu) \sum_{k=i}^{N-1} p_k^{i*} \sum_{j=N-k}^\infty p_j \sum_{r=0}^\infty \beta_r(\mu) \sum_{l=r}^\infty p_l^{r*} Q_{k+j+l}^O(m, \mu) \\ & \quad + I\{m \geq N\} \sum_{i=0}^{N-1} \delta_{i+1}(\mu) \sum_{k=i}^{N-1} p_k^{i*} \sum_{j=N-k}^{m-k} p_j \sum_{r=0}^{m-k-j} p_{m-k-j}^{r*} \gamma_r(\mu) \\ & \quad + I\{m \leq N - 1\} \sum_{k=0}^m p_m^{k*} \alpha_k(\mu), \end{aligned} \tag{33}$$

where $\mu > 0$, $m = 0, 1, 2, \dots$, sequences $(\alpha_k(\mu))$, $(\beta_k(\mu))$, $(\gamma_k(\mu))$ and $(\delta_k(\mu))$ are defined in (20)–(23), respectively, and the representation for $Q_{k+j+l}^O(m, \mu)$ for fixed k, j and l is given in (9).

In the case of the system with individual arrivals ($p_1 = 1$) from the last Theorem we obtain the following corollary:

Corollary 1. For the “special” system with individual arrivals the following formula for the LT of the queue-size distribution on the first busy cycle C_1 is true:

$$Q_S(m, \mu) = I\{m \geq 1\} \delta_N(\mu) \sum_{r=0}^{\infty} \beta_r(\mu) Q_{N+r}^O(m, \mu) + I\{m \geq N\} \delta_N(\mu) \gamma_{m-N}(\mu) + I\{m \leq N-1\} \alpha_m(\mu). \quad (34)$$

Similarly, for the system with group arrivals but without the N -policy (with $N = 1$) we get

Corollary 2. For the “special” system with batch arrivals and without the N -policy ($N = 1$): the formula for the LT of the queue-size distribution on the first busy cycle C_1 is following:

$$Q_S(m, \mu) = I\{m \geq 1\} \delta_1(\mu) \left(\sum_{j=1}^{\infty} p_j \sum_{r=0}^{\infty} \beta_r(\mu) \sum_{l=r}^{\infty} p_l^{r*} Q_{j+l}^O(m, \mu) + \sum_{j=1}^m p_j \sum_{r=0}^{m-j} p_{m-j}^{r*} \gamma_r(\mu) \right) + I\{m = 0\} \alpha_0(\mu). \quad (35)$$

In the probabilistic sense the behavior of the queue-size distribution on each of busy cycles C_1, C_2 etc. is the same. Hence, applying Theorem 3 and the renewal process defined by successive cycles, in the next section we obtain a general result for the original system.

5. Queue-size distribution in the original system

Let us return to the original system of the $M^X/G/1$ type with the N -policy and setup times. From the memoryless property of interarrival times it follows that, taking into consideration the evolution of the process $X(t)$, start epochs of successive busy cycles C_0, C_1, \dots are renewal moments. Denote by $B_0(\cdot)$ and $B(\cdot)$ d.fs of random variables C_0 and C_i , $i \geq 1$, respectively. Similarly, let for $\mu > 0$

$$b_0(\mu) = \int_0^{\infty} e^{-\mu t} dB_0(t), \quad b(\mu) = \int_0^{\infty} e^{-\mu t} dB(t). \quad (36)$$

It is obvious that the busy cycle C_0 in the original system coincides with the first busy period τ_1 in the corresponding ordinary system that begins working in the standard way (it is empty before the initializing and comes into operation at $t = 0$ when the first group of customers arrives). In [3], the following representation can be found:

$$b_0(\mu) = \mathbf{E}\{e^{-\mu C_0}\} = \mathbf{E}_{std}^O\{e^{-\mu \tau_1}\} = 1 - f_+(0, \mu), \quad (37)$$

where $f_+(0, \mu)$ was defined in (3) and the subscript *std* denotes standard initial conditions for the ordinary system.

Similarly, in [4] the following formula is derived:

$$\mathbf{E}_n^O\{e^{-\mu\tau_1}\} = (f(\mu))^n - f_+(0, \mu) \int_0^\infty e^{-\mu t} \int_{-0}^t (1 - e^{-\lambda(t-x)}) dP_+^{(0)}(x, \mu) dF^{n*}(t). \quad (38)$$

The representation for $b(\mu)$ can be found using the formula of total probability. Indeed, we have

$$\begin{aligned} b(\mu) &= \mathbf{E}\{e^{-\mu C_1}\} \\ &= \sum_{j=N}^\infty p_j \int_0^\infty \lambda e^{-\lambda y} dy \sum_{r=0}^\infty \sum_{l=r}^\infty p_l^{r*} \int_0^\infty \frac{(\lambda u)^r}{r!} e^{-\lambda u} e^{-\mu(y+u)} \mathbf{E}_{j+l}^O\{e^{-\mu\tau_1}\} dT(u) \\ &\quad + \sum_{i=1}^{N-1} \sum_{k=i}^{N-1} p_k^{i*} \int_0^\infty \frac{\lambda^i}{(i-1)!} x^{i-1} e^{-\lambda x} dx \sum_{j=N-k}^\infty p_j \int_0^\infty \lambda e^{-\lambda y} dy \\ &\quad \times \sum_{r=0}^\infty \sum_{l=r}^\infty p_l^{r*} \int_0^\infty \frac{(\lambda u)^r}{r!} e^{-\lambda u} e^{-\mu(x+y+u)} \mathbf{E}_{k+j+l}^O\{e^{-\mu\tau_1}\} dT(u) \\ &= \sum_{i=0}^{N-1} \delta_{i+1}(\mu) \sum_{k=i}^{N-1} p_k^{i*} \sum_{j=N-k}^\infty p_j \sum_{r=0}^\infty \beta_r(\mu) \sum_{l=r}^\infty p_l^{r*} \mathbf{E}_{k+j+l}^O\{e^{-\mu\tau_1}\}, \end{aligned} \quad (39)$$

where $\mathbf{E}_{k+j+l}^O\{e^{-\mu\tau_1}\}$ is defined in (38). The interpretation of (39) is similar to (16)–(18).

Introduce the following notation:

$$Q_{std}^O(m, \mu) = \int_0^\infty e^{-\mu t} \mathbf{P}_{std}^O\{X(t) = m, t \in \tau_1\} dt, \quad \mu > 0, m = 1, 2, \dots \quad (40)$$

Let us note that

$$\mathbf{P}_{std}^O\{\cdot\} = \sum_{n=1}^\infty p_n \mathbf{P}_n^O\{\cdot\} \quad (41)$$

and hence

$$Q_{std}^O(m, \mu) = \sum_{n=1}^\infty p_n Q_n^O(m, \mu), \quad (42)$$

where the representation for $Q_n^O(m, \mu)$ was found in (9).

The main theorem below gives a representation for the LT of the queue-size distribution in the original system with the N -policy and setup times.

Theorem 4. For $m \geq 0$ and $\mu > 0$ the following representation holds true:

$$\int_0^\infty e^{-\mu t} \mathbf{P}\{X(t) = m\} dt = Q_{std}^O(m, \mu) + Q_S(m, \mu) \frac{b_0(\mu)}{1-b(\mu)}, \quad (43)$$

where the formulae for $Q_{std}^O(m, \mu)$, $Q_S(m, \mu)$, $b_0(\mu)$ and $b(\mu)$ are given in (42), (33), (37) and (39), respectively.

Proof. The proof follows immediately from the fact that successive busy cycles C_0, C_1, \dots form a delayed renewal process. Indeed, we have

$$\begin{aligned}
 & \int_0^\infty e^{-\mu t} \mathbf{P}\{X(t) = m\} dt \\
 &= \sum_{k=0}^\infty \int_0^\infty e^{-\mu t} \mathbf{P}\{X(t) = m, t \in C_k\} dt \\
 &= \int_0^\infty e^{-\mu t} \mathbf{P}_{std}^O\{X(t) = m, t \in \tau_1\} dt \\
 &\quad + \sum_{k=1}^\infty \int_0^\infty e^{-\mu t} \mathbf{P}\{X(t) = m, t \in C_k\} dt \\
 &= \int_0^\infty e^{-\mu t} \mathbf{P}_{std}^O\{X(t) = m, t \in \tau_1\} dt \\
 &\quad + \int_0^\infty e^{-\mu t} \sum_{k=1}^\infty \int_0^t \mathbf{P}_S\{X(t-y) = m, t-y \in C_1\} d(B_0 * B^{(k-1)*})(y) dt \\
 &= Q_{std}^O(m, \mu) + Q_S(m, \mu) \sum_{k=1}^\infty b_0(\mu) b^{k-1}(\mu) \\
 &= Q_{std}^O(m, \mu) + Q_S(m, \mu) \frac{b_0(\mu)}{1-b(\mu)}. \tag{44}
 \end{aligned}$$

□

6. Numerical examples

In this section, we present sample numerical computations which illustrate the main result (43) from Theorem 4. In particular, we are interested in what is the influence of “input” parameters of the system (e.g. the arrival and service rates, and the mean of the setup time) for the transient queue-size distribution. The representations for successive components of the right side of (43) are very complex, but it is easy to note that the main problem in numerical treatment of all theoretical results stated above is connected with the components $f_\pm(s, \mu)$ (and next the functions $P_\pm(x, \mu)$) of the factorization identity (3). It is clear that, since the left side of (3) depends on the arrival rate λ and the distributions of service times and batch sizes, it is impossible to find the universal formulae for $f_+(s, \mu)$ and $f_-(s, \mu)$.

As an example, let us consider the system with individual arrivals ($p_1 = 1$) in which all “input” distributions are exponential (i.e. apart from interarrival times, service times and setup times are exponentially distributed random variables). Let us denote the means of service times and setup times by $\mathbf{E}F = \lambda_F^{-1}$ and $\mathbf{E}T = \lambda_T^{-1}$, respectively. Moreover, let us assume that the threshold level needed to activate the service process after the idle time is $N = 2$. Since

$$f(s) = \frac{\lambda_F}{\lambda_F + s}, \tag{45}$$

then the left side of the factorization identity (3) takes the form

$$1 - \frac{\lambda}{\lambda+s} \cdot \frac{\lambda_F}{\lambda_F+\mu-s} = \frac{-s^2+(\lambda_F+\mu-\lambda)s+\lambda\mu}{(\lambda+s)(\lambda_F+\mu-s)}. \tag{46}$$

The numerator on the right side of (46) has the following roots:

$$s_{1,2}(\lambda, \lambda_F, \mu) = \frac{(\lambda_F+\mu-\lambda) \pm \sqrt{(\lambda_F+\mu-\lambda)^2+4\lambda\mu}}{2}. \tag{47}$$

Since $f_{\pm}(s, \mu)$ should be regular and non-zero in the half-planes $\text{Re}(s) > 0$ and $\text{Re}(s) < \mu$, respectively, and $f_{\pm}(\infty, \lambda) = 1$ (see Theorem 1), then we obtain the following unique representations:

$$f_+(s, \mu) = \frac{s - \frac{(\lambda_F+\mu-\lambda) - \sqrt{(\lambda_F+\mu-\lambda)^2+4\lambda\mu}}{2}}{\lambda+s}, \tag{48}$$

$$f_-(s, \mu) = \frac{-\left(s - \frac{(\lambda_F+\mu-\lambda) + \sqrt{(\lambda_F+\mu-\lambda)^2+4\lambda\mu}}{2}\right)}{\lambda_F + \mu - s}. \tag{49}$$

In fact, in all analytical formulae we need only $f_+(s, \mu)$ and $P_+(s, \mu)$, for which the LT can be written as follows (see (4)):

$$\int_0^\infty e^{-sx} P_+(x, \mu) dx = \frac{1-f_+(s, \mu)}{sf_+(s, \mu)}. \tag{50}$$

Let us note that for the considered queueing system the following simplified forms of some functionals hold true:

$$b_0(\mu) = 1 - f_+(0, \mu) = \frac{\lambda_F + \mu + \lambda - \sqrt{(\lambda_F + \mu - \lambda)^2 + 4\lambda\mu}}{2\lambda}, \tag{51}$$

$$b(\mu) = \delta_2(\mu) \sum_{r=0}^\infty \beta_r(\mu) \mathbf{E}_{2+r}^O \{e^{-\mu\tau_1}\}, \tag{52}$$

where

$$\beta_r(\mu) = \lambda_T \int_0^\infty e^{-(\mu+\lambda)t} \frac{(\lambda t)^r}{r!} e^{-\lambda\tau t} dt, \quad \delta_2(\mu) = \lambda^2 \int_0^\infty t e^{-(\mu+\lambda)t} dt \tag{53}$$

and

$$\begin{aligned} \mathbf{E}_n^O \{e^{-\mu\tau_1}\} &= \left(\frac{\lambda_F}{\lambda_F + \mu}\right)^n - f_+(0, \mu) \\ &\times \int_0^\infty e^{-\mu t} \left(\int_0^t (1 - e^{-\lambda(t-x)}) dP_+(x, \mu) + 1 - e^{-\lambda t}\right) \\ &\times \frac{\lambda_F^n}{(n-1)!} t^{n-1} e^{-\lambda_F t} dt. \end{aligned} \tag{54}$$

Let us take into consideration the probability that the system is empty ($m = 0$) at time t . Since then the component $Q_{std}^O(m, \mu)$ in the formula (43) vanishes and moreover (see (34)) $Q_S(0, \mu) = \alpha_0(\mu) = \frac{1}{\mu+\lambda}$, then from (43) we obtain

$$\int_0^\infty e^{-\mu t} \mathbf{P}\{X(t) = 0\} dt = \frac{b_0(\mu)}{(\mu+\lambda)(1-b(\mu))}. \tag{55}$$

To obtain $\mathbf{P}\{X(t) = 0\}$ as a function of t , we approximate the right side of (55) using the algorithm of numerical Laplace transform inversion described in [1]. The algorithm is based on the Bromwich inversion integral of the form

$$\varphi(t) = \frac{1}{2\pi i} \int_{\Delta-i\infty}^{\Delta+i\infty} e^{st} \widehat{\varphi}(s) ds, \tag{56}$$

which allows for finding the value of the function φ at a fixed point t by means of its transform $\widehat{\varphi}$. In (56) $\Delta \in \mathbf{R}$ is located on the right to all singularities of the transform $\widehat{\varphi}$.

Introducing the following notation:

$$q_k(t) = \frac{e^{A/2L}}{2Lt} \omega_k(t), \quad k \geq 0, \tag{57}$$

where

$$\omega_0(t) = \widehat{\varphi}\left(\frac{A}{2Lt}\right) + 2 \sum_{j=1}^L \operatorname{Re} \left[\widehat{\varphi}\left(\frac{A}{2Lt} + \frac{ij\pi}{Lt}\right) e^{ij\pi/L} \right] \tag{58}$$

and

$$\omega_k(t) = 2 \sum_{j=1}^L \operatorname{Re} \left[\widehat{\varphi}\left(\frac{A}{2Lt} + \frac{ij\pi}{Lt} + \frac{ik\pi}{t}\right) e^{ij\pi/L} \right], \quad k \geq 1, \tag{59}$$

we obtain the following approximation (see [1] for more details):

$$\varphi(t) \approx \sum_{k=0}^m \binom{m}{k} \frac{1}{2^m} \sum_{j=0}^{n+k} (-1)^j q_j(t), \tag{60}$$

where typical values of parameters are the following (see [1]): $m = 38, n = 11, A = 19, L = 1$.

A precise evaluation of the error of approximation (60) is difficult. In practice, we obtain a sufficiently good estimation of the error executing the calculation twice, changing by 1 one of the parameters: m, n, A or L . The difference between results gives a rough error of approximation (60) (see discussion on this topic in [1] or [6]).

6.1. Queue-size distribution dependence on the setup duration

In the considered example of the $M/M/1$ queueing system with the threshold $N = 2$ and exponentially distributed setup times, let us fix $\lambda = \lambda_F = 1$ and take three different values of the parameter $\lambda_T : 1, 2$ and 5 which give the means \mathbf{ET} of the setup duration equal to $1, 0.5$ and 0.2 , respectively. In Table 1, we present the values of probabilities $\mathbf{P}\{X(t) = 0\}$ for five different time moments: $0.05, 1, 5, 20$ and 100 .

Means \mathbf{ET} of setup duration	t				
	0.05	1	5	20	100
1.0	0.0464111	0.265251	0.0839149	0.00966544	0.00976325
0.5	0.0464111	0.26549	0.0967537	0.0282864	0.0145028
0.2	0.0464111	0.265941	0.107777	0.0399178	0.0187605

Table 1: Probabilities $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_F = 1$ and different means of the setup duration

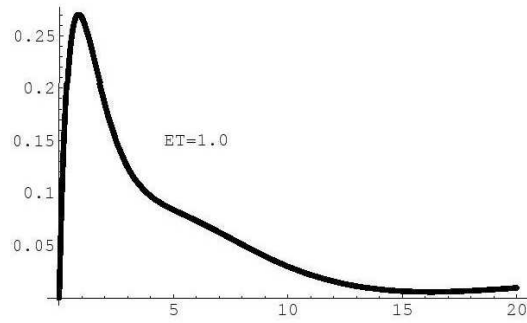


Figure 1: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_F = 1$ and $\mathbf{ET} = 1.0$

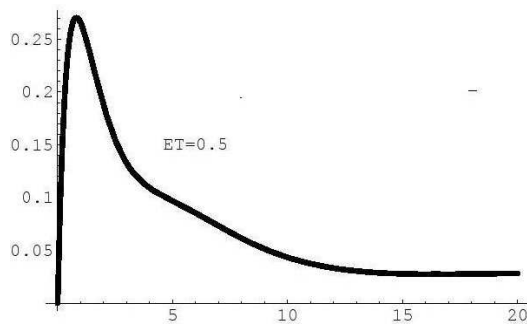


Figure 2: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_F = 1$ and $\mathbf{ET} = 0.5$

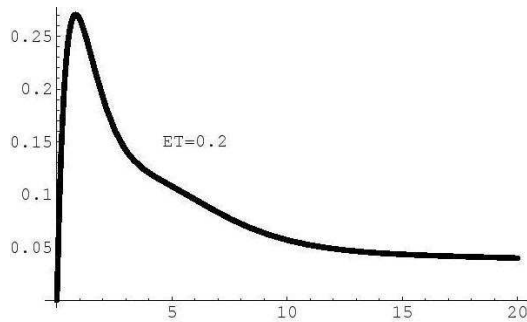


Figure 3: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_F = 1$ and $\mathbf{ET} = 0.2$

Transient distributions $\mathbf{P}\{X(t) = 0\}$ for $\mathbf{ET} = 1, 0.5$ and 0.2 are presented in Figures 1, 2 and 3, respectively.

It is easy to note that the probability that the system is empty, for small t , is similar in all three cases. Moreover, as the time increases, the smaller the mean of the setup time, the greater the probability $\mathbf{P}\{X(t) = 0\}$.

6.2. Queue-size distribution dependence on the arrival rate

To investigate the impact of the arrival rate on the probability $\mathbf{P}\{X(t) = 0\}$ let us fix now the values of parameters of exponentially distributed service times and setups, taking $\lambda_F = \lambda_T = 2$, and let us leave the threshold for the N -policy on the same level, i.e. $N = 2$. In Table 2, probabilities $\mathbf{P}\{X(t) = 0\}$ for $t = 0.05, 1, 5, 20$ and 100 are given for three different values of the arrival rate $\lambda : 1, 2$ and 8.

Arrival rates λ	t				
	0.05	1	5	20	100
1	0.090595	0.391514	0.210829	0.203117	0.203022
2	0.0862467	0.183933	0.0627603	0.0266888	0.0122143
8	0.0642058	0.0109123	0.00753159	0.0022155	0.000000330912

Table 2: Probabilities $\mathbf{P}\{X(t) = 0\}$ for $\lambda_F = \lambda_T = 2$ and different arrival rates

Distribution functions $\mathbf{P}\{X(t) = 0\}$ for $\lambda = 1, 2$ and 8 are presented in Figures 4, 5 and 6, respectively.

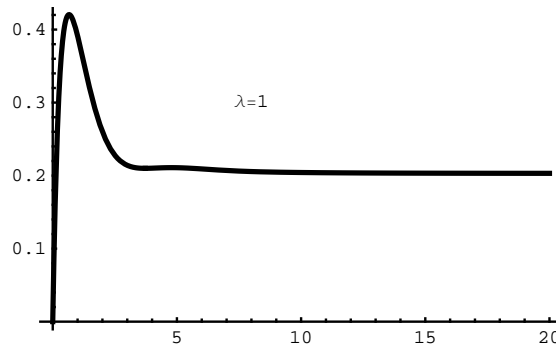


Figure 4: Distribution function $\mathbf{P}\{X(t) = 0\}$ for $\lambda_F = \lambda_T = 2$ and $\lambda = 1$

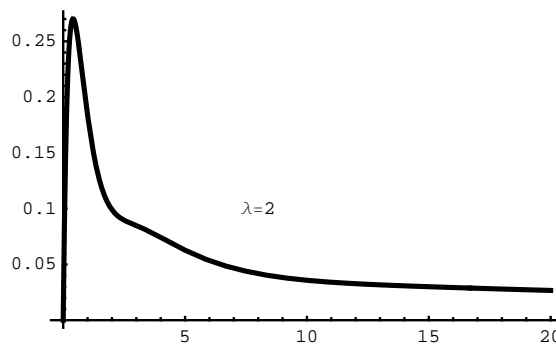


Figure 5: Distribution function $\mathbf{P}\{X(t) = 0\}$ for $\lambda_F = \lambda_T = 2$ and $\lambda = 2$

As one can observe, if the arrival rate λ is greater than the service rate λ_F , the probability $\mathbf{P}\{X(t) = 0\}$ decreases as t increases (Fig. 6). It is intuitively clear:

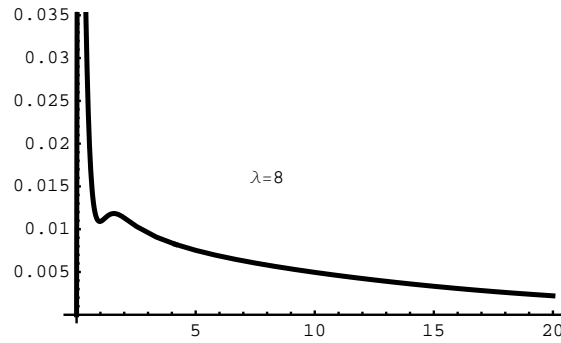


Figure 6: Distribution function $\mathbf{P}\{X(t) = 0\}$ for $\lambda_F = \lambda_T = 2$ and $\lambda = 8$

in the case of the increasing number of packets waiting for service ($\lambda > \lambda_F$), the probability that the system is empty at time t is getting smaller as the time increases. For $\lambda = 2$ (when the arrival and service rates are equal), the decrease of values of the function $\mathbf{P}\{X(t) = 0\}$ is smaller (Fig. 5). If the service “overtakes” the arrival process ($\lambda = 1$), then the equilibrium of the system exists and the probability that the system is empty at time t stabilizes as t increases, tending to the stationary probability (Fig. 4).

6.3. Queue-size distribution dependence on the service rate

Finally, let us examine the influence of the service rate on the probability $\mathbf{P}\{X(t) = 0\}$, fixing the values of the arrival rate and the mean of setup time. Let us assume that $\lambda = \lambda_T = 3$ and take three different values of the service rate $\lambda_F : 1, 3$, and 6 (which are equivalent to the values $\mathbf{E}F = 1, 0.333$ and 0.167 of the mean of the service time, respectively). Taking the same level $N = 2$ we obtain the values of the function $\mathbf{P}\{X(t) = 0\}$ for five moments $t = 0.05, 1, 5, 20$ and 100 , which are given in Table 3.

Service rate λ_F	t				
	0.05	1	5	20	100
1	0.0420291	0.0485438	0.0135558	0.00301106	0.000000950186
3	0.120324	0.124706	0.0584651	0.029244	0.013274
6	0.224636	0.186035	0.183104	0.182741	0.182737

Table 3: Probabilities $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_T = 3$ and different service rates

In Figures 7, 8 and 9, transient distributions $\mathbf{P}\{X(t) = 0\}$ are presented for $\mathbf{E}F = 1, 0.333$ and 0.167 , respectively.

For $\lambda_F = 1$ and $\lambda_F = 3$ (or, equivalently, for $\mathbf{E}F = 1$ and $\mathbf{E}F = 0.333$), as the time increases, the distribution function $\mathbf{P}\{X(t) = 0\}$ decreases (see Fig. 7 and 8), slower in the latter case (in which the arrival and service rates are equal - Fig. 8). If the stationary state of the system exists, for $\lambda_F = 6$ (or $\mathbf{E}F = 0.167$), the probability that the system is empty at time t stabilizes, tending (as t tends to infinity) to the stationary probability (Fig. 9).

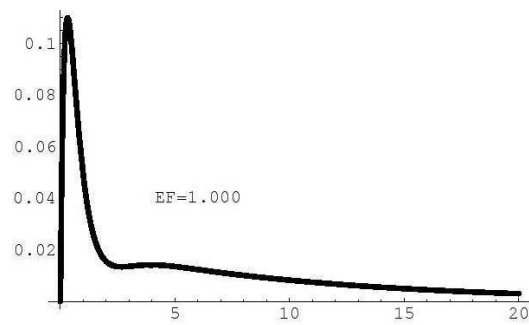


Figure 7: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_T = 3$ and $\mathbf{EF} = 1$

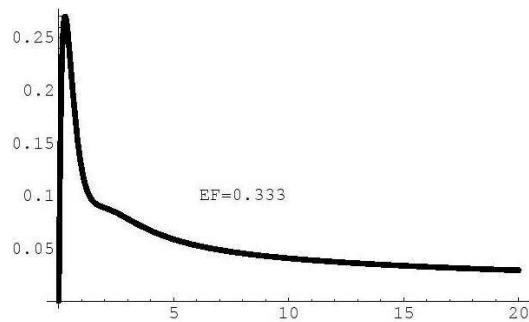


Figure 8: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_T = 3$ and $\mathbf{EF} = 0.333$

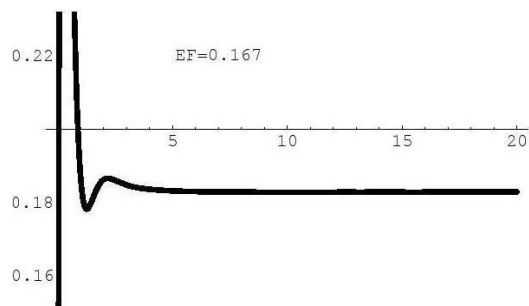


Figure 9: Transient distribution $\mathbf{P}\{X(t) = 0\}$ for $\lambda = \lambda_T = 3$ and $\mathbf{EF} = 0.167$

Acknowledgement

The author would like to thank the anonymous referees for their helpful suggestions which have contributed to the improvement of the manuscript.

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