MATHEMATICAL COMMUNICATIONS
Math. Commun. **17**(2012), 303–312

A simple nonlinear model of a generic axisymmetric wave energy converter*

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Received October 31, 2011; accepted February 20, 2012

Abstract. The aim of this work is to develop a simple nonlinear model of a wave energy converter (WEC) for capturing power from ocean waves and converting it into electrical power. A generic axisymmetric device is considered, which consists of a vertical circular cylinder surrounded by a circular annulus. The nonlinear system of equations of motion of this generic WEC are derived; these include the nonlinear term arising from viscous drag due to boundary layer separation. The expressions for radiation damping and added mass are determined by dimensional analysis. These equations are then solved numerically and the results are displayed graphically in a number of figures. Consideration of these graphs leads to conclusions that should be taken into account by the design engineer.

AMS subject classifications: 76B15, 76M20, 76M55

Key words: ocean wave energy device, mathematical model, numerical model, nonlinear system, dimensional analysis, added mass, radiation damping, power take-off, viscous drag

1. Introduction

The generic axisymmetric wave energy converter (WEC) considered here consists of a floating buoy-like device that is loosely moored to the seabed. It can freely move up and down in response to the rising and falling of the waves. Since it is assumed that the horizontal dimensions of the buoy are small compared to the wavelength of the incident wave, it follows that it operates primarily in heave mode. It also has a power take–off device, an electrical generator, a power electronics system and a control system, all of which are sealed in the device. The WEC converts mechanical energy, due to the heaving, into electrical energy, by means of an electrical generator. The power electronics system then conditions the output from the generator into useable electric power. The heaving motion resonates with the ocean waves, which is vitally important in a resonating energy absorber, if it is to recover useful power from ocean wave energy. In the next section, the mathematical model is discussed and the corresponding nonlinear system of two equations of motion are derived. The treatment is novel, because viscous drag is taken into account. In Section 3, dimensional analysis is used to derive, in a new manner, expressions for the added

*Presented at the Seventh Conference on Applied Mathematics and Scientific Computing, Trogir, Croatia, June 2011.

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mass and the radiation damping coefficient. The second order equations of Section 2 may be reduced to first order equations in a standard manner, which is described in Section 4. This results in a nonlinear system of four first order equations, which can be solved by a standard numerical solver contained in, for example, Matlab. In Section 5, graphs of the numerical results, obtained by a systematic exploration of the parameter space of the buoy, are presented and discussed. The goal of this exploration is to determine optimal parameter values, which correspond approximately to those required for maximum power output. Since the equations are nonlinear, an analytic solution is not available. However, it should be emphasised that, even if an analytic solution were available, the dimension of the parameter space is such that it would not be feasible to determine analytically these optimal parameter values.

The present approach is made feasible by the careful formulation of the mathematical model and the availability of standard packages containing fast nonlinear solvers. Because of the nonlinearity of the mathematical model, its solution is beyond the scope of standard approaches based on frequency domain models. The approach described here enables the WEC designer to derive approximate optimal values of the design parameters in a few minutes using a desktop computer. Each point in the graphs presented in Section 5 requires the solution of the nonlinear system. Nevertheless, the three-dimensional graphs can all be obtained in a few minutes. This low cost rapid determination of approximate optimal values of the parameters is a useful supplement, at the early stages of a WEC design, to the high cost slow determination of accurate values by methods using a three-dimensional boundary element or CFD techniques.

2. Mathematical model

Consider first the heaving motion of the cylinder in isolation. Let L be its total length, d its draft, a its cross-sectional area, z the distance measured vertically upwards from the draft level to the mean level of the water surface, so that z is positive when the draft level is below the mean level of the water surface. In addition, let x be the height of the wave above the mean water level and y the distance from the wave to the top of the cylinder. It follows that

$$x + y + z + d = L,$$

and that

$$\ddot{x} + \ddot{y} + \ddot{z} = 0.$$

Now the vertical acceleration of the top of the cylinder, referred to the constant mean water level, is

$$\ddot{x} + \ddot{y},$$

so that the vertical acceleration of the cylinder is

$$-\ddot{z}.$$

Neglecting damping, the vertical forces on the cylinder consist of the upward pressure force on the bottom of the cylinder, namely

$$\rho g a(x + z + d),$$

and the downward weight of the dead mass, m, of the cylinder,

$$mg,$$

where ρ is the density of the water and g is the gravitational acceleration. Now, the dead weight of the cylinder is equal to the weight of the water displaced when the cylinder is at rest, so that

$$mg = \rho gad.$$

It follows that, neglecting damping, the net vertical force on the cylinder is

$$\rho ga(x + z).$$

Damping forces are of two types. First, the linear radiation damping force, which is given by

$$-b(\text{vertical velocity}) = \text{b}\dot{\text{z}},$$

where b is a constant. The determination of its value is discussed in the next section. Secondly, there is a nonlinear viscous drag force, which is caused by boundary layer separation and is proportional to the square of the velocity. This force is

$$-c\dot{z}|\dot{z}|,$$

where c is the constant of proportionality. The determination of its value is also discussed in the next section. To summarise: taking damping into account, the net vertical force on the cylinder is

$$\rho ga(x + z) + b\dot{z} + c\dot{z}|\dot{z}|.$$

Now, as the cylinder moves through the water, it drags a volume of water behind it of mass μ, say, so that the total inertial mass of the cylinder is $m + \mu$. The quantity μ is called the added mass and its determination is discussed in the next section. It follows that the equation of motion of the cylinder is

$$(m + \mu)(-\ddot{z}) = \rho ga(x + z) + b\dot{z} + c\dot{z}|\dot{z}|,$$

or, rearranging,

$$(m + \mu)\ddot{z} + b\dot{z} + c\dot{z}|\dot{z}| + \rho gaz = -\rho gax. \tag{1}$$

Assuming that the water wave is monochromatic, x can be written in the form

$$x = \alpha sin(\omega t),$$

where t is the time, ω the angular frequency and α the amplitude. The use of *sin* rather than *cos* here permits the convenient choice in Section 4 of homogeneous initial conditions for the numerical solutions. In deep water, where the depth is much larger than the wavelength λ,

$$\omega = \sqrt{\frac{2\pi g}{\lambda}}.$$

See [1] for an introduction to the classical theory of monochromatic waves.

The equation of motion of the annulus in isolation is similar to (1). In isolation, therefore, the two bodies of the WEC satisfy the following uncoupled system of two second order nonlinear ordinary differential equations

$$(m_1 + \mu_1)\ddot{z}_1 + b_1\dot{z}_1 + c_1\dot{z}_1|\dot{z}_1| + \rho g a_1 z_1 = -\rho g a_1 x, \tag{2}$$
$$(m_2 + \mu_2)\ddot{z}_2 + b_2\dot{z}_2 + c_2\dot{z}_2|\dot{z}_2| + \rho g a_2 z_2 = -\rho g a_2 x, \tag{3}$$

where the subscript 1 refers to the cylinder and the subscript 2 to the annulus. The undamped natural angular frequencies of the two separate bodies are then

$$f_1 = 2\pi\sqrt{\frac{m_1 + \mu_1}{\rho g a_1}}, \quad f_2 = 2\pi\sqrt{\frac{m_2 + \mu_2}{\rho g a_2}}. \tag{4}$$

However, to generate power, both cylinder and annulus must interact. For example, electric power may be generated by constructing the cylinder out of a solid magnet and the annulus containing a cylindrical coil of electrical conductor. In the present case, the instantaneous force between the two parts of the device is proportional to their relative velocity

$$\dot{z}_1 - \dot{z}_2,$$

the constant of proportionality being denoted by pto, an abbreviation for "power take–off". It follows that the WEC satisfies the following coupled system of two second order nonlinear ordinary differential equations

$$(m_1 + \mu_1)\ddot{z}_1 + b_1\dot{z}_1 + c_1\dot{z}_1|\dot{z}_1| + pto(\dot{z}_1 - \dot{z}_2) + \rho g a_1 z_1 = -\rho g a_1 x, \tag{5}$$
$$(m_2 + \mu_2)\ddot{z}_2 + b_2\dot{z}_2 + c_2\dot{z}_2|\dot{z}_2| - pto(\dot{z}_1 - \dot{z}_2) + \rho g a_2 z_2 = -\rho g a_2 x. \tag{6}$$

Note that the instantaneous power output is equal to the product of instantaneous force and velocity.

$$pto(\dot{z}_1 - \dot{z}_2)^2.$$

3. Added mass, radiation damping and drag

In this section expressions for the added mass and the radiation damping coefficient, up to an unknown multiplicative constant, are derived using standard techniques of dimensional analysis, see, for example, [5].

Consider first the added mass μ; this is a function of ρ, g, a, λ and may be expressed in the form

$$f_1(\mu, \rho, g, a, \lambda) = 0. \tag{7}$$

In the Table, the dimensions of these quantities are given in terms of the fundamental dimensions mass $\tilde{M}$, length $\tilde{L}$ and time $\tilde{T}$.

Quantity	μ	ρ	g	a	λ
Dimensions	$\tilde{M}$	$\frac{\tilde{M}}{\tilde{L}^3}$	$\frac{\tilde{L}}{\tilde{T}^2}$	$\tilde{L}^2$	$\tilde{L}$

Thus, there emerge two dimensionless variables, namely $\frac{\mu}{\rho a^{3/2}}$ and $\frac{a^{1/2}}{\lambda}$. Consequently, the functional relationship (7) reduces to

$$f_2(\frac{\mu}{\rho a^{3/2}}, \frac{a^{1/2}}{\lambda}) = 0.$$

Solving for $\frac{\mu}{\rho a^{3/2}}$ then gives

$$\frac{\mu}{\rho a^{3/2}} = f_3(\frac{a^{1/2}}{\lambda}).$$

In the limiting case of small acceleration, which corresponds to low frequency and therefore large wavelength, the added mass remains finite; see, for example, [6]. Therefore, if $\frac{a^{1/2}}{\lambda} << 1$,

$$\frac{\mu}{\rho a^{3/2}} \approx K,$$

where K is a dimensionless constant, which depends on the shape of the WEC. For a cylinder, K can be uniquely determined by either analytic methods or experiment. In the case of an annulus, the value of K depends on the ratio of the inner and outer radii, and can be similarly determined.

On the other hand, the radiation damping coefficient b has dimensions of mass/time, and there are again five dimensional quantities in the functional relationship, namely

$$b, \rho, g, a, \lambda.$$

Again, out of these five dimensional quantities emerge two dimensionless quantities, which are related by

$$\frac{b}{\rho a^{3/2}}\sqrt{\frac{\lambda}{g}} = F(\frac{a^{1/2}}{\lambda}).$$

The above formulae are in agreement with the work of [2] for the case of a sphere half submerged in water. Havelock shows that, when $\frac{a^{1/2}}{\lambda}$ is small, then the latter relationship is linear, that is

$$\frac{b}{\rho a^{3/2}}\sqrt{\frac{\lambda}{g}} = \beta\frac{a^{1/2}}{\lambda},$$

where β is a constant which depends on whether the shape is a solid cylinder or an annulus. The value of β can be determined by analogous methods to those used to determine K.

Next, the coefficient c is considered. It can be shown (see [6]) that, in developed turbulent flows for which the Reynolds number is greater than 10^6, $c = \frac{1}{2}\rho a C_D$, where C_D is the non-dimensional drag coefficient. C_D depends on the body shape and for a sphere it is of order 10^{-1}.

4. Numerical solution

Observe that the nonlinear system of two second order differential equations (5), (6) can be written in the form

$$M\ddot{Z} + B\dot{Z} + PTO.\dot{Z} + C.diag(|\dot{Z}|).\dot{Z} + DZ = F, \tag{8}$$

where Z and F are the column 2-vectors

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad F = \begin{pmatrix} -\rho g a_1 x \\ -\rho g a_2 x \end{pmatrix}$$

and $M, B, C, D, PTO, diag|\dot{Z}|$ are the 2×2 matrices

$$M = \begin{pmatrix} m_1 + \mu_1, & 0 \\ 0, & m_2 + \mu_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1, & 0 \\ 0, & b_2 \end{pmatrix}, \quad C = \begin{pmatrix} c_1, & 0 \\ 0, & c_2 \end{pmatrix},$$
$$D = \begin{pmatrix} \rho g a_1, 0 & 0 \\ 0, & \rho g a_2 \end{pmatrix}, \quad PTO = pto \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}, \quad diag|\dot{Z}| = \begin{pmatrix} |\dot{z}_1|, & 0 \\ 0, & |\dot{z}_2| \end{pmatrix}.$$

Introducing the derivatives $\dot{Z}$ of the displacements Z as new dependent variables, the second order system (8) can be rewritten as a first order system of four first order equations in a standard way. Specifying homogeneous initial values the resulting initial-value problem is then solved numerically with the ODE45 code in Matlab.

In the following section some results of extensive numerical experiments are described. Values of the parameters are chosen, which reflect typical values arising from typical WEC.

5. Discussion of the results

In the graphs of this section all quantities are made dimensionless by using the physical parameters of the WEC device. The four components of the numerical solution, as functions of the dimensionless time T, are shown in Fig.1. $Y(1), Y(2)$ are the dimensionless displacements and $Y(3), Y(4)$ are the corresponding dimensionless velocities of the cylinder and annulus respectively. Similarly the dimensionless stroke and dimensionless power versus T are seen in Fig.2. In Fig.3, it is seen that the WEC resonates with the waves at a particular frequency, which is close, but not equal, to the frequency of the waves. It is observed that the dimensionless mean power generated by the WEC falls off rapidly on both sides of the resonant frequency.

In Fig.4, the dimensionless mean power is plotted against the frequency and the pto. It is easy to see that mean power at the resonant frequency increases with increasing pto. In Fig.5, the dimensionless mean power is plotted against the frequency and the dimensionless mass of the cylinder. It can be seen that the mean power does not change significantly with varying mass. Fig.6 shows the dependence of the dimensionless mean power on the frequency and the dimensionless cross sectional area of the annulus. This demonstrates that there are significant variations in the power output with changing cross section. In Fig.7, the dimensionless mean power is plotted against the frequency and the dimensionless mass of the cylinder. It is clear from this graph that the resonant frequency depends strongly on this mass.

Finally, Fig.8 shows clearly that the mean power output is highly dependent on the position of the WEC in its parameter space. The power output changes rapidly due to small variations in the parameter values. Indeed it can be seen that a catastrophic fall-off in power occurs in certain regions of the parameter space. The designer of a WEC must take such properties into account in order to ensure that the device is not constructed in such a way that it lies within or near such a region of parameter space.

The simple mathematical model discussed here provides qualitative guidance to the design engineer, because it identifies finite regions of the parameter space in which the design parameters should lie. This is important at the early stages of the design process, since it gives rapid, inexpensive guidance, which protects the engineer from gross design errors. Later, if more accurate values of the design parameters are required, more complex, mathematical models may be employed, which involve boundary element techniques or full three-dimensional CFD methods. These are, of course, orders of magnitude slower and more expensive than the simple models employed in the present paper.

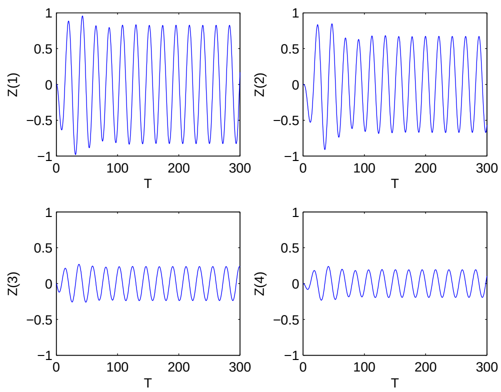

Figure 1: *Numerical solution of the initial value problem*

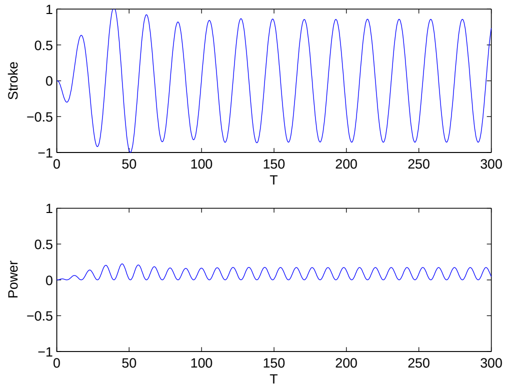

Figure 2: *Stroke and power versus time*

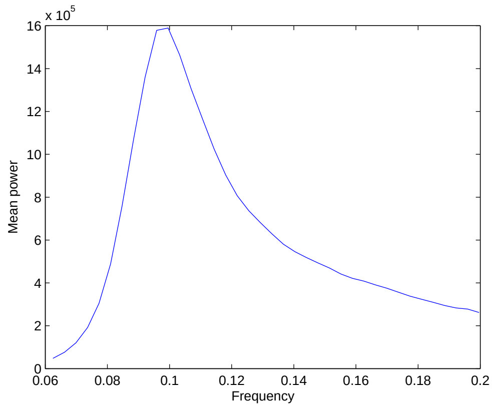

Figure 3: *Mean power versus frequency*

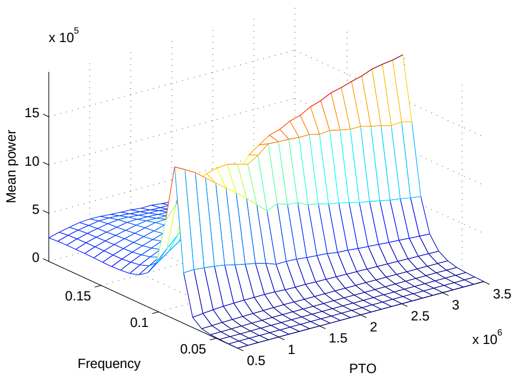

Figure 4: *Mean power versus frequency and pto*

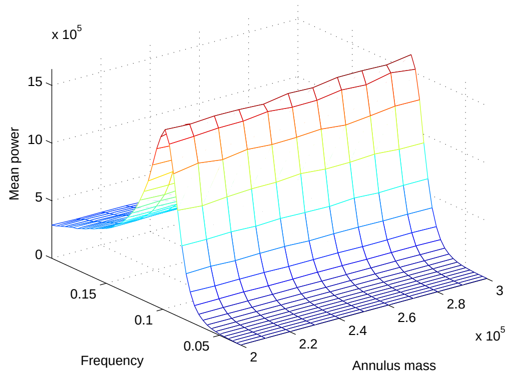

Figure 5: *Mean power versus frequency and annulus mass*

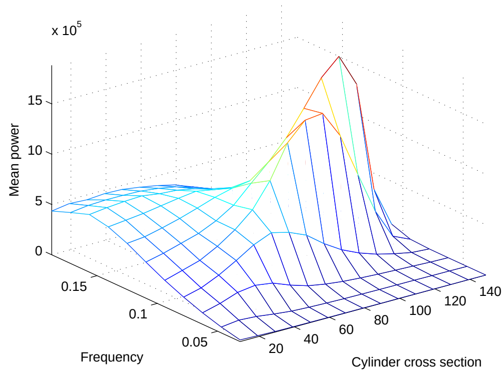

Figure 6: *Mean power versus frequency and cylinder cross section*

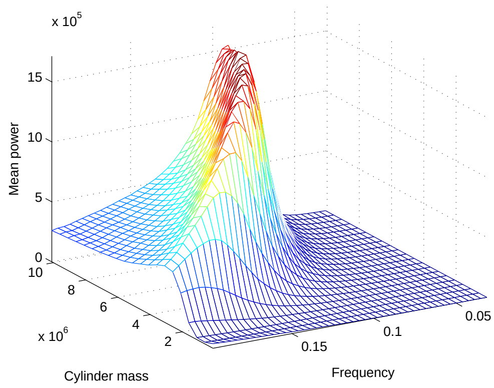

Figure 7: *Mean power versus frequency and cylinder mass*

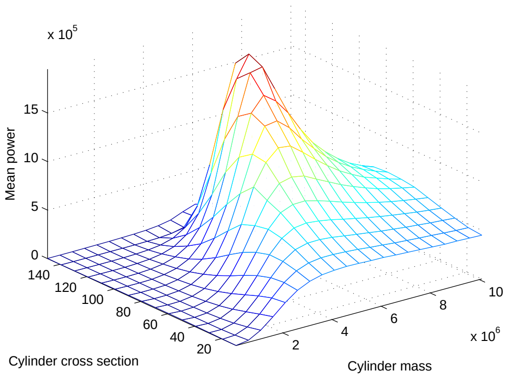

Figure 8: *Mean power versus annulus cross section and cylinder mass*

Acknowledgement

The authors are grateful to the unknown referees whose comments led to significant improvements in the exposition.

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