On some algebraic equations in connection with one kind of tangential polygons

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Abstract. *This article can be considered as an appendix to article [1]. Here article [1] is completed and extended, where some new relations concerning one kind of tangential polygons and algebraic equations are established.*

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The whole article is in some way connected with the following definition which is a completion of Definition 1 in [1].

Definition 1. *Let $A_1 \dots A_n$ be a tangential n-gon and let C be a denoted center of the inscribed circle into $A_1 \dots A_n$. Then we say that $A_1 \dots A_n$ is a k-inscribed tangential n-gon or, shortly, k-tangential n-gon if it has the following properties:*

1. *No two of its consecutive vertices are the same, that is, $A_i \neq A_{i+1}$ for each $i = 1, \dots, n$.*
2. *All of the angles*
$$\measuredangle CA_iA_{i+1}, \quad i = 1, \dots, n \tag{1}$$
have the same orientation, that is, all of them are positively or negatively oriented.
3. *It holds*
$$\beta_1 + \dots + \beta_n = (n-2k)\frac{\pi}{2}, \tag{2a}$$
where
$$\beta_i = \text{ measure of } \measuredangle CA_iA_{i+1}, \quad i = 1, \dots, n \tag{2b}$$
$$k \in \left\{1, 2, \dots, \left[\frac{n-1}{2}\right]\right\}, \tag{2c}$$
$$\left[\frac{n-1}{2}\right] = \frac{n-1}{2} \text{ if } n \text{ is odd, } \left[\frac{n-1}{2}\right] = \frac{n-2}{2} \text{ if } n \text{ is even.} \tag{2d}$$

Of course, indices of vertices A_i, A_{i+1} are calculated modulo n. Thus, $A_{n+1} = A_1$.

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Notice 1. *In the following we shall take the measure of an oriented angle with sign + or − depending on whether the angle is positively or negatively oriented. Also, let us remark that measures will be expressed by radians.*

From the following examples it will be easily seen that for a k-tangential n-gon it holds

$$|\varphi_1 + \cdots + \varphi_n| = 2k\pi, \tag{3}$$

where $\varphi_i =$ measure of oriented $\measuredangle A_iCA_{i+1}$, $\quad i = 1, \ldots, n$.

Example 1. *Pentagon $A_1 \ldots A_5$ shown in* Figure 1 *has a property that all of the angles $\measuredangle CA_iA_{i+1}$, $i = 1, \ldots, n$, are positively oriented and there holds*

$$2\beta_1 + \cdots + 2\beta_5 = (5-2)\pi$$

or

$$\beta_1 + \cdots + \beta_5 = (5 - 2 \cdot 1)\frac{\pi}{2}.$$

Thus, $A_1 \ldots A_5$ is a 1-tangential pentagon.

In this connection let us remark that all of the angles $\measuredangle A_iCA_{i+1}$, $i = 1, \ldots, n$, are negatively oriented and that

$$\begin{aligned} |\varphi_1 + \cdots + \varphi_5| &= \sum_{i=1}^{5} [\pi - (\beta_i + \beta_{i+1})] \\ &= 5\pi - (2\beta_1 + \cdots + 2\beta_5) \\ &= 5\pi - 3\pi = 2\pi = 2 \cdot 1 \cdot \pi. \end{aligned}$$

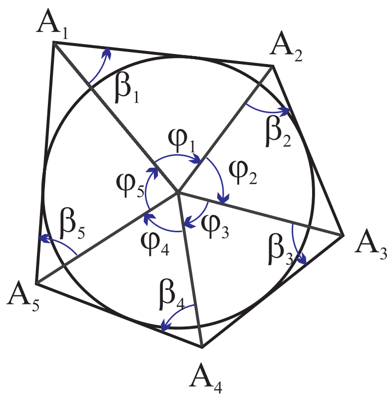

Figure 1.

Now let us consider the pentagon shown in Figure 2. *As can be seen, all of the angles $\measuredangle CA_iA_{i+1}$, $i = 1, \cdots, 5$, are negatively oriented, and all of the angles $\measuredangle A_iCA_{i+1}$, $i = 1, \cdots, 4$, are positively oriented. Therefore, for example,*

$$\beta_1 + \beta_2 = -(\pi - \varphi_1) \text{ since } \beta_1 < 0,\ \beta_2 < 0,\ \varphi_1 > 0.$$

Hence

$$2\beta_1 + \cdots + 2\beta_5 = \sum_{i=1}^{5} (\beta_i + \beta_{i+1}) = \sum_{i=1}^{5} (-\pi + \varphi_i) = -5\pi + 4\pi$$

which can be written as

$$|\beta_1| + \cdots + |\beta_5| = -(-5\pi + 4\pi) = (5 - 2 \cdot 2)\frac{\pi}{2}.$$

Thus, the pentagon shown in Figure 2 *is a 2-tangential pentagon. In this connection let us remark that* $\varphi_1 + \cdots + \varphi_5 = 2 \cdot 2\pi$.

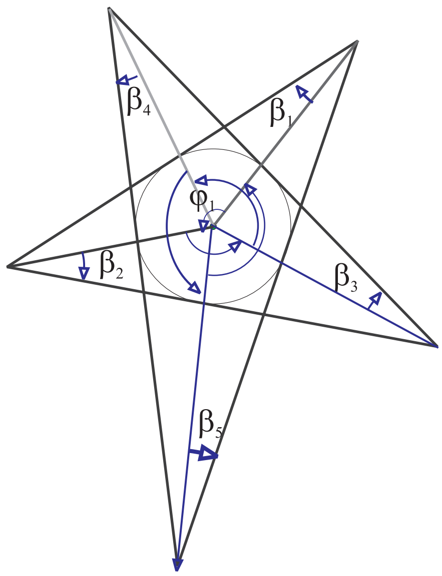

Figure 2.

Now let us remark that the numbering of vertices of the pentagon shown in Figure 1 *is such that the pentagon (as oriented one) is negatively oriented, and that the numbering of vertices of the pentagon shown in* Figure 2 *is such that the pentagon (as oriented one) is positively oriented.*

It is easy to see that generally there holds: If the numbering of the vertices of a k-tangential n-gon is such that the n-gon (as oriented one) is positively oriented, then $\varphi_i > 0,\ \beta_i < 0,\ i = 1, \ldots, n$, but if the n-gon is negatively oriented, then $\varphi_i < 0,\ \beta_i > 0,\ i = 1, \ldots, n$.

For convenience, in the following, where we shall mostly deal with $\beta_1, \ldots, \beta_n$, we shall suppose that the considered n-gon is negatively oriented, that is, all $\beta_1, \ldots, \beta_n$ are positive.

It can be easily proved that

$$\sum_{i=1}^{n} \varphi_i = -2k\pi \iff \sum_{i=1}^{n} \beta_i = (n - 2k)\frac{\pi}{2}.$$

So, we can write

$$-2k\pi = \sum_{i=1}^{n} \varphi_i = \sum_{i=1}^{n} [-\pi + (\beta_i + \beta_{i+1}] = -n\pi + 2(\beta_1 + \cdots + \beta_n),$$

from which follows that $\beta_1 + \cdots + \beta_n = (n-2k)\frac{\pi}{2}$.

First, we prove the following theorem which is a completion of Theorem 1 in [1].

Theorem 1. *Let $t_1, \dots, t_n$ be any given lengths (in fact, positive numbers), where $n \geq 3$. Then for each*

$$k \in \left\{1, 2, \dots \left[\frac{n-1}{2}\right]\right\} \tag{4}$$

there is a k-tangential n-gon $A_1^{(k)} \dots A_n^{(k)}$ such that

$$\left|A_i^{(k)} A_{i+1}^{(k)}\right| = t_i + t_{i+1}, \quad i = 1, \dots, n.$$

Proof. We need to prove that there are $\beta_1^{(k)}, \dots, \beta_n^{(k)}$ and length (radius) r_k such that for each k given by (4) it holds

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = (n-2k)\frac{\pi}{2}, \quad \tan \beta_i^{(k)} = \frac{r_k}{t_i}, \quad i = 1, \dots, n.$$

First, it is clear that for r enough large it holds

$$\sum_{i=1}^{n} \arctan \frac{r}{t_i} \approx n\frac{\pi}{2}.$$

Thus, there are lengths (radii) $r_1, r_2, \dots, r_m$, where $m = \left[\frac{n-1}{2}\right]$, such that

$$\begin{aligned}
\sum_{i=1}^{n} \arctan \frac{r_1}{t_i} &= (n-2\cdot 1)\frac{\pi}{2},\\
\sum_{i=1}^{n} \arctan \frac{r_2}{t_i} &= (n-2\cdot 2)\frac{\pi}{2},\\
\dots \quad & \quad \dots\\
\sum_{i=1}^{n} \arctan \frac{r_{m-1}}{t_i} &= (n-2(m-1))\frac{\pi}{2},\\
\sum_{i=1}^{n} \arctan \frac{r_m}{t_i} &= (n-2m)\frac{\pi}{2}.
\end{aligned}$$

Here let us remark that

$$\begin{aligned}
(n-2(m-1))\,\frac{\pi}{2} &= 3\frac{\pi}{2} \ \text{ if } n \text{ is odd}, \qquad & (n-2(m-1))\,\frac{\pi}{2} &= 2\pi \ \text{ if } n \text{ is even},\\
(n-2m)\,\frac{\pi}{2} &= \frac{\pi}{2} \ \text{ if } n \text{ is odd}, \qquad & (n-2m)\,\frac{\pi}{2} &= \pi \ \text{ if } n \text{ is even}.
\end{aligned}$$

□

In the following we shall use fundamental symmetric functions S_j^n and $\hat{S}_j^n$, first of $t_1, \dots, t_n$ and second of $\cot\beta_1^{(k)}, \dots, \cot\beta_n^{(k)}$, that is

$$S_j^n = \sum_{1 \le i_1 < \dots < i_j \le n}^{n} t_{i_1} \cdot \ldots \cdot t_{i_j}, \quad j =, 1, \dots, n$$

$$\hat{S}_j^n = \sum_{1 \le i_1 < \dots < i_j \le n}^{n} \cot\beta_{i_1}^{(k)} \cdot \ldots \cdot \cot\beta_{i_j}^{(k)}, \quad j =, 1, \dots, n.$$

For example

$$S_1^3 = t_1 + t_2 + t_3,\ S_2^3 = t_1t_2 + t_2t_3 + t_3t_1,\ S_3^3 = t_1t_2t_3.$$

Theorem 2. *The radii of these* $\left[\frac{n-1}{2}\right]$ *tangential n-gones described in* Theorem 1 *are positive roots of the equation*

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \dots + (-1)^{s_1} S_n^n = 0, \quad n \textit{ is odd} \tag{5a}$$

or

$$S_1^n x^{n-2} - S_3^n x^{n-4} + S_5^n x^{n-6} - \dots + (-1)^{s_2} S_{n-1}^n = 0, \quad n \textit{ is even,} \tag{5b}$$

where

$$s_1 = (1 + 3 + 5 + \dots + n) + 1, \quad s_2 = (1 + 3 + 5 + \dots + (n-1)) + 1. \tag{5c}$$

Proof. Let $\beta_i^{(k)} = \arctan\frac{r_k}{t_i}$, $i = 1, \dots, n$. Then

$$\beta_1^{(k)} + \dots + \beta_n^{(k)} = (n - 2k)\frac{\pi}{2}.$$

Hence

$$\cot\left(\beta_1^{(k)} + \dots + \beta_n^{(k)}\right) = 0 \quad \text{ if } n \text{ is odd.} \tag{6a}$$

$$\tan\left(\beta_1^{(k)} + \dots + \beta_n^{(k)}\right) = 0 \quad \text{ if } n \text{ is even.} \tag{6b}$$

Relation (6a) can be written as

$$\hat{S}_1^n - \hat{S}_3^n + \hat{S}_5^n - \dots + (-1)^{s_1}\hat{S}_n^n = 0, \tag{7}$$

from which, replacing $\cot\beta_i^{(k)}$ by $\frac{t_i}{r_k}$, we get equality

$$S_1^n r^{n-1} - S_3^n r^{n-3} + S_5^n r^{n-5} - \dots + (-1)^{s_1} S_n^n = 0. \tag{8}$$

Thus, in the case when n is odd, each r_k, $k = 1, \dots \frac{n-1}{2}$, is a root of equation (5a).

Similarly holds in the case when n is even, namely, relation (5b) can be written as

$$\hat{S}_1^n - \hat{S}_3^n + \hat{S}_5^n - \dots + (-1)^{s_2}\hat{S}_{n-1}^n = 0, \tag{9}$$

from which, replacing $\cot\beta_i^{(k)}$ by $\frac{t_i}{r_k}$, we get equality

$$S_1^n r^{n-2} - S_3^n r^{n-4} + S_5^n r^{n-6} - \cdots + (-1)^{s_2} S_{n-1}^n = 0. \tag{10}$$

So *Theorem 2* is proved. □

Here is an example.

Example 2. *Let $t_i = i,\ i = 1, \ldots, 5$. Then*

$$S_1^5 = 15, S_3^5 = 225, S_5^5 = 120$$

and

$$15r_k^4 - 225r_k^2 + 120 = 0, \quad k = 1, 2$$

for $r_1 \approx 3.800818595$, $r_2 \approx 0.74416262$.

It can be checked that

$$\sum_{i=1}^{5} \arctan\frac{r_1}{t_i} = 3\frac{\pi}{2}, \quad \sum_{i=1}^{5} \arctan\frac{r_2}{t_i} = \frac{\pi}{2}.$$

Theorem 3. *Let $t_1, \ldots, t_n, t_{n+1}, \ldots, t_{2n}$ be any given lengths such that*

$$t_{n+i} = t_i, \quad i = 1, \ldots, n. \tag{11}$$

Let $F_1(x)$, $F_2(x)$ and $F(x)$ be polynomials given by

$$\begin{aligned} F_1(x) &= S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \cdots + (-1)^{s_1} S_n^n, \quad n \text{ is odd}, \\ F_2(x) &= S_1^n x^{n-2} - S_3^n x^{n-4} + S_5^n x^{n-6} - \cdots + (-1)^{s_2} S_{n-1}^n, \quad n \text{ is even} \\ F(x) &= S_1^{2n} x^{2n-2} - S_3^{2n} x^{2n-4} + S_5^{2n} x^{2n-6} - \cdots + (-1)^s S_{2n-1}^{2n}, \end{aligned}$$

where s_1 and s_2 are given by (5c)*, and s is given by*

$$s = (1 + 3 + 5 + \cdots + (2n-1)) + 1.$$

Then

$$F_1(x)\Big|F(x) \quad \text{if } n \text{ is odd,}$$

$$F_2(x)\Big|F(x) \quad \text{if } n \text{ is even,}$$

where $\big|$ *stands for* is the divisor of.

Proof. By *Theorem 1* there are $m = \left[\frac{n-1}{2}\right]$ tangential n-gons whose tangent lengths are $t_1, \ldots, t_n$. Let

$$r_1^{(n)}, r_2^{(n)}, \ldots, r_m^{(n)}$$

denote radii of these m tangential n-gons, and let

$$r_1^{(2n)}, r_2^{(2n)}, \ldots, r_m^{(2n)}$$

denote radii of $2m\left(= n-1 = \frac{2n-2}{2}\right)$ tangential $2n$-gons whose tangent lengths are $t_1, \ldots, t_{2n}$ for which (11) holds. Then

$$r_i^{(n)} = r_{2i}^{(2n)}, \quad i = 1, \ldots, m.$$

It is because $r_{2i}^{(2n)}$ is the radius of a $2i$-tangential $2n$-gon which is a double i-tangential n-gon whose radius is $r_i^{(n)}$. For example, if $n = 5$, then

$$r_2^{(5)} = r_4^{(10)},$$

since a 4-tangential 10-gon (in the case when $t_{5+i} = t_i,\ i = 1, \ldots, 5$) is a double 2-tangential 5-gon. (See *Figure 3*)

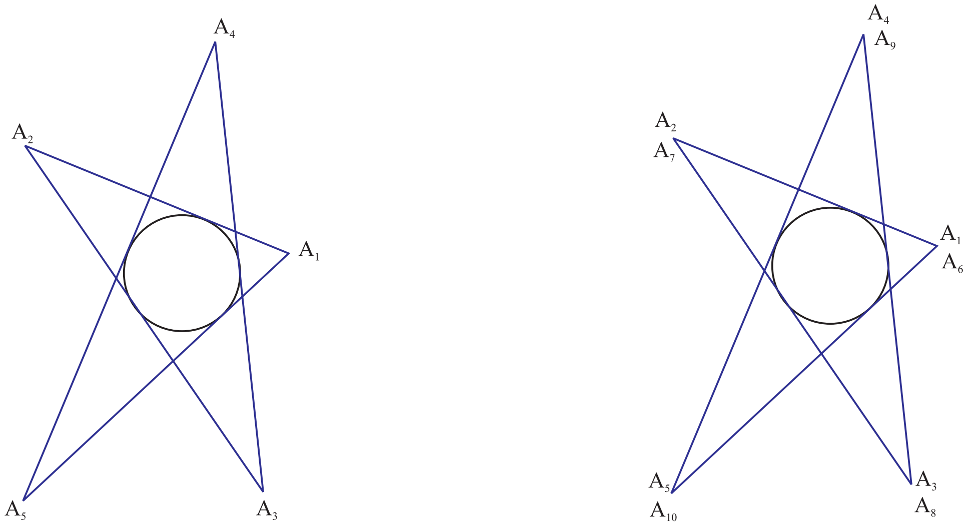

Figure 3.

□

Here let us remark that

$$\sum_{i=1}^{10} \arctan \frac{r_4^{(10)}}{t_i} = (10 - 2 \cdot 4)\frac{\pi}{2},$$

or

$$2\sum_{i=1}^{5} \arctan \frac{r_4^{(10)}}{t_i} = (10 - 2 \cdot 4)\frac{\pi}{2},$$

from which there follows

$$\sum_{i=1}^{5} \arctan \frac{r_4^{(10)}}{t_i} = (5 - 2 \cdot 2)\frac{\pi}{2}.$$

But also

$$\sum_{i=1}^{5} \arctan \frac{r_2^{(5)}}{t_i} = (5 - 2 \cdot 2)\frac{\pi}{2}.$$

Corollary 1. *Let* $t_1, \ldots, t_{3n}$ *be any given lengths such that*

$$t_{n+i} = t_{2n+i} = t_i, \quad i = 1, \ldots, n.$$

Let $F_1(x)$ *and* $F_2(x)$ *be as in* Theorem 3, *and let* $G(x)$ *and* $H(x)$ *be given by*

$$G(x) = S_1^{3n} x^{3n-1} - S_3^{3n} x^{3n-3} + S_5^{3n} x^{3n-5} - \cdots + (-1)^u S_{3n}^{3n} \quad \text{if } n \text{ is odd} \tag{12}$$
$$H(x) = S_1^{3n} x^{3n-2} - S_3^{3n} x^{3n-4} + S_5^{3n} x^{3n-6} - \cdots + (-1)^v S_{3n-1}^{3n} \quad \text{if } n \text{ is even,} \tag{13}$$

where

$$u = (1 + 3 + 5 + \cdots + 3n) + 1, \quad v = (1 + 3 + 5 + \cdots + (3n - 1)) + 1.$$

Then

$$F_1(x) \Big| G(x) \qquad \text{if } n \text{ is odd,} \tag{14}$$

$$F_2(x) \Big| H(x) \qquad \text{if } n \text{ is even.} \tag{15}$$

Proof. It holds

$$r_i^{(n)} = r_{3i}^{(3n)}, \quad i = 1, \ldots, n.$$

□

Notice 2. *Concerning* Theorem 3, *it can be easily seen that there are many other cases where one equation is the divisor of the other, which may be interesting in the theory of algebraic equations.*

References

[1] M. RADIĆ, *Some relations and properties concerning tangential polygons*, Mathematical Communications **4**(1999), 197-206.