Sensitivity analysis of the proportionate change of a subset of outputs or/and inputs in DEA*

JÜRGEN GUDDAT†, LUKA NERALIĆ‡ AND OLIVER STEIN§

Abstract. *Sensitivity analysis in data envelopment analysis (DEA) is studied for the case of the proportionate change of a subset of outputs or/and of a subset of inputs of an efficient decision making unit (DMU) according to the Charnes–Cooper–Rhodes (CCR) ratio model. Sufficient conditions for an efficient DMU to preserve its efficiency under the proportionate change of a subset of inputs or/and of outputs are obtained. An illustrative numerical example is provided.*

Key words: *data envelopment analysis, efficiency, proportionate change of a subset of outputs or/and of inputs, sensitivity analysis, linear programming*

AMS subject classifications: 90C05, 90C031, 90C50

Received May 17, 2006 Accepted January 14, 2007

1. Introduction

Sensitivity analysis in Data Envelopment Analysis (DEA) for the Charnes–Cooper–Rhodes (CCR) ratio model (see Charnes and Cooper [1], Cooper *et al.* [5]) for the cases (a) with the proportionate change (increase) of all inputs and (b) with the proportionate change (decrease) of all outputs were studied by Charnes and Neralić [3]. In (a) sufficient conditions for an efficient DMU to preserve its efficiency after the proportionate change (increase) of all inputs were obtained and in (b) sufficient conditions for an efficient DMU after the proportionate change (decrease) of all outputs were given. The case of the simultaneous proportionate change (increase) of all inputs and proportionate change (decrease) of all outputs of an efficient DMU

*This research was partly supported by the MZOS grant 067-0000000-1076 of the Republic of Croatia. The paper was presented at the Eight International Conference on Approximation and Optimization in the Caribbean, Santo Domingo, Republica Dominicana, April 3 - 7, 2006.

†Humboldt-University Berlin, Institute of Mathematics, Unter den Linden 6, D-10099 Berlin, Germany, e:mail: `guddat@mathematik.hu-berlin.de`

‡University of Zagreb, Faculty of Economics, Trg J. F. Kennedya 6, HR-10 000 Zagreb, Croatia, e-mail: `lneralic@efzg.hr`

§University of Karlsruhe, Faculty of Economics, D-76 128 Karlsruhe, Germany, e-mail: `stein@wior.uni-karlsruhe.de` This author gratefully acknowledges financial support through a Heisenberg grant of the *Deutsche Forschungsgemeinschaft.*

preserving its efficiency for the CCR ratio model was studied by Charnes and Neralić [4] and sufficient conditions for efficiency preservation of efficient DMU were established. Similar results for the additive model were obtained by Neralić [7]. Sufficient conditions for an efficient DMU to preserve efficiency for the case of the proportionate change (increase) of all inputs or/and the proportionate change (decrease) of all outputs for the CCR ratio model and for the Additive model with different coefficients of proportionality for all inputs or/and for all outputs were obtained by Neralić and Sexton [8].

The aim of this paper is firstly to study the case of the simultaneous proportionate change (decrease) of a subset of outputs or/and the proportionate change (increase) of a subset of inputs of an efficient DMU preserving its efficiency for the CCR ratio model. Secondly, the case of the proportionate change (decrease) of a subset of outputs or/and the proportionate change (increase) of a subset of inputs for the CCR ratio model with different coefficients of proportionality for outputs and/or for inputs will be studied too. Sufficient conditions for an efficient DMU according to the CCR ratio model to preserve efficiency after the simultaneous proportionate change (decrease) of a subset of outputs or/and of the proportionate change (increase) of a subset of inputs are given for the first case. (Similar results can be obtained for the second case.) In that way a measure of stability of efficiency for an efficient DMU is obtained.

The paper is organized as follows. Results in sensitivity analysis for the proportionate change (increase) of a subset of inputs or/and the proportionate change (decrease) of outputs for two considered cases are contained in *Section 2*. *Section 3* gives an illustrative example. The last Section contains some conclusions and suggestions for further research.

2. Sensitivity analysis of a subset of outputs or/and of a subset of inputs

2.1. Let us suppose that there are n Decision Making Units (DMUs) with m inputs and s outputs. Let x_{ij} be the observed amount of ith type of input of the jth DMU ($x_{ij} > 0$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$) and let y_{rj} be the observed amount of output of the rth type for the jth DMU ($y_{rj} > 0$, $r = 1, 2, \ldots, s, j = 1, 2, \ldots, n$). Let $Y_j,\ X_j$ be the observed vectors of outputs and inputs of the DMU_j, respectively, $j = 1, 2, \ldots, n$. Let e be the column vector of ones and let T as a superscript denote the transpose. In order to see if the $\mathrm{DMU}_{j0} = \mathrm{DMU}_0$ is efficient according to the CCR ratio model the following linear programming problem should be solved:

$$\min 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - \varepsilon e^T s^+ - \varepsilon e^T s^- + \theta$$

subject to

$$\begin{array}{rcccl} Y_1\lambda_1 + \cdots + Y_0\lambda_0 + \cdots + Y_n\lambda_n & - & s^+ & & = Y_0 \\ -X_1\lambda_1 - \cdots - X_0\lambda_0 - \cdots - X_n\lambda_n & - & s^- & + \ X_0\theta & = \ 0 \\ & & \lambda_1,\, \ldots,\, \lambda_n,\, s^+, & s^- & \geq \ 0, \end{array} \tag{1}$$

with $Y_0 = Y_{j_0}, X_0 = X_{j_0}, \lambda_0 = \lambda_{j_0}$ and θ unconstrained. The symbol ε represents the infinitesimal we use to generate the non-Archimedean ordered extension field.

In this extension field ε is less than every positive number in our base field, but greater than zero. DMU_0 is DEA efficient if and only if for the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of the linear programming problem (1) both of the following are satisfied (for details see Charnes and Cooper [1]):

$$\begin{aligned} \theta^* &= 1 \\ s^{+*} &= s^{-*} = 0, \quad \text{in all alternative optima.} \end{aligned} \tag{2}$$

Linear programming problem (1), with discussion about its use, can be found in Cooper *et al* [5], pp 73-75. As it is pointed out in that book on p. 73, representing the non-Archimedean infinitesimal $\varepsilon > 0$ by a small real number could get erroneous results. Because of that, a two-phase procedure for solving (1) is suggested (see Cooper *et al* [5], p. 43-45, 50-51). In Phase I we solve the LP problem

$$\min \theta \tag{3}$$

subject to the constraints in (1). Using the optimal value $\theta^* = \min \theta$ of the LP problem (3) from Phase I, in Phase II we solve the LP problem

$$\min(-e^T s^+ - e^T s^-) \tag{4}$$

subject to the constraints in (1) with the substitution $\theta = \theta^*$. DMU_0 is called CCR efficient if an optimal value of (3) and (4) satisfies $\theta^* = 1$ and $s^{+*} = 0, s^{-*} = 0$ respectively (see Cooper *et al* [5], p. 45).

2.2. We are interested in the proportionate change of a subset of outputs or/and of a subset of inputs of an efficient DMU_0 preserving its efficiency. An increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU_0. Hence, without loss of generality, we can restrict attention to the proportionate decrease of the subset of the first $\bar{s}$ ($\bar{s} < s$) outputs which can be written as

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, \bar{s}, \tag{5}$$

with the last $s - \bar{s}$ outputs fixed

$$\widehat{y}_{r0} = y_{r0}, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \tag{6}$$

Similarly, a decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient DMU_0. Hence we can restrict attention to upward variations of inputs of an efficient DMU_0. Without loss of generality, let us consider the proportionate increase of the subset of the first $\overline{m}$ ($\overline{m} < m$) inputs, with the last $m - \overline{m}$ inputs fixed. It can be written as

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, \overline{m}, \tag{7}$$

and

$$\widehat{x}_{i0} = x_{i0}, \quad i = \overline{m} + 1, \overline{m} + 2, \ldots, m. \tag{8}$$

Let us introduce the following substitution

$$\widehat{\alpha} = 1 - \alpha, \quad 0 \leq \alpha < 1. \tag{9}$$

Using (9) in (5) we have

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad r = 1, 2, \ldots, \bar{s}, \tag{10}$$

with

$$\alpha_r = \alpha y_{r0}, \quad \alpha_r \geq 0, \quad r = 1, 2, \ldots, \bar{s}. \tag{11}$$

Because of (6) we have

$$\widehat{\alpha} = 1, \quad \alpha = 0, \quad \alpha_r = 0, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \tag{12}$$

Let us also introduce the substitution

$$\widehat{\beta} = 1 + \beta, \quad \beta \geq 0. \tag{13}$$

Using (13) in (7) we have

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \quad i = 1, 2, \ldots, \overline{m}, \tag{14}$$

with

$$\beta_i = \beta x_{i0}, \quad \beta_i \geq 0, \quad i = 1, 2, \ldots, \overline{m}. \tag{15}$$

Because of (8) we have

$$\widehat{\beta} = 1, \quad \beta = 0, \quad \beta_i = 0, \quad i = \overline{m} + 1, \overline{m} + 2, \ldots, m. \tag{16}$$

It means that the proportionate change of a subset of outputs (5), with the other outputs fixed (6), can be considered as the additive change (10), with α_r in (11) and (12). Similarly, the proportionate change of a subset of inputs (7), with the other inputs fixed (8), can be considered as the additive change (14) with β_i in (15) and (16). Because of that we will consider additive changes (10) of outputs together with (11) - (12) or/and additive changes (14) of inputs together with (15) - (16).

2.3. We will also consider the proportionate decrease of a subset of the first $\bar{s}$ ($\bar{s} < s$) outputs with different coefficients of proportionality which can be written as

$$\widehat{y}_{r0} = \widehat{\alpha}_r y_{r0}, \quad 0 < \widehat{\alpha}_r \leq 1, \quad r = 1, 2, \ldots, \bar{s}, \tag{17}$$

with the last $s - \bar{s}$ outputs fixed

$$\widehat{y}_{r0} = y_{r0}, \quad r = \bar{s} + 1, \bar{s} + 2, \ldots, s. \tag{18}$$

Similarly, we will consider the proportionate increase of the subset of the first $\overline{m}$ ($\overline{m} < m$) inputs with different coefficients of proportionality and the last $m - \overline{m}$ inputs fixed. It can be written as

$$\widehat{x}_{i0} = \widehat{\beta}_i x_{i0}, \quad \widehat{\beta}_i \geq 1, \quad i = 1, 2, \ldots, \overline{m}, \tag{19}$$

and

$$\widehat{x}_{i0} = x_{i0}, \quad i = \overline{m}+1, \overline{m}+2, \ldots, m. \tag{20}$$

Let us introduce the following notation

$$\widehat{\alpha}_r = 1 - \bar{\alpha}_r, \quad 0 \le \bar{\alpha}_r < 1. \tag{21}$$

Using (21) in (17) we have

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad r = 1, 2, \ldots, \bar{s}, \tag{22}$$

with

$$\alpha_r = \bar{\alpha}_r y_{r0}, \quad \alpha_r \ge 0, \quad r = 1, 2, \ldots, \bar{s}. \tag{23}$$

Because of (18) we have

$$\widehat{\alpha}_r = 1, \quad \bar{\alpha}_r = 0, \quad \alpha_r = 0, \quad r = \bar{s}+1, \bar{s}+2, \ldots, s. \tag{24}$$

Similarly, let use introduce notation

$$\widehat{\beta}_i = 1 + \bar{\beta}_i, \quad \bar{\beta}_i \ge 0. \tag{25}$$

Using (25) in (19) we have

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \quad i = 1, 2, \ldots, \overline{m}, \tag{26}$$

with

$$\beta_i = \bar{\beta}_i x_{i0}, \quad \beta_i \ge 0, \quad i = 1, 2, \ldots, \overline{m}. \tag{27}$$

Because of (20) we have

$$\widehat{\beta}_i = 1, \quad \bar{\beta}_i = 0, \quad \beta_i = 0, \quad i = \overline{m}+1, \overline{m}+2, \ldots, m. \tag{28}$$

So, the proportionate change of a subset of outputs with different coefficients of proportionality (17), with the other outputs fixed (18), can be considered as the additive change (22), with α_r in (23) and (24). Similarly, the proportionate change of a subset of inputs with different coefficients of proportionality (19), with the other inputs fixed (20), can be considered as the additive change (26) with β_i in (27) and (28). Because of that we will consider additive changes (22) of outputs together with (23) - (24) or/and additive changes (26) of inputs together with (27) - (28).

2.4. For an efficient DMU$_0$ because of (2) vectors $[\ Y_0 \quad -X_0\ ]^T$ and $[\ 0 \quad X_0\ ]^T$ must occur in some optimal basis, which means that there is a basic optimal solution to (1) with $\lambda_0^* = 1$ and $\theta^* = 1$. Similarly to Charnes and Neralić [2], simultaneous changes (10) - (12), and changes (14) - (16), are then accompanied by alterations in the inverse B^{-1} of the optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 & 0 \\ -X_B & 0 & -I_B^- & X_0 \end{bmatrix}, \tag{29}$$

which corresponds to the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of (1) with $\lambda_0^* = 1$ and $\theta^* = 1$. Let

$$B^{-1} = \left[b_{ij}^{-1}\right], \quad i,\ j = 1, 2, \ldots, s+m,$$

be the inverse of the optimal basis matrix B in (29). Let $P_j,\ j = 1, 2, \ldots, n+s+m+1$ be the columns of the matrix and let P_0 be the right-hand side of the linear programming problem (1). We will use the following notation:

$$\begin{aligned} \Gamma_j &= B^{-1}P_j,\ j = 0, 1, \ldots, n+s+m+1, \\ \omega^T &= c_B^T B^{-1}, \\ z_j &= c_B^T B^{-1} P_j \\ &= \omega^T P_j,\ j = 0, 1, \ldots, n+s+m+1. \end{aligned}$$

The simultaneous change of outputs (10) together with (11) - (12) and of inputs (14) together with (15) - (16) leads to the following change of the optimal basis matrix B

$$\widehat{B} = B + \triangle B \tag{30}$$

with

$$\triangle B = \begin{array}{c} \begin{array}{ccccccc} & & & k & & & s+m \\ & & & \downarrow & & & \downarrow \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_{\bar{s}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_1 & 0 & \cdots & \beta_1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\beta_{\overline{m}} & 0 & \cdots & \beta_{\overline{m}} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \tag{31}$$

and the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + \left[-\alpha_1\ -\alpha_2\ \ldots - \alpha_{\bar{s}}\ 0\ \ldots 0\ \right]^T, \tag{32}$$

where indices k and $s+m$ correspond to the optimal basic variables $\lambda_0^* = 1$ and

$\theta^* = 1$, respectively. Using matrices

$$U = U_{(s+m)\times 2} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \vdots & \vdots \\ \alpha_{\bar{s}} & \alpha_{\bar{s}} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \beta_1 & 0 \\ \vdots & \vdots \\ \beta_{\overline{m}} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \tag{33}$$

and

$$V^T = V^T_{2\times(s+m)} = \begin{array}{c} \begin{array}{ccccccccc} & & & k & & & & s+m & \\ & & & \downarrow & & & & \downarrow & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \end{array} \tag{34}$$

we can write the perturbation matrix (31) as

$$\triangle B = UV^T. \tag{35}$$

As in Charnes and Neralić [2], because of (30) and (35) we can use the Sherman-Morrison-Woodbury formula (see, for example, Golub and Van Loan [6], p. 3) to get the following perturbed basis matrix inverse

$$\begin{aligned} (\widehat{B})^{-1} &= (B + UV^T)^{-1} \\ &= B^{-1} - B^{-1}U(I + V^T B^{-1} U)^{-1} V^T B^{-1}. \end{aligned} \tag{36}$$

Using the abbreviation

$$D = U(I + V^T B^{-1} U)^{-1} V^T \tag{37}$$

we can write (36) as

$$\begin{aligned} (\widehat{B})^{-1} &= B^{-1} - B^{-1} D B^{-1} \\ &= B^{-1}(I - DB^{-1}) \\ &= (I - B^{-1}D)B^{-1}. \end{aligned} \tag{38}$$

Also, using (33), (34) we can get $V^T B^{-1} U$ and

$$M = I + V^T B^{-1} U \tag{39}$$

with

$$\det M = 1 - \sum_{t=1}^{\bar{s}} b_{k,t}^{-1}\alpha_t + \sum_{t=1}^{\overline{m}}(-b_{k,s+t}^{-1} + b_{s+m,s+t}^{-1})\beta_t + \\ + (\sum_{t=1}^{\bar{s}} b_{s+m,t}^{-1}\alpha_t)(\sum_{t=1}^{\overline{m}} b_{k,s+t}^{-1}\beta_t) - (\sum_{t=1}^{\bar{s}} b_{k,t}^{-1}\alpha_t)(\sum_{t=1}^{\overline{m}} b_{s+m,s+t}^{-1}\beta_t). \quad (40)$$

It is easy to get inverse M^{-1} of the matrix M and matrix $D = UM^{-1}V^T$ in the form

$$D = \begin{bmatrix} 0 & \dots & 0 & d_{1k} & 0 & \dots & 0 & d_{1,s+m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & d_{\bar{s}k} & 0 & \dots & 0 & d_{\bar{s},s+m} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & d_{s+1,k} & 0 & \dots & 0 & d_{s+1,s+m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & d_{s+\overline{m},k} & 0 & \dots & 0 & d_{s+\overline{m},s+m} \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad (41)$$

with

$$d_{t,k} = -\frac{1}{\det M}(1 + \sum_{t=1}^{\overline{m}} b_{s+m,s+t}^{-1}\beta_t)\alpha_t, \ t = 1, 2, \dots, \bar{s}, \quad (42)$$

$$d_{s+t,k} = \frac{1}{\det M}(-1 + \sum_{t=1}^{\bar{s}} b_{s+m,t}^{-1}\alpha_t)\beta_t, \ t = 1, 2, \dots, \overline{m}, \quad (43)$$

$$d_{t,s+m} = \frac{1}{\det M}(\sum_{t=1}^{\overline{m}} b_{k,s+t}^{-1}\beta_t)\alpha_t, \ t = 1, 2, \dots, \bar{s}, \quad (44)$$

$$d_{s+t,s+m} = \frac{1}{\det M}(1 - \sum_{t=1}^{\bar{s}} b_{k,t}^{-1}\alpha_t)\beta_t, \ t = 1, 2, \dots, \overline{m}. \quad (45)$$

Now we can prove the following

Theorem 1. *Let us suppose that DMU_0 is efficient. Conditions*

$$\omega^T D\Gamma_j \geq z_j - c_j, \quad j \ \textit{an index of nonbasic variables}, \quad (46)$$

are sufficient for DMU_0 to continue to be efficient after the simultaneous proportionate changes of a subset of outputs (5) - (6) and of a subset of inputs (7) - (8).

The proof is omitted because it is similar to the proof of Theorem 1 in Charnes and Neralić [2].

Remark 1. *In conditions (46) there is* $\det M$ *in the denominator of elements of matrix D. In order to get the system of inequalities in* $\alpha_1, \dots, \alpha_{\bar{s}}, \beta_1, \dots, \beta_{\overline{m}}$ *we have to multiply (46) by* $\det M$*. So, if we suppose that* $\det M > 0$*, multiplying (46) by* $\det M > 0$ *will keep the sign* $\geq$ *in (46). If we suppose that* $\det M < 0$*, because of multiplying (46) by* $\det M < 0$ *the sign in (46) has to be changed to* $\leq$*. Besides, constraints (10) - (12) and (14) - (16), should be added to conditions (46). The solution set of the corresponding system of inequalities* S_{j0} *will be a set of points* $(\alpha_1, \dots, \alpha_{\bar{s}}, 0, \dots, 0, \beta_1, \dots, \beta_{\overline{m}}, 0, \dots, 0)$ *in* $\mathbb{R}^{s \times m}$*. Because of (11) and (15) we can get the corresponding system of inequalities in* α, β *and the solution set* $\bar{S}_{j0}$*. Using substitutions (9) and (13) we can also get the corresponding system of inequalities in* $\widehat{\alpha}, \widehat{\beta}$ *and the solution set* S^*_{j0}*. For all points* $(\widehat{\alpha}, \widehat{\beta})$ *in the solution set* S^*_{j0} *after the changes of outputs according to (5) - (6) and the changes of inputs according to (7) - (8) efficiency of* DMU_0 *will be preserved. The solution set* S^*_{j0} *gives an area* A^*_{j0} *in the plane with the coordinate system* $\widehat{\alpha}\widehat{O}\widehat{\beta}$*.*

Remark 2. *Using points* $(\widehat{\alpha}, \widehat{\beta})$ *from the set* S^*_{j0} *in (5), (7) with (6), (8) we can get the corresponding region of efficiency* R_{j0} *around* DMU_{j0}*. The size of the region of efficiency around efficient point, within which perturbations (5) - (6) and (7) - (8) keep it efficient, is an important property of the (empirical) efficient production function at this point. It is a measure of stability of efficiency at that point. If for efficient* DMU_1 *and* DMU_2 *holds* $R_1 > R_2$ *it means that* DMU_1 *is more stable than* DMU_2 *in preserving efficiency at the simultaneous proportionate changes (5) - (6) and (7) - (8).*

For the case with $\det M > 0$ (see also Theorem 2 in Charnes and Neralić [4]) it is easy to get from *Theorem 1* the following

Corollary 1. *Let us suppose that* DMU_0 *is efficient and let*

$$\det\ M = 1 - a_1(1-\widehat{\alpha}) + (-b_1+b_2)(\widehat{\beta}-1) + (a_2b_1 - a_1b_2)(1-\widehat{\alpha})(\widehat{\beta}-1) > 0, \quad (47)$$

with

$$a_1 = \sum_{t=1}^{\bar{s}} b^{-1}_{kt} y_{t0}, \quad a_2 = \sum_{t=1}^{\bar{s}} b^{-1}_{s+m,t} y_{t0}, \quad b_1 = \sum_{t=1}^{\overline{m}} b^{-1}_{k,s+t} x_{t0}, \quad b_2 = \sum_{t=1}^{\overline{m}} b^{-1}_{s+m,s+t} x_{t0}. \quad (48)$$

Let

$$a_3 = \sum_{t=1}^{\bar{s}} \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^{\overline{m}} \omega_{s+t} x_{t0}, \quad (49)$$

$$d_j = -a_3\Gamma_{kj} + a_1\bar{c}_j, \quad e_j = -b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1+b_2)\bar{c}_j, \quad (50)$$

$$f_j = (a_2b_3 - a_3b_2)\Gamma_{kj} + (a_3b_1 - a_1b_3)\Gamma_{s+m,j} - (a_2b_1 - a_1b_2)\bar{c}_j, \quad (51)$$

$$j = 1, 2, \dots, n+s+m+1,$$

with $\bar{c}_j = z_j - c_j$*. Then the conditions*

$$d_j(1-\widehat{\alpha}) + e_j(\widehat{\beta}-1) + f_j(1-\widehat{\alpha})(\widehat{\beta}-1) \geq \bar{c}_j, \quad (52)$$

$$j \text{ an index of nonbasic variables},$$

are sufficient for DMU_0 to preserve efficiency after the simultaneous proportionate changes of a subset of outputs (5) - (6) and of a subset of inputs (7) - (8).

For fixed inputs, we can consider the proportionate change (decrease) of a subset of outputs (5) with the other outputs fixed (6). In that case it is easy to get the matrix D_1 from the matrix D with elements in (42) - (45) taking $\beta_t = 0, t = 1, 2, \ldots, \overline{m}$. With the matrix D_1 instead of matrix D from *Theorem 1* we have the following

Corollary 2. *Conditions*

$$\omega^T D_1 \Gamma_j \geq z_j - c_j, \quad j \ \ an\ index\ of\ nonbasic\ variables, \tag{53}$$

are sufficient for DMU_0 to be efficient after the proportionate changes of a subset of outputs (5) with the other outputs and all inputs fixed.

For fixed outputs, we can consider the proportionate change (increase) of a subset of inputs (7) with the other inputs fixed (8). In that case it is easy to get the matrix D_2 from the matrix D with elements in (42) - (45) taking $\alpha_t = 0, t = 1, 2, \ldots, \overline{s}$. With the matrix D_2 instead of matrix D from *Theorem 1* we have the following

Corollary 3. *Conditions*

$$\omega^T D_2 \Gamma_j \geq z_j - c_j, \quad j \ \ an\ index\ of\ nonbasic\ variables, \tag{54}$$

are sufficient for DMU_0 to be efficient after the proportionate change of a subset of inputs (7) with other inputs and all outputs fixed.

Remark 3. *In the case of the proportionate change (decrease) of a subset of outputs with different coefficients of proportionality (17) - (18) or/and the proportionate change (increase) of inputs with different coefficients of proportionality (19) - (20) of an efficient DMU_0 similar results on its efficiency preservation as in* Subsection 2.4. *(*Theorem 1*,* Corollaries 1, 2 *and* 3*) can be obtained.*

3. Illustrative example

3.1. We will consider the following example taken from Rhodes [9] (see also Charnes and Neralić [3]) with five DMUs, one output, two inputs and data in *Table 1.*

Output/Input \ DMU_j	1	2	3	4	5
y_{1j}	2	4	2	3	2
x_{1j}	4	12	8	6	2
x_{2j}	6	8	2	6	8

Table 1. *Data for the example*

We are interested in the efficency of DMU_4, with $X_0 = [6 \;\; 6]^T$ and $Y_0 = [3]$. In order to see if DMU_4 is efficient, the following linear programming problem should be solved:

$$\min 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 + 0\lambda_5 - \varepsilon s_1^+ - \varepsilon s_1^- - \varepsilon s_2^- + \theta$$

subject to

$$\begin{array}{rlllr} 2\lambda_1 + 4\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 & -s_1^+ & & & = 3 \\ -4\lambda_1 - 12\lambda_2 - 8\lambda_3 - 6\lambda_4 - 2\lambda_5 & & -s_1^- & +6\theta & = 0 \\ -6\lambda_1 - 8\lambda_2 - 2\lambda_3 - 6\lambda_4 - 8\lambda_5 & & \quad -s_2^- & +6\theta & = 0 \end{array} \tag{55}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, s_1^+, s_1^-, s_2^- \geq 0.$$

According to (3), in Phase I the following LP problem should be solved:

$$\min \theta \tag{56}$$

subject to the constraints of (55). The optimal solution of the LP problem (56) is $\lambda_4^* = 1, \theta^* = 1, \lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = s_2^{-*} = 0$. Using $\theta^* = 1$ from the optimal solution of the LP problem (56) in Phase II we solve the LP problem

$$\min(-s_1^+ - s_1^- - s_2^-) \tag{57}$$

subject to the constraints of (55) with the substitution $\theta = \theta^* = 1$. Optimal solution of (57) is $\lambda_4^* = 1, \lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_5^* = 0, s_1^{+*} = s_1^{-*} = s_2^{-*} = 0$, which means that DMU_4 is efficient.

Using these results it is easy to reconstruct the optimum tableau for the problem (55) which is given in *Table 2* below. Namely, it is slightly changed optimum tableau for the LP (56), where $z_6' - c_6' = -(1/3)$, $z_7' - c_7' = -(1/9)$ and $z_8' - c_8' = -(1/18)$ are changed to $z_6 - c_6 = -(1/3) + \varepsilon$, $z_7 - c_7 = -(1/9) + \varepsilon$ and $z_8 - c_8 = -(1/18) + \varepsilon$.

The optimal solution of problem (55) is $\lambda_0^* = \lambda_4^* = 1,\ \theta^* = 1,\ \lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_5^* = 0,\ s_1^{+*} = s_1^{-*} = s_2^{-*} = 0$ with optimal basic variables $\lambda_3^* = 0, \lambda_4^* = 1, \theta^* = 1$. The optimal basis matrix is

$$B = \begin{bmatrix} 2 & 3 & 0 \\ -8 & -6 & 6 \\ -2 & -6 & 6 \end{bmatrix},$$

with the inverse

$$B^{-1} = \begin{bmatrix} 0 & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{9} & -\frac{1}{9} \\ \frac{1}{3} & \frac{1}{18} & \frac{1}{9} \end{bmatrix}, \tag{58}$$

and the corresponding optimum tableau in *Table 2*. (Because of degeneracy, some other optimal solutions can be obtained, as, for example, the one with basic variables $\lambda_4^* = 1, \lambda_5^* = 0, \theta^* = 1$.)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_0
λ_3	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	-1	0	$\frac{1}{6}$	$-\frac{1}{6}$	0	0
λ_4	$\frac{8}{9}$	$\frac{8}{9}$	0	1	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{9}$	$\frac{1}{9}$	0	1
θ	$-\frac{2}{9}$	$-\frac{2}{9}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{18}$	$-\frac{1}{9}$	1	1
$z_j - c_j$	$-\frac{2}{9}$	$-\frac{2}{9}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3} + \varepsilon$	$-\frac{1}{18} + \varepsilon$	$-\frac{1}{9} + \varepsilon$	0	1

Table 2. *Optimum tableau*

3.2. Let us consider the simultaneous proportionate change (decrease) of output

$$\widehat{y}_{10} = 3\widehat{\alpha}, \quad 0 < \widehat{\alpha} \le 1, \tag{59}$$

and proportionate change (increase) of input 1

$$\widehat{x}_{10} = 6\widehat{\beta}, \quad \widehat{\beta} \ge 1 \tag{60}$$

with input 2 fixed

$$\widehat{x}_{20} = 6 \tag{61}$$

of DMU_4 preserving its efficiency. So, we consider proportionate change of output and of a subset of inputs of DMU_4.

Using substitutions (9), with (10) - (12), (13) with (14) - (16) it is easy to get

$$\widehat{y}_{10} = 3 - \alpha_1 > 0, \quad \alpha_1 = 3\alpha, \quad 0 \le \alpha_1 < 3. \tag{62}$$

and

$$\widehat{x}_{10} = 6 + \beta_1, \quad \beta_1 = 6\beta, \quad \beta_1 \ge 0, \tag{63}$$

$$\widehat{x}_{20} = 6. \tag{64}$$

Because of $s = 1,\ \bar{s} = 1,\ m = 2,\ \overline{m} = 1,\ k = 2,\ s + m = 3,$ the optimal basis perturbation matrix is

$$\triangle B = \begin{bmatrix} 0 & -\alpha_1 & 0 \\ 0 & -\beta_1 & \beta_1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{65}$$

and the change of the right-hand side vector is

$$\widehat{P}_0 = P_0 + [-\alpha_1 \; 0 \; 0]^T. \tag{66}$$

Using matrices

$$U = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \beta_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{67}$$

and

$$V^T = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \tag{68}$$

and (65) we can write the perturbed optimal basis matrix as

$$\widehat{B} = B + \triangle B = B + UV^T. \tag{69}$$

It is easy to get

$$M = I + V^T B^{-1} U = \begin{bmatrix} 1 - (1/18)\beta_1 & 0 \\ (-1/18)(6\alpha_1 + \beta_1) & 1 - (1/3)\alpha_1 \end{bmatrix},$$

$$\det M = 1 - (1/3)\alpha_1 - (1/18)\beta_1 + (1/54)\alpha_1\beta_1, \tag{70}$$

$$M^{-1} = \frac{1}{\det M} \begin{bmatrix} 1 - (1/3)\alpha_1 & 0 \\ (1/18)(6\alpha_1 + \beta_1) & 1 + (1/18)(-\beta_1) \end{bmatrix},$$

$$D = UM^{-1}V^T = \frac{1}{\det M}\begin{bmatrix} 0 & -[1+(1/18)\beta_1]\alpha_1 & [(1/9)\beta_1]\alpha_1 \\ 0 & [-1+(1/3)\alpha_1]\beta_1 & [1-(1/3)\alpha_1]\beta_1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{71}$$

$$\omega^T = c_B^T B^{-1} = [0\ 0\ 1]\, B^{-1} = [1/3\ 1/18\ 1/9]\,, \tag{72}$$

$$\omega^T D = \frac{1}{\det M}\,[0\ \ (-1/18)(6\alpha_1+\beta_1)\ \ (1/54)(\alpha_1\beta_1+3\beta_1)]\,. \tag{73}$$

Let us suppose that

$$\det M = 1-(1/3)\alpha_1-(1/18)\beta_1+(1/54)\alpha_1\beta_1 > 0. \tag{74}$$

Using (73) with (74) and elements of *Table 2* in (46) it is easy to get sufficient conditions for DMU_4 to preserve its efficiency as the system of inequalities in α_1, β_1 in the form

$$5\alpha_1+\beta_1 \le 3,\ 0 \le \alpha_1 < 3,\ \beta_1 \ge 0. \tag{75}$$

The other constraints, including (74), are redundant. Using (62), (63) in (75) we can get the corresponding system of inequalities in α, β

$$5\alpha+2\beta \le 1,\ 0 \le \alpha < 1,\ \beta \ge 0. \tag{76}$$

Using substitutions (9), (13) it is easy to get from (76) the corresponding system of inequalities in $\widehat{\alpha}, \widehat{\beta}$

$$5\widehat{\alpha}-2\widehat{\beta} \ge 2,\ 0 < \widehat{\alpha} \le 1,\ \widehat{\beta} \ge 1. \tag{77}$$

Let us point out that in order to get the system of inequalities (77) we could use conditions (52) from *Corollary 1.*

It is easy to see that the solution set S_4^* of the system of inequalities (77) is the triangle $\widehat{A}\widehat{B}\widehat{C}$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$, with $\widehat{A}(1,\ 1.5)$, $\widehat{B}(0.8, 1)$ and $\widehat{C}(1, 1)$. For every point $(\widehat{\alpha}, \widehat{\beta})$ which belongs to the triangle $\widehat{A}\widehat{B}\widehat{C}$ the efficiency of DMU_4 will be preserved after the simultaneous proportionate change (59) of output with the coefficient $\widehat{\alpha}$ and proportionate change (60) of input 1 with the coefficient $\widehat{\beta}$ and input 2 fixed (61). The point $\widehat{C}(1,\ 1)$ means that there are no changes of output and of input 1, the point $\widehat{A}(1,\ 1.5)$ means the maximal proportionate increase of input 1 of DMU_4 for 50% preserving its efficiency and the point $\widehat{B}(0.8,\ 1)$ means the maximal proportionate decrease of output of DMU_4 for 20% preserving efficiency of DMU_4. Let us point out that these maximal changes cannot be done simultaneously.

Using the solution set S_4^* we can get the corresponding region of efficiency R_4 around DMU_4 as the triangle ABC in the coordinate system $Ox_1x_2y_1$ ($x_1 =$ input 1, $x_2 =$ input 2, $y_1 =$ output 1) with A(9,6,3), B(6,6, 2.4), C(6,6,3). (Because input 2 is fixed, we can project triangle ABC into the plane x_1Oy_1 and get the triangle $A'B'C'$, with $A'(9,3), B'(6,2.4)$ and $C'(6,3)$.) The size of the region of efficiency R_4 is 0.9 (which is the area of the triangle ABC or triangle $\widehat{A}\widehat{B}\widehat{C}$) and it is a measure of stability of efficiency of DMU_4.

3.3. Let us consider the case of the proportionate change of a subset of outputs or/and of a subset of inputs with different coefficients of proportionality in the example from *Subsection 3.1.* For efficient $\mathrm{DMU}_0 = \mathrm{DMU}_4$ we can consider the proportionate change (decrease) of output

$$\widehat{y}_{10} = 3\widehat{\alpha}_1, \quad 0 < \widehat{\alpha}_1 \leq 1, \tag{78}$$

and the proportionate change (increase) of input 1

$$\widehat{x}_{10} = 6\widehat{\beta}_1, \quad \widehat{\beta}_1 \geq 1 \tag{79}$$

with input 2 fixed

$$\widehat{x}_{20} = 6 \tag{80}$$

preserving its efficiency. So, we consider the proportionate change of output and of a subset of inputs of DMU_4 with different coefficients of proportionality. But, this is just the case considered in *Subsection 3.2.* with $\widehat{\alpha} = \widehat{\alpha}_1$ and $\widehat{\beta} = \widehat{\beta}_1$. It means that the results in that case are the same as results obtained in the case in *Subsection 3.2.* with $\widehat{\alpha} = \widehat{\alpha}_1$ and $\widehat{\beta} = \widehat{\beta}_1$.

4. Summary and conclusions

The proportionate change of a subset of inputs and/or proportionate change of a subset of outputs of an efficient DMU_0 preserving efficiency for the case of the CCR ratio model in DEA is studied in the paper. Sufficient conditions for an efficient DMU_0 to preserve efficiency are established for the case of the proportionate decrease of a subset of outputs with the coefficient of proportionality $\widehat{\alpha}$ or/and the proportionate increase of a subset of inputs with the coefficient of proportionality $\widehat{\beta}$. Similar results could be obtained for the case of the proportionate decrease of a subset of outputs with different coefficients of proportionality $\widehat{\alpha}_r, r = 1, 2, \ldots, \overline{s} < s$ or/and the proportionate increase of a subset of inputs with different coefficients of proportionality $\widehat{\beta}_i, i = 1, 2, \ldots, \overline{m} < m$. In the case with $\widehat{\alpha}$ and $\widehat{\beta}$ sufficiency conditions give for each efficient DMU_0 region of efficiency and the area the size of which is a measure of stability of efficiency at the proportionate change of a subset of inputs or/and of a subset of outputs. A numerical example illustrating the results is provided.

Sensitivity and stability analysis for the case of the proportionate change of a subset of inputs or/and of a subset of outputs with two parameters (one for outputs and the other for inputs) preserving efficiency of an efficient DMU_0 according to the BCC or Additive model is an interesting open question. The same holds for the case of proportionate changes with different coefficients of proportionality. Also, the question of efficiency preservation of all efficient DMUs according to the CCR ratio model under the proportionate (or additive) changes of all data is open. An application of the results in the paper using data for a real world problem seems to be a challenge too.

References

[1] A. CHARNES, W. W. COOPER, *Preface to topics in data envelopment analysis*, Annals of Operations Research **2**(1985), 59-94.

[2] A. CHARNES, L. NERALIĆ, *Sensitivity analysis in data envelopment analysis 3*, Glasnik Matematički **27**(1992), 191-201.

[3] A. CHARNES, L. NERALIĆ, *Sensitivity analysis of the proportionate change of inputs (or outputs) in data envelopment analysis*, Glasnik Matematički **27** (1992), 393-405.

[4] A. CHARNES, L. NERALIĆ, *Sensitivity Analysis of the Simultaneous Proportionate Change of Inputs and Outputs in Data Envelopment Analysis*, Research Report CCS 667, Center for Cybernetic Studies, University of Texas at Austin, 1991.

[5] W. W. COOPER, L. M. SEIFORD, K. TONE, *Data Envelopment Analysis. A Comprehensive Text with Models, Applications, References and DEA-Solver Software*, Kluwer Academic Publishers, Boston, 2000.

[6] G. H. GOLUB, C. F. VAN LOAN, *Matrix Computation*, John Hopkins University Press, Baltimore, Maryland, 1983.

[7] L. NERALIĆ, *Sensitivity analysis of the proportionate change of outputs or/and inputs of the additive model in data envelopment analysis*, in: *Proceedings of the International Conference "Restructuring Transitional Economies"*, (S. Sharma and P. Sikavica, Eds.), Part I, University of Zagreb, Faculty of Economics, Zagreb, 1995, 221-233.

[8] L. NERALIĆ, T. R. SEXTON, *Sensitivity analysis in data envelopment analysis for the case of different coefficients of proportionality*, in: *Parametric Optimization and Related Topics IV*, (J. Guddat, H. Th. Jongen, F. Nožička and F. Twilt, Eds.), Peter Lang Publ., Frankfurt am Main, 1996, 261-280.

[9] E. RHODES, *DEA and Related Approaches for Measuring the Efficiency of Decision Making Units with an Application to Program Follow Through in U. S. Education*, Ph. D. Thesis, Carnegie Mellon University, 1978.