Multipliers and factorizations for bounded $\mathcal{I}-$convergent sequences

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Abstract. *Connor, Demirci and Orhan* [5] *studied multipliers for bounded statistically convergent sequences. In this paper we get analogous results for $\mathcal{I}-$convergent sequences.*

Key words: *statistical convergent sequence, $\mathcal{I}-$convergent sequence, $\mathcal{I}-$limit point of sequences of real numbers*

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1. Introduction

Kostyrko, Mačaj and Šalát [12] , [13] introduced the concept of $\mathcal{I}-$convergence of sequences of real numbers based on the notion of the ideal of subsets of $\mathbb{N}$. Some results on $\mathcal{I}$-convergence may be found in [6] , [13]. In this paper we study multipliers for $\mathcal{I}$-convergence. So we show that our results are a non-trivial generalization of well-known results in classical convergence, statistical convergence, A-statistical convergence and uniform statistical convergence.

If K is a subset of natural numbers $\mathbb{N}$, K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ will denote the cardinality of K_n. The natural density of K [17], is given by $\delta(K) := \lim_n \frac{1}{n} |K_n|$, if it exists. Fast introduced the definition of statistical convergence using the natural density of a set. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero [7]. Hence x is statistically convergent to L iff $(C_1 \chi_{K(\varepsilon)})_n \to 0$, (as $n \to \infty$, for ever $\varepsilon > 0$), where C_1 is the Cesáro mean of order one and χ_K is the characteristic function of the set K. Properties of statistically convergent sequences have been studied in [2] , [3] , [9] , [10] , [15] , [18].

Statistical convergence can be generalized by using a nonnegative regular summability matrix A in place of C_1.

Following Freedman and Sember [8] , we say that a set $K \subseteq \mathbb{N}$ has $A-$density if $\delta_A(K) := \lim_n (A\chi_K)_n = \lim_n \sum_{k \in K} a_{nk}$ exists where $A = (a_{nk})$ is a nonnegative regular matrix.

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The number sequence $x = (x_k)$ is $A-$statistically convergent to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon)$ has $A-$density zero [2], [8], [11], [15].

Also Connor has introduced $\mu-$statistical analogue of these concepts using a finitely additive set function μ taking values in $[0, 1]$ defined on a field Γ of subsets of $\mathbb{N}$ such that if $|A| < \infty$, then $\mu(A) = 0$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$ and $\mu(\mathbb{N}) = 1$ [4].

The number sequence $x = (x_k)$ is $\mu-$statistically convergent to L provided that $\mu(\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$ [4].

2. Definition and notations

Now we introduce some notation and basic definitions used in this paper. A sequence space is a linear subspace of the collection of all scalar valued sequences. we let c, c_0, and ℓ^∞ denote the normed by $\|x\| = \sup_n |x_n|$.

We first recall the concepts of an ideal and filter of sets.

Definition 1 [see [12]**,** [13]**,** [14, *p*.34]**].** *Let $X \neq \phi$. A class $S \subseteq 2^X$ of subsets of X is said to be an ideal in X provided that S is additive and hereditary , i.e if S satisfies the conditions:*

(i) $\phi \in S$,

(ii) $A, B \in S \Rightarrow A \cup B \in S$,

(iii) $A \in S$, $B \subseteq A \Rightarrow B \in S$.

An ideal is called non-trivial if $X \notin S$.

Definition 2 [see [12], [13], [16, *p*.44]**].** *Let $X \neq \phi$. A non-empty class $\mathcal{F} \subseteq 2^X$ of subsets of X is said to be a filter in X provided that:*

(i) $\phi \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,

(iii) $\phi \in \mathcal{F}$, $A \subseteq B \Rightarrow B \in \mathcal{F}$.

The following proposition expresses a relation between the notions of an ideal and a filter :

Proposition 1 [see [6]**,** [12], [13]**].** *Let S be a non-trivial in X, $X \neq \phi$. Then the class*

$$\mathcal{F}(S) = \{M \subseteq X : \exists A \in S : M = X \setminus A\} \tag{1}$$

is a filter on X (we will call $\mathcal{F}(S)$ the filter associated with S).

Definition 3 [see [6]**,** [12], [13]**].** *A non- trivial ideal S in X is called admissible if $\{x\} \in S$ for each $x \in X$.*

As usual, $\mathbb{R}$ will denote the real numbers.

Definition 4 [see [6]**,** [12], [13]**].** *Let $\mathcal{I}$ be a non- trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_n)$ of real numbers is said to be $\mathcal{I}-$convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$ the set $A(\varepsilon) = \{n \ : \ |x_n - L| \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case we write $\mathcal{I}-\lim x = L$.*

By $F_{\mathcal{I}}$ and $F_{\mathcal{I}}(b)$ we denote the set of all $\mathcal{I}$-convergent sequences and all $\mathcal{I}$-convergent bounded sequences. And by $F_{\mathcal{I}}^0(b)$ we denote all $\mathcal{I}$-convergent bounded null sequences. Throughout the paper $\mathcal{I}$ will be an admissible ideal.

3. Multipliers

Assume that two sequence spaces, E and F are given. A multiplier from E into F is a sequence u such that $ux = (u_n x_n) \in F$ whenever $x \in E$. The linear space of all such multipliers will be denoted by $m(E, F)$.

Bounded multipliers will be denoted by $M(E, F)$. Hence $M(E, F) = \ell^\infty \cap m(E, F)$. If $E = F$, then we write $m(E)$ and $M(E)$ instead of $m(E, F)$ and $M(E, F)$, respectively.

Connor, Demirci and Orhan [5] studied multipliers for bounded statistically convergent sequences.

This section is devoted to multipliers on or into $F_{\mathcal{I}}(b)$ and $F_{\mathcal{I}}^0(b)$. Before we begin, we note that if E and F are subspaces of ℓ^∞ that contain c_o, then $c_0 \subset m(E, F) \subset \ell^\infty$. The first inclusion follows from noting that if $u \in c_0$ and $x \in \ell^\infty$, then $ux \in c_0 \subset F$. The second inclusion follows from noting that $u \in m(E, F)$, then $ux \in F \subset \ell^\infty$ for all $x \in c_0 \subset E$, and hence $u \in \ell^\infty$.

We have the following

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal in N. Then*

(i) $m(F_{\mathcal{I}}^0(b)) = M(F_{\mathcal{I}}^0(b)) = \ell^\infty$,

(ii) $m(F_{\mathcal{I}}(b)) = F_{\mathcal{I}}(b)$.

Proof. (i) We show that $m(F_{\mathcal{I}}^0(b)) = \ell^\infty$. The observation preceding the theorem yields that $m(F_{\mathcal{I}}^0(b)) \subset \ell^\infty$. Note that if $u \in \ell^\infty$ and $z \in F_{\mathcal{I}}^0(b)$, then

$$\{k \ : \ |u_k \ z_k| \geq \varepsilon\} \subseteq \left\{k \ : \ |z_k| \geq \frac{\varepsilon}{\|u\|_\infty + 1}\right\}$$

and the right set belongs to $\mathcal{I}$, so $\{k \ : \ |u_k \ z_k| \geq \varepsilon\} \in \mathcal{I}$. Also note uz is bounded and hence $\ell^\infty \subseteq m(F_{\mathcal{I}}^0(b))$.

(ii) First observe that $\chi_N \in F_{\mathcal{I}}(b)$ implies that $m(F_{\mathcal{I}}(b)) \subset F_{\mathcal{I}}(b)$. Conversely, if $u \in F_{\mathcal{I}}(b)$, then $ux \in F_{\mathcal{I}}(b)$ for any $x \in F_{\mathcal{I}}(b)$. Hence $u \in m(F_{\mathcal{I}}(b))$, which proves the claim. □

Before proving theorem 3, we observe that, in general $c_0 \subset m(F_{\mathcal{I}}(b),\ c) \subseteq c$. The first inclusion follows from noting $ux \in c_0 \subseteq F_{\mathcal{I}}(b)$ for any $u \in c_0$ and $x \in \ell^\infty$. The second inclusion follows from $\chi_N \in F_{\mathcal{I}}(b)$. Note that if $F_{\mathcal{I}}(b) = c$, then $m(c, F_{\mathcal{I}}(b)) = c$. The next theorem shows that this the only situation for which $m(F_{\mathcal{I}}(b), c) = c$.

Theorem 2. *Let $\mathcal{I}$ be an admissible ideal in N. Then*

(i) If c is a proper subset of $F_{\mathcal{I}}(b)$, then $m(\ F_{\mathcal{I}}(b),\ c) = c$,

(ii) $m(c,\ F_{\mathcal{I}}(b)) = F_{\mathcal{I}}(b)$.

Proof. (i) Given the remarks preceding the theorem, all we need to establish is that if $u \in c$ and $\lim x = \ell \neq 0$, then $u \in m(\mathcal{F}_{\mathcal{I}}(b), c)$. Let $z \in \mathcal{F}_{\mathcal{I}}(b)$, $z \notin c$, and, without loss of generality, suppose z is $\mathcal{I}-$convergent to 1. Then there is an $\varepsilon > 0$ such that $A := \{k : |z_k - 1| \geq \varepsilon\} \in \mathcal{I}$. Define x by $x_k = \chi_{A^c}(k)$ and observe that x is $\mathcal{I}-$convergent to 1, hence $x \in \mathcal{F}_{\mathcal{I}}(b)$. Also note xu converges to $\ell \neq 0$ along A^c and to 0 along A, hence $xu \notin c$ and thus $u \notin m(\mathcal{F}_{\mathcal{I}}(b), c)$.

(ii) As $\chi_N \notin c$, $m(c, \mathcal{F}_{\mathcal{I}}(b)) \subseteq \mathcal{F}_{\mathcal{I}}(b)$. The reverse inclusion follows from noting that if $u \in \mathcal{F}_{\mathcal{I}}(b)$ and $x \in c \subseteq \mathcal{F}_{\mathcal{I}}(b)$, then ux is $\mathcal{I}-$convergent. □

Theorem 3. *Let $\mathcal{I}$ be an admissable ideal in $\mathbb{N}$. Then* $m\left(c_0, \mathcal{F}^0_{\mathcal{I}}(b)\right) = \ell^\infty$

Proof.

$$\ell^\infty = m\,(c_0, c_0) = \{u : ux \in c_0 \text{ for all } x \in c_0\} \tag{2}$$

in general $c_0 \subset \mathcal{F}^0_{\mathcal{I}}(b)$ and hence

$$\ell^\infty = m\,(c_0, c_0) \subseteq m\left(c_0, \mathcal{F}^0_{\mathcal{I}}(b)\right) \subset \ell^\infty. \tag{3}$$

It follows

$$m\left(c_0, \mathcal{F}^0_{\mathcal{I}}(b)\right) = \ell^\infty. \tag{4}$$

□

Further we will give some special cases.

4. Special cases

Case 1. Let $\mathcal{I}$ be a class of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}$-convergence reduces to a classical convergence.
Case 2. Let $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta\,(K) = 0\}$. Then $\mathcal{I}$-convergence reduces to a statisticall convergence.
Case 3. Let $\mathcal{I} = \{K \in \Gamma : \mu\,(K) = 0\}$. Then $\mathcal{I}$-convergence reduces to a $\mu-$statistical convergence.
Case 4. Let $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta_A\,(K) = 0\}$(see [12] , [13]). Then $\mathcal{I}$-convergence reduces to an A-statisticall convergence.
Case 5. Furthermore, let $\mathcal{I} = \{K \subseteq \mathbb{N} : u\,(K) = 0\}$(see[1]). Then Then $\mathcal{I}$-convergence reduces to an $\mathcal{I}_u$- convergence.

These special cases show that our results are non-trivial generations of well-known results.

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