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ON GENERIC IRREDUCIBLE REPRESENTATIONS OF $Sp(n,F)$ AND $SO(2n+1,F)$

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Abstract. In this paper the author gives the complete classification of irreducible generic representations of symplectic and odd–orthogonal groups in terms of supercuspidals.

Introduction

In this paper we give the complete classification of generic (i.e. having Whittaker model) representations of the series of groups $G_n = Sp(n,F)$ or $G_n = SO(2n+1,F)$, where F is a local non–archimedean field of characteristic zero. Our result is analogous to the well-known result of A.V. Zelevinsky ([Ze], Theorem 9.7) about classification of nondegenerate representations of $GL(n,F)$.

Here is an outline of the paper. In the first section we recall notation and basic structure results for these groups and basic facts from the representation theory of classical groups.

In ([Mu], Theorem 3.1) we found necessary and sufficient conditions on a representation of G_n, parabolically induced from supercuspidal generic representation of a standard Levi factor, to contain generic square integrable subquotient. The proof of ([Mu], Theorem 3.1) is based mostly on general results on L–functions and its connections with Plancherel measures obtained by F. Shahidi in ([Sh1]). On the other hand, Tadić ([T4]) has constructed a large family of nonsupercuspidal square integrable representations of groups G_n, using his method of Jacquet modules. In the second section we recall his result, and, combining with ([Mu], Theorem 3.1), we prove that Tadić has constructed, among many others, all generic nonsupercuspidal square integrable representations (cf. Proposition 2.1). We refer to ([Mu], Section 3) for the interpretation of that result in terms of Local Langlands Conjecture.

In third section we recall some results about tempered representations of classical groups related to reducibility of unitary generalized principal series ([Go]) and elliptic tempered representations ([He]). They are needed in the proof of our main result in the fourth section (cf. Theorem 4.1). This theorem follows mainly from the one of our main results in ([Mu]). That is the characterization of a generic representation (realised as a Langlands quotient) in terms of the irreducibility of the corresponding standard representation. (See Theorem 5.1 in [Mu].)

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We hope that these results will be useful in theory of automorphic forms as well as the results of Zelevinsky ([Ze]) are.

I would like to express my gratitude to Marko Tadić for introducing me to several aspects of representation theory involved here.

1. Preliminaries

Let F be a nonarchimedean field of characteristic zero. Let ν be the character given by normalized absolute value of F. We fix ψ_F nontrivial additive character of F. Let $\mathbb{Z}_+$, $\mathbb{R}$, and $\mathbb{C}$ be the set of non-negative rational integers, the field of real numbers, and the field of complex numbers, respectively.

Let G_n denote one of the groups $Sp(n, F)$ or $SO(2n+1, F)$. (For more details see [T2] and [T3].) Denote by $T_n U_n$ the standard Borel subgroup in G_n. Then

$$T_n \cong F^\times \times \cdots \times F^\times = (F^\times)^n$$

Denote by $\{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots. We have $\alpha_i(x_1, \ldots, x_n) = x_i x_{i+1}^{-1}$, for $1 \leqslant i \leqslant n-1$, and $\alpha_n(x_1, \ldots, x_n) = x_n^2$ if $G_n = Sp(n)$, and $\alpha_n(x_1, \ldots, x_n) = x_n$ if $G_n = SO(2n+1)$.

The groups G_n have proper standard parabolic subgroups parametrised by ordered partitions $\alpha = (m_1, \ldots, m_k)$ of $1 \leqslant m \leqslant n$. For a given partition α, denote the corresponding parabolic subgroup by $P_\alpha = M_\alpha N_\alpha$, where

$$M_\alpha = GL(m_1, F) \times \cdots \times GL(m_k, F) \times G_{n-m}. \tag{1.1}$$

Through this paper w_0 denotes the longest element of the Weyl group of T_n in G_n modulo that of the Weyl group T_n in M_α. We choose its representative as in ([Sh1]), and denote by the same letter.

Let π_i be an admisible representation of $GL(m_i, F)$, $i = 1, \ldots, k$, and π an admisible representation of G_n, then we write

$$\pi_1 \times \cdots \times \pi_k \rtimes \pi$$

for a representation obtained by normalized parabolic induction $\mathrm{Ind}_{P_\alpha}^{G_n}(\pi_1 \otimes \cdots \otimes \pi_k \otimes \pi)$.

For any positive (nonnegative) integer m we denote by Irr_m (Irr'_m) the set of classes of equivalence of irreducible representations of $GL(m, F)$ (G_m), respectively. Set

$$\mathrm{Irr} = \cup_m \mathrm{Irr}_m \ \text{ and } \ \mathrm{Irr}' = \cup_m \mathrm{Irr}'_m .$$

For an essentially tempered representation $\delta \in \mathrm{Irr}$ there exists a unique $e(\delta) \in \mathbb{R}$ such that $\delta^u = \nu^{-e(\delta)}\delta$ is unitary.

Now, we recall Langlands quotient theorem in G_n–setting ([BW]). Here we follow Tadić ([T3]).

LEMMA 1.1.

(i) Assume that $\delta_i \in \mathrm{Irr}_{m_i}$, $i = 1, \ldots, k$, *are essentially square integrable representations,* $e(\delta_1) \geqslant \cdots \geqslant e(\delta_k) > 0$, *and* $\mathcal{T}$ *is a tempered representation of* G_{n-m}. *Then the induced representation* $\delta_1 \times \cdots \times \delta_k \rtimes \mathcal{T}$ *has a unique irreducible quotient* $L(\delta_1, \ldots, \delta_k; \mathcal{T})$.

(ii) Assume that π *is an irreducible representation of* G_n. *Then there exists a unique datum as in (i) such that* $\pi \cong L(\delta_1, \ldots, \delta_k; \mathcal{T})$.

Denote by $\mathfrak{g}_n$ the Lie algebra of G_n. Denote by X_i a non-zero element of a root space in $\mathfrak{g}_n$ which belongs to α_i, for $i = 1, \ldots, n$ (Chevalley basis). Denote by U_n^D the derived group of U_n. Then we have a canonical isomorphism of groups

$$U_n^{ab} = U_n/U_n^D \cong FX_1 \oplus \cdots \oplus FX_n.$$

We can (and will) regard U_n^{ab} as a vector space. For $f \in \mathrm{Hom}_F(U_n^{ab}, F)$, we obtain character $\psi_F \circ f$ of U_n. It is non-degenerate if and only if $f(X_i) \neq 0$, for all i. T_n acts on the set of all non-degenerate characters in the usual way. For $\mu \in F^\times$ we define

$$\chi_\mu = \psi_F \circ f_\mu,$$

where $f_\mu(X_i) = 1$, $i = 1, \ldots, n-1$, $f_\mu(X_n) = \mu$. All of these characters are compatible with all w_0 as above ([Sh1], page 282). This follows from the following discussion. Note that χ_1 is in fact χ_0 from ([Sh1], Section 3). For $\mu \in F^\times$ we set $t_\mu = (\mu', \ldots, \mu') \in T_n(\bar{F})$, where $\bar{F}$ denotes algebraic closure of F, $(\mu')^2 = \mu$ if $G_n = Sp(n)$, and $\mu' = \mu$ if $G_n = SO(2n+1)$. Then $u \longmapsto t_\mu u t_\mu^{-1}$ is F–rational on $U_n(\bar{F})$, and $\chi_\mu(u) = \chi_1(t_\mu u t_\mu^{-1})$, $u \in U_n$. Assume that w_0 is associated to the parabolic subgroup $P_\alpha = M_\alpha N_\alpha$ with M_α given by (1.1). The action of w_0 on T_n is given by

$$w_0(x_1, \ldots, x_n) =$$

$$(x_{m_1}^{-1}, \ldots, x_1^{-1}, x_{m_1+m_2}^{-1}, \ldots, x_{m_1+1}^{-1}, \ldots, x_m^{-1}, \ldots, x_{m_1+\cdots m_{k-1}+1}^{-1}, x_{m+1}, \ldots, x_n).$$

Then we see $w_0(t_\mu)t_\mu^{-1}$ is in the centre of M_α. Discussion in ([Sh1], pages 282–283) implies assertion.

In the case of $G_n = SO(2n+1)$ all non-degenerate characters are T_n–equivalent. In the case of $G_n = Sp(n)$ orbits are parametrised by χ_μ, $\mu \in F^\times/(F^\times)^2$.

Suppose that (π, V) is an admissible representation of G_n. We write $V(U_n)_\chi$ for a $\mathbb{C}$–span of all $\pi(u)v - \chi(u)v$, $u \in U_n$, $v \in V$. Write $(r_\chi(\pi), r_\chi(V))$ for the corresponding quotient representation of U_n. The functor $V \longmapsto r_\chi(V)$ is exact. (π, V) is χ– generic if

$$\mathrm{Hom}_{U_n}(\pi|_{U_n}, \chi) = \mathrm{Hom}_{\mathbb{C}}(r_\chi(\pi), \mathbb{C}) \neq 0.$$

We may assume (and will) that π is χ_μ–generic for some $\mu \in F^\times$. If π is irreducible then ([Ro1])

$$\dim_{\mathbb{C}} \mathrm{Hom}_{U_n}(\pi|_{U_n}, \chi) \leqslant 1. \tag{1.2}$$

Finally, we will several times use the following result that follows from a more general result of F. Rodier ([Ro1]) using above discussion on compatibility.

LEMMA 1.2. *Suppose that $\pi_1, \dots, \pi_k \in \mathrm{Irr}$ are generic, and π is an irreducible representation of G_n. Then we have an isomorphism of vector spaces*

$$r_{\chi_\mu}(\pi_1 \times \cdots \times \pi_k \rtimes \pi) \cong r_{\chi_\mu}(\pi).$$

In particular, $\pi_1 \times \cdots \times \pi_k \rtimes \pi$ is χ_μ–generic if and only if π is. $\pi_1 \times \cdots \times \pi_k \rtimes \pi$ satisfies (1.2).

Later on we will need an elementary lemma.

LEMMA 1.3. *Suppose that (π, V) is a χ–generic admissible representation of finite length such that (1.2) is valid, and $V_1, \dots, V_k \subseteq V$ are χ–generic subrepresentations. Then*

$$r_\chi(V_1 \cap \ldots \cap V_k) \neq 0.$$

In particular, a unique irreducible subquotient of π, which is χ–generic, is an irreducible subquotient of $V_1 \cap \ldots \cap V_k$.

Proof. If $W \subseteq V$ is χ–generic subrepresentation, then we have $r_\chi(W) \hookrightarrow r_\chi(V)$. Then (1.2) and assumptions of the lemma imply $r_\chi(W) = r_\chi(V)$. This implies $\dim_{\mathbb{C}} r_\chi(W) = 1$.

It is enough to prove the lemma for $k = 2$. Now, $(V_1 + V_2)/V_1 \cong V_2/V_1 \cap V_2$ and the first part of the proof imply $\dim_{\mathbb{C}} r_\chi(V_1 \cap V_2) = 1$. This completes the proof of the lemma. □

About reducibility in the supercuspidal case we have the following result of F. Shahidi ([Sh1], [Sh2], [Si]).

LEMMA 1.4. *Suppose that $\rho \in \mathrm{Irr}$ is supercuspidal, σ is a generic supercuspidal representation of G_n and $\alpha \in \mathbb{R}$. If $\rho \ncong \tilde{\rho}$, then $\nu^\alpha \rho \rtimes \sigma$ is irreducible. If $\rho \cong \tilde{\rho}$, then there exists $\alpha_0 \in \{0, 1/2, 1\}$ such that $\nu^{\pm\alpha_0} \rho \rtimes \sigma$ reduces, and $\nu^\alpha \rho \rtimes \sigma$ is irreducible for $|\alpha| \neq \alpha_0$. (Then we say that the pair (ρ, σ) satisfies $(C\alpha_0)$).*

At the end of this section we will recall some results about representations of general linear groups. (For more details see [Ze] and [Ro2].) If $\rho \in \mathrm{Irr}$ is supercuspidal and $k \in \mathbb{Z}_+$, then we form a segment ([Ze]) $\Delta = [\rho, \nu^k \rho]$ as a set $\{\rho, \nu\rho, \cdots, \nu^k \rho\}$. The segment Δ has uniquely associated essentially square integrable representation $\delta(\Delta)$ as a unique irreducible subrepresentation of $\nu^k \rho \times \cdots \times \nu\rho \times \rho$. We say that Δ is balanced if $\delta(\Delta)$ is square integrable. Furthermore, ([Ze], Theorem 9.7) for every generic representation $\Pi \in \mathrm{Irr}$ there exists a unique multiset of segments $\{\Delta_1, \cdots, \Delta_k\}$ (Δ_i and Δ_j are not linked, for $i \neq j$) such that Π is isomorphic to the induced representation $\delta(\Delta_1) \times \cdots \times \delta(\Delta_k)$. (Segments Δ and Δ' are linked if $\Delta \cup \Delta'$ is segment, and $\Delta \not\subset \Delta'$, $\Delta' \not\subset \Delta$.)

2. Square integrable representations

Fix a generic supercuspidal representation σ of $G_{n'}$. As in ([T4], Proposition 9.4), we consider sequences of segments

$$\Delta_i' = [\nu^{-n_i} \rho_i, \nu^{m_i} \rho_i], \;\; 2m_i \in \mathbb{Z}_+, \;\; 2n_i \in \mathbb{Z}, \;\; \rho_i \cong \tilde{\rho}_i, \;\; i = 1, \dots, k, \tag{2.1}$$

satisfying:

(i') $m_i > n_i$.

(ii')

(1) If (ρ_i, σ) satisfies $(C1/2)$, then $m_i \in 1/2 + \mathbb{Z}$, $n_i \geqslant -1/2$.

(2) If (ρ_i, σ) satisfies $(C0)$, then $m_i \in \mathbb{Z}$, $n_i \geqslant 0$.

(3) If (ρ_i, σ) satisfies $(C1)$, then $m_i \in \mathbb{Z}$, $n_i \geqslant -1$, $n_i \neq 0$.

(iii') If $\rho_i \cong \rho_j$, $i \neq j$, then either $m_i < n_i$ or $m_j < n_i$.

If $\Delta = [\rho, \nu^k\rho]$ is a segment, then we define a new segment $\widetilde{\Delta}$ by $\widetilde{\Delta} = [\nu^{-k}\widetilde{\rho}, \widetilde{\rho}]$. Note that $\Delta'_i \cap \widetilde{\Delta}'_i$ is equal to $[\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i]$ if $n_i \geqslant 0$, and is empty, otherwise. Hence, if $n_i \geqslant 0$, then $\delta(\Delta'_i \cap \widetilde{\Delta}'_i)$ is square integrable. If we denote by l_0 the number of all i such that $n_i \geqslant 0$, then ([T4], Proposition 9.1 and Proposition 9.4) the induced representation

$$\delta(\Delta'_1 \cap \widetilde{\Delta}'_1) \times \cdots \times \delta(\Delta'_k \cap \widetilde{\Delta}'_k) \rtimes \sigma \tag{2.2}$$

(we omit empty segments) is a multiplicity one representation of length 2^{l_0}. Clearly, any irreducible subquotient of (2.2) is tempered. Now, we have

THEOREM 2.1. (Tadić, [T4]). *Under the above assumptions, we have*

(i) Suppose that τ is an irreducible subrepresentation of (2.2). Then the induced representation

$$\delta(\Delta'_1 \setminus \widetilde{\Delta}'_1) \times \cdots \times \delta(\Delta'_k \setminus \widetilde{\Delta}'_k) \rtimes \tau$$

has a unique irreducible subrepresentation $\delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau$. $\delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau$ is a subrepresentation of $\delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k) \rtimes \sigma$.

(ii) If $\tau \not\cong \tau'$, then $\delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau \not\cong \delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_{\tau'}$.

(iii) $\delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau$ is square integrable.

(iv) Suppose that π is an irreducible subrepresentation of $\delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k) \rtimes \sigma$. Then there exists τ, an irreducible subrepresentation of (2.2), such that

$$\pi \cong \delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau.$$

REMARK 2.1. *By (i')–(iii') segments Δ'_i and Δ'_j are not linked for all i, j. Hence, the induced representation $\delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k)$ is irreducible ([Ze], Theorem 9.7).*

The following lemma is a consequence of ([T4]) which is not drawn in ([T4]). So, we shall sketch the proof.

LEMMA 2.1. *Suppose that σ is χ_μ–generic and τ is a unique irreducible χ_μ–generic subquotient of (2.2). Then $\delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau$ is χ_μ–generic.*

Proof. We prove the lemma by induction. Assume $k = 1$. Then by ([T4], Propositions 4.4, 5.9 and 7.6) every square integrable subquotient of $\delta(\Delta'_1) \rtimes \sigma$ is given by Theorem 2.1. Now, ([Mu], Theorem 3.1) and Lemma 1.2 imply the lemma. More precisely, the lemma follows from the following fact. Since the induced representation $\nu^{-n_1}\rho_1 \times \ldots \times \nu^{m_1}\rho_1 \rtimes \sigma$ contains square integrable subquotients, all its generic irreducible subquotients are square integrable([Mu], Theorem 3.1 (ii)), and by Lemma 1.2 they are subquotients of $\delta(\Delta'_1) \rtimes \sigma$.

Suppose that the claim of the lemma holds when we have i segments, $1 \leqslant i \leqslant k-1$. Then we look at the collection of all representations Π' that are obtained in the following way. Consider a permutation $(\Delta''_1, \ldots, \Delta''_k)$ of $(\Delta'_1, \ldots, \Delta'_k)$. For each i', $1 \leqslant i' \leqslant k$, we denote by τ'' a unique χ_μ–generic subquotient of

$$\delta(\Delta''_{i'+1} \cap \widetilde{\Delta}''_{i'+1}) \times \cdots \times \delta(\Delta''_k \cap \widetilde{\Delta}''_k) \rtimes \sigma,$$

and set (see Remark 2.1)

$$\Pi' = \delta(\Delta''_1) \times \cdots \times \delta(\Delta''_{i'}) \rtimes \delta(\Delta''_{i'+1}, \ldots, \Delta''_k, \sigma)_{\tau''} \hookrightarrow \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k) \rtimes \sigma.$$

By the inductive assumption we have $\dim_{\mathbb{C}} r_{\chi_\mu}(\Pi') = 1$, for all Π'. Then Lemma 1.3 implies that the intersection Π of all Π' satisfies the same. Therefore, a unique χ_μ–generic irreducible subquotient of $\delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k) \rtimes \sigma$ is an irreducible subquotient of Π.

To continue, we need the following general result. Let G and Z be reductive F–group and maximal F–split torus in the centre of G, respectively. An admissible representation of finite length Π is tempered if every irreducible subquotient of Π is tempered in the usual sense ([Si]). Then we have ([Si], Lemma 5.4.1.4)

LEMMA 2.2. *Assume that Π is a tempered representation of G with central character. If π is any irreducible square integrable (modulo Z) subquotient of Π, then* $\mathrm{Hom}_G(\pi, \Pi) \neq 0$.

Now, we continue with the proof of Lemma 2.1. Denote by π any irreducible subquotient of Π. As in the proof of Lemma 9.6 and Lemma 9.9 in ([T4]), we see that π is square integrable. (We are thankful to M. Tadić for explaining this to us.) Hence, Π is tempered representation which has square integrable representations as irreducible subquotients. If we denote by π_μ a unique χ_μ–generic subquotient of Π, then, according to Lemma 2.2, we have

$$\pi_\mu \hookrightarrow \Pi \subseteq \delta(\Delta'_1) \times \cdots \times \delta(\Delta'_k) \rtimes \sigma.$$

Then, by Theorem 2.1 (iv), there exists an irreducible subquotient τ' of (2.2) such that

$$\pi_\mu \cong \delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_{\tau'}. \tag{2.3}$$

By Theorem 2.1 (i) and Lemma 1.2, the right–hand side in (2.3) cannot be χ_μ–generic if $\tau' \ncong \tau$. The lemma follows. □

Now, we are ready to relate Theorem 2.1 and ([Mu], Theorem 3.1).

PROPOSITION 2.1. *Suppose that $\mathcal{T}$ is a χ_μ–generic square integrable representation of G_n. Then there exist a unique σ and a unique set of segments $\{\Delta'_1, \ldots, \Delta'_k\}$, satisfying (i')–(iii'), such that*

$$\mathcal{T} \cong \delta(\Delta'_1, \ldots, \Delta'_k, \sigma)_\tau,$$

where τ is a unique χ_μ–generic irreducible subrepresentation of (2.2).

Proof. Let us begin the proof by recalling some results from ([Mu]). Assume that $\mathcal{T}$ is an irreducible subquotient of

$$\omega_1 \times \cdots \times \omega_k \rtimes \sigma,$$

where all ω_i and σ are supercuspidal. (Then σ is χ_μ–generic by Lemma 1.2.). Any ω_i we can write as a twist of a unitary supercuspidal representation ω_i^u with a positive character $\nu^{e(\omega_i)}$ ($e(\omega_i) \in \mathbb{R}$), uniquely. Then ([T1], Theorem 6.2 (i), (ii)), $\widetilde{\omega}_i^u \cong \omega_i^u$ and $2e(\omega_i) \in \mathbb{Z}$, for all $i = 1, \ldots, k$. Furthermore, let X be the set of all mutually non–isomorphic ω_i^u such that the pair (ω_i^u, σ) satisfies $(C1)$. Let Θ be the unique generic component of

$$\omega_1 \times \cdots \times \omega_k \times (\times_{\omega' \in X} \omega') \times \widetilde{\omega}_k \times \cdots \times \widetilde{\omega}_1.$$

Then we have the following result.

LEMMA 2.3. ([MU], Theorem 3.1 (i)). *Under the above assumptions, Θ is an irreducible tempered representation. If we write Θ as the induced representation*

$$\Theta \cong \delta(\Delta_1) \times \cdots \times \delta(\Delta_l),$$

where segments Δ_i are given by $\Delta_i = [\nu^{-k_i}\rho_i, \nu^{k_i}\rho_i]$ (ρ_i unitary supercuspidal, $2k_i \in \mathbb{Z}$), then we have $\widetilde{\rho}_i \cong \rho_i$, $i = 1, \ldots, l$, and

(1) If (ρ_i, σ) satisfies $(C0)$ or $(C1)$, then $k_i \in \mathbb{Z}$.
(2) If (ρ_i, σ) satisfies $(C1/2)$, then $k_i \in 1/2 + \mathbb{Z}$.
(3) If $\rho_i \cong \rho_j$, for $i \neq j$, then $k_i \neq k_j$.

Further, if we set $S_\rho = \{i\,;\, \rho_i \cong \rho\}$, for any supercuspidal $\rho \in$ Irr, then

(4) If (ρ, σ) satisfies $(C0)$, then the cardinality of the set S_ρ is zero or is even.
(5) If (ρ, σ) satisfies $(C1)$, then the cardinality of the set S_ρ is zero or is odd. Further, if $S_\rho = \{i\}$, then $k_i > 0$.

Now, we are ready to prove the proposition. Suppose that $\rho \in$ Irr is a selfcontragredient supercuspidal representation. Then we consider the set S of all i, $1 \leqslant i \leqslant l$, such that $\rho \cong \rho_i$ (ρ_i are given by Lemma 2.3). Assume that S is nonempty. Let $S = \{i_1, \ldots, i_{l'}\}$. By the above lemma, we may assume

$$0 \leqslant k_{i_1} < k_{i_2} < \cdots < k_{i_{l'}}.$$

Now, we have several cases:

(i) (ρ, σ) satisfies $(C1/2)$. If l' is even, then we define the segments $[\nu^{-k_{i_{2j-1}}}\rho, \nu^{k_{i_{2j}}}\rho]$, $j = 1, \ldots, l'/2$. If l' is odd, then we define the segments $[\nu^{1/2}\rho, \nu^{k_1}\rho]$ and $[\nu^{-k_{i_{2j-2}}}\rho, \nu^{k_{i_{2j-1}}}\rho]$, $j = 2, \ldots, (l'+1)/2$.

(ii) (ρ, σ) satisfies $(C0)$. Then l' is even, and we define the segments $[\nu^{-k_{i_{2j-1}}}\rho, \nu^{k_{i_{2j}}}\rho]$, $j = 1, \ldots, l'/2$.

(iii) (ρ, σ) satisfies $(C1)$. Then l' is odd and we define segments $[\nu\rho, \nu^{k_1}\rho]$ (we omit this segment if $k_1 = 0$), and $[\nu^{-k_{i_{2j-2}}}\rho, \nu^{k_{i_{2j-1}}}\rho]$, $j = 2, \ldots, (l'+1)/2$.

By repeating this procedure for all possible ρ (i.e. when S is non empty), we obtain a sequence of segments (2.1) such that (i')–(iii') are valid. It is clear that $\mathscr{T}$ is a subquotient of $\delta(\Delta_1') \times \cdots \times \delta(\Delta_k') \rtimes \sigma$. Now, Lemma 2.1 implies $\mathscr{T} \cong \delta(\Delta_1', \ldots, \Delta_k', \sigma)_\tau$, with τ χ_μ–generic. The part about uniquenes follows from ([T4], Proposition 9.4 and Proposition 9.11). $\square$

3. Elliptic representations

The purpose of this section is to recall some results about tempered representations. We start with the following particular case of the more general result of Harish–Chandra ([Si], Chapter 5).

LEMMA 3.1. *Assume that $\Upsilon_1, \dots, \Upsilon_m$ is a sequence of balanced segments (cf. Section 1), and $\mathscr{T}$ is a square integrable representation of G_n. Then we have*

(i) *Any irreducible subrepresentation of $\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T}$ is tempered.*

(ii) *Let $\Upsilon'_1, \dots, \Upsilon'_{m'}$ be also a sequence of balanced segments, and $\mathscr{T}'$ square integrable representation. Assume*

$$\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T} \text{ and } \delta(\Upsilon'_1) \times \cdots \times \delta(\Upsilon'_{m'}) \rtimes \mathscr{T}'$$

have a common irreducible subquotient. Then $m' = m$, $\mathscr{T}' \cong \mathscr{T}$, and there exists a permutation p of the set $\{1, \dots, m\}$ such that $\Upsilon'_i = \Upsilon_{p(i)}$ or $\Upsilon'_i = \widetilde{\Upsilon}_{p(i)}$, for all i.

Now, we continue with the following result of Goldberg ([Go]).

LEMMA 3.2. *Assume that $\Upsilon_1, \dots, \Upsilon_m$ is a sequence of balanced segments (cf. Section 1), and $\mathscr{T}$ is a square integrable representation of G_n. Let l be the number of mutually different segments Υ_i, $i = 1, \dots, m$, such that the induced representation $\delta(\Upsilon_i) \rtimes \mathscr{T}$ reduces. Then the induced representation*

$$\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T} \tag{3.1}$$

is a multiplicity one representation of length 2^l.

A representation $\mathscr{T}'$ of G_n is elliptic if its character does not vanish on the set of all regular elliptic elements in G_n (cf. [He]). It is well–known that any discrete series is elliptic. Now, we will recall the following result of Herb ([He])

LEMMA 3.3.

(i) *Assume that $\Upsilon_1, \dots, \Upsilon_m$ is a sequence of mutually different balanced segments (cf. Section 1), and $\mathscr{T}$ is a square integrable representation of G_n. Assume that $\delta(\Upsilon_i) \rtimes \mathscr{T}$ reduces for all i. Then all irreducible subquotients of the induced representation (3.1) are elliptic.*

(ii) *Assume that $\mathscr{T}'$ is an elliptic representation of G_n. Then there exists a datum as in (i) such that $\mathscr{T}'$ is a subrepresentation of (3.1).*

REMARK 3.1. *We refer to ([Mu], Theorem 3.2) for reducibility of generalized principal series $\delta(\Delta) \rtimes \mathscr{T}$, where Δ is a balanced segment, and $\mathscr{T}$ is a square integrable representation with generic support.*

Finally, we will prove a corollary.

COROLLARY 3.1. *Assume that $\mathscr{T}'$ is an elliptic tempered representation of G_n. Assume that $\mathscr{T}'$ is a subrepresentation of the induced representation (3.1). Let Δ be a balanced segment. Then the induced representation $\delta(\Delta) \rtimes \mathscr{T}'$ is reducible if and only if*

(i) $\Delta \notin \{\Upsilon_1, \ldots, \Upsilon_m\}$.
(ii) $\delta(\Delta) \rtimes \mathscr{T}$ *is reducible.*

Proof. Assume first that (i) and (ii) hold. Then, by Lemma 3.2, the induced representation

$$\delta(\Delta) \times \delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T} \tag{3.2}$$

is a multiplicity one representation of length 2^{l+1}. Furthermore, every irreducible subquotient of (3.2) is elliptic. Since, $\delta(\Delta) \rtimes \mathscr{T}'$ is a subrepresentation of (3.2), it cannot be irreducible. (Otherwise, $\delta(\Delta) \rtimes \mathscr{T}$ is induced irreducible representation, and, consequently, it is not elliptic (see for example [Ar]).)

If (i) or (ii) does not hold, then the induced representation (3.2) is of length 2^l. Hence, for any of 2^l irreducible subquotients $\mathscr{T}''$ of (3.1), $\delta(\Delta) \rtimes \mathscr{T}'$ is irreducible. □

4. Main result

In this section we give the final step of the classification of generic representations for G_n in terms of supercuspidals. Theorem 4.1 is analogous to the well-known result of A. V. Zelevinsky ([Ze], Theorem 9.7).

THEOREM 4.1.
(i) Let $\Upsilon_1, \ldots, \Upsilon_m$ *,be a sequence of segments, and* $\mathscr{T} \in \mathrm{Irr}'$ *an elliptic tempered* χ_μ*–generic representation. Then the representation*

$$\pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T}) = \delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T} \tag{4.1}$$

is χ_μ*–generic. The representation (4.1) is irreducible if and only if*

(1) *Segments* Υ_i, Υ_j *are not linked, for all* $1 \leqslant i < j \leqslant m$.
(2) *Segments* $\Upsilon_i, \widetilde{\Upsilon}_j$ *are not linked, for all* $1 \leqslant i < j \leqslant m$.
(3) $\delta(\Upsilon_i) \rtimes \mathscr{T}$ *is irreducible,* $i = 1, \ldots, m$.

(ii) Let $\pi \in \mathrm{Irr}'$ *be* χ_μ*–generic representation. Then there exists a sequence of segments* $\Upsilon_1, \ldots, \Upsilon_m$*, and elliptic tempered representation* $\mathscr{T}$ *such that*

$$\pi \cong \pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T}).$$

Furthermore, if π *is isomorphic to* $\pi(\Upsilon'_1, \ldots, \Upsilon'_{m'}, \mathscr{T}')$*, then* $m' = m$*,* $\mathscr{T}' \cong \mathscr{T}$*, and there exists a permutation* p *of* $\{1, \ldots, m\}$*, such that* $\Upsilon'_i = \Upsilon_{p(i)}$ *or* $\Upsilon'_i = \widetilde{\Upsilon}_{p(i)}$*, for all i.*

First, for the sake of completeness, we will state some reducibility results. The first follows easily from ([Mu], Theorem 5.1).

THEOREM 4.2. *Assume that* $\Sigma_1, \ldots, \Sigma_l$ *is a sequence of balanced segments, and* $\mathscr{T}$ *is generic square integrable representation, given by Theorem 2.1 and Proposition 2.1. Denote by* $\mathscr{T}_0$ *a generic irreducible subrepresentation of* $\delta(\Sigma_1) \times \cdots \times \delta(\Sigma_l) \rtimes \mathscr{T}$. *(*$\mathscr{T}_0$ *is tempered.) Let* Δ *be a segment,* $e(\delta(\Delta)) \neq 0$*. Then the induced representation* $\delta(\Delta) \rtimes \mathscr{T}_0$ *is irreducible if and only if*

(i) *Segments* Δ *and* Σ_i *are not linked, for all* $1 \leqslant i \leqslant l$.
(ii) *Segments* Δ *and* $\widetilde{\Sigma}_i$ *are not linked, for all* $1 \leqslant i \leqslant l$.
(iii) $\delta(\Delta) \rtimes \mathscr{T}$ *is irreducible.*

Furthermore, $\delta(\Delta) \rtimes \mathscr{T}$ *is irreducible if and only if*

(i') *Segments* Δ *and* $[\nu^{-m_i}\rho_i, \nu^{m_i}\rho_i]$, $i = 1, \ldots, k$, *are not linked.*
(ii') *Segments* Δ *and* $[\nu^{-n_i}\rho_i, \nu^{n_i}\rho_i]$, $i = 1, \ldots, k$, $n_i \geqslant 0$, *are not linked.*
(iii') $\delta(\Delta) \rtimes \sigma$ *is irreducible or there exists* $\rho' \in X'$ *such that segments* $[\rho']$ *and* Δ *are linked. (*X' *is the set of all* ρ' *such that there exists* i, $1 \leqslant i \leqslant k$, *such that* $\rho' \cong \rho_i$ *and* $n_i = -1$.*)*

REMARK 4.1. *If* $\mathscr{T}$ *is a nongeneric square integrable representation, given by Theorem 2.1, then the reducibility of nonunitary generalized principal series* $\delta(\Delta) \rtimes \mathscr{T}$ *can differ from those given by Theorem 4.2 ([Jan2]).*

Finally, the second is ([T5], Theorem 13.2).

THEOREM 4.3. *Assume that* Δ *is a segment, and* σ *is a generic supercuspidal representation of* G_n. *Then the induced representation* $\delta(\Delta) \rtimes \sigma$ *is reducible if and only if there exists* $\rho'' \in \Delta$ *such that* $\rho'' \rtimes \sigma$ *is reducible.*

Proof of Theorem 4.1. First, we will show (i). By Lemma 1.2,

$$\dim_{\mathbb{C}} r_{\chi_\mu}(\pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T})) = 1.$$

This means that the representation given by (4.1) is generic.

Furthermore, if p is a permutation of the set $\{1, \ldots, m\}$, and if for the segments Υ_i', $i = 1, \ldots, m$, we have $\Upsilon_i' = \Upsilon_{p(i)}$ or $\Upsilon_i' = \widetilde{\Upsilon}_{p(i)}$, for all i, then ([BDK], Lemma 5.4 (iii))

$$\pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T}) = \pi(\Upsilon_1', \ldots, \Upsilon_m', \mathscr{T}),$$

in the corresponding Grothendieck group. Now, we can easily see that the induced representation (4.1) is reducible if one of the conditions (1)–(3) is not valid. Let us prove that these conditions are sufficient for irreducibility. Let $S \subseteq \{1, \ldots, m\}$ be the set of all i such that $i \in S \Rightarrow e(\Upsilon_i) = 0$. (In the rest of the proof we will write $e(\Delta)$ instead of $e(\delta(\Delta))$ for simplicity.)

First, let us consider $S = \emptyset$. Then we define segments in the following way. $\Upsilon_i' = \Upsilon_i$ if $e(\Upsilon_i) > 0$, and $\Upsilon_i' = \widetilde{\Upsilon}_i$ if $e(\Upsilon_i) < 0$, for all $i = 1, \ldots, m$. Furthermore, choose a permutation p of $\{1, \ldots, m\}$ such that

$$e(\Upsilon_{p(1)}') \geqslant \cdots \geqslant e(\Upsilon_{p(m)}') > 0.$$

As above, in the corresponding Grothendieck group,

$$\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T} = \delta(\Upsilon_{p(1)}') \times \cdots \times \delta(\Upsilon_{p(m)}') \rtimes \mathscr{T}.$$

Hence, the representation (4.1) reduces if and only if $\pi(\Upsilon_{p(1)}', \ldots, \Upsilon_{p(m)}', \mathscr{T})$ reduces. The segments $\Upsilon_{p(1)}', \ldots, \Upsilon_{p(m)}'$ satisfy (1)–(3) if and only if $\Upsilon_1, \ldots, \Upsilon_m$ satisfy (1)–(3). Furthermore, by Lemma 1.1, $\pi(\Upsilon_{p(1)}', \ldots, \Upsilon_{p(m)}', \mathscr{T})$ is a standard representation. Then one can use a factorisation of the corresponding long intertwining operator to finish the proof of (i). (See for example the proof of Theorem 7.1 in [T3] or the proof of Theorem 3.3 in [Jan1] for such type of arguments.)

Let us now consider the case $S \neq \emptyset$. We can assume $S = \{1, \ldots, m_1\}$, $m_1 \leqslant m$. Let us prove that the induced representation

$$\mathscr{T}_1 = \delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_{m_1}) \rtimes \mathscr{T} \tag{4.2}$$

is irreducible. By ([He]) or Lemma 3.3, there exist mutually different balanced segments $\Sigma_1, \ldots, \Sigma_l$, and a discrete series $\mathscr{T}_0 \in \mathrm{Irr}'$ such that

$$\mathscr{T} \hookrightarrow \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_l) \rtimes \mathscr{T}_0. \tag{4.3}$$

Any irreducible subquotient of (4.3) is elliptic. Moreover, the length of (4.3) is 2^l. To prove that the induced representation (4.2) is irreducible (under the conditions (1), (2), and (3)), it is enough to prove that the length of

$$\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_{m_1}) \times \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_l) \rtimes \mathscr{T}_0$$

is 2^l. By ([Go]) or Lemma 3.2, it is enough to prove that for any i, we have

$$\Upsilon_i \in \{\Sigma_1, \ldots, \Sigma_l\} \quad \text{or} \quad \delta(\Upsilon_i) \rtimes \mathscr{T}_0 \ \text{ is irreducible.}$$

Now, since $\delta(\Upsilon_i) \rtimes \mathscr{T}$ is irreducible, the assertion follows from Corollary 3.1. Since

$$\pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T}) = \delta(\Upsilon_{m_1+1}) \times \cdots \times \delta(\Upsilon_m) \rtimes \mathscr{T}_1,$$

in the corresponding Grothendieck group, we can finish the proof as in the case when $S = \emptyset$. The proof of (i) is finished.

The part of (ii) refering to existence follows from the following discussion. First, the following result is proved in ([Mu]).

LEMMA 4.1 ([Mu], Theorem 5.1). *Assume that $\delta_1, \ldots, \delta_k \in \mathrm{Irr}$ are essentially square integrable representations such that $e(\delta_1) \geqslant \cdots \geqslant e(\delta_k) > 0$, and $\mathscr{T}$ is a tempered generic representation of $G_{n'}$. Then the standard induced representation*

$$\delta_1 \times \cdots \times \delta_k \rtimes \mathscr{T}$$

reduces if and only if the Langlands quotient $L(\delta_1, \ldots, \delta_k; \mathscr{T})$ is not generic.

Since, in the case of symplectic and odd–orthogonal groups every tempered representation is fully induced from a elliptic tempered representation ([He]), the existence in (ii) follows from the Langlands classification, using Lemma 4.1.

To prove (ii), it remains to consider equivalence among representations defined by (4.1). Let us assume that we have an isomorphism of induced irreducible representations

$$\pi(\Upsilon_1, \ldots, \Upsilon_m, \mathscr{T}) \cong \pi(\Upsilon'_1, \ldots, \Upsilon'_{m'}, \mathscr{T}'). \tag{4.4}$$

Let $S' \subseteq \{1, \ldots, m'\}$ be the set of all i such that $i \in S' \Rightarrow e(\Upsilon'_i) = 0$. As in the proof of (i), we can assume

$$e(\Upsilon_m) \geqslant \cdots \geqslant e(\Upsilon_{m_1+1}) > e(\Upsilon_{m_1}) = \cdots = e(\Upsilon_1) = 0$$

and

$$e(\Upsilon'_{m'}) \geqslant \cdots \geqslant e(\Upsilon'_{m'_1+1}) > e(\Upsilon'_{m'_1}) = \cdots = e(\Upsilon'_1) = 0.$$

Let

$$\mathscr{T}'_1 = \delta(\Upsilon'_1) \times \cdots \times \delta(\Upsilon'_{m'_1}) \rtimes \mathscr{T}', \tag{4.5}$$

where

$$\mathscr{T}' \hookrightarrow \delta(\Sigma_1') \times \cdots \times \delta(\Sigma_{l'}') \rtimes \mathscr{T}_0', \tag{4.6}$$

where $\Sigma_1', \ldots, \Sigma_{l'}'$ are mutually different balanced segments, and $\mathscr{T}_0'$ is square integrable. As in the proof of part (i), $\mathscr{T}_1'$ is irreducible. Now, the isomorphism given in (4.4) is in fact an isomorphism of the standard representations

$$\delta(\Upsilon_m) \times \cdots \times \delta(\Upsilon_{m_1+1}) \rtimes \mathscr{T}_1 \cong \delta(\Upsilon_{m'}') \times \cdots \times \delta(\Upsilon_{m_1'+1}') \rtimes \mathscr{T}_1'.$$

Then, by Lemma 1.1 (ii), $m' - m_1' = m - m_1$, $\mathscr{T}_1' \cong \mathscr{T}_1$, and, up to a permutation, $\Upsilon_{i+m_1} = \Upsilon_{i+m_1'}'$, $i = 1, \ldots, m - m_1$. It remains to consider $\mathscr{T}_1' \cong \mathscr{T}_1$. (See (4.2), (4.3), (4.5), and (4.6).) This implies that the induced representations

$$\delta(\Upsilon_1) \times \cdots \times \delta(\Upsilon_{m_1}) \times \delta(\Sigma_1) \times \cdots \times \delta(\Sigma_l) \rtimes \mathscr{T}_0$$
$$\text{and} \quad \delta(\Upsilon_1') \times \cdots \times \delta(\Upsilon_{m_1'}') \times \delta(\Sigma_1') \times \cdots \times \delta(\Sigma_{l'}') \rtimes \mathscr{T}_0'$$

have a common irreducible subquotient. Now, Lemma 3.1 implies $m_1' + l' = m_1 + l$ and $\mathscr{T}_0 \cong \mathscr{T}_0'$. Furthermore, we can assume the following equality of multisets

$$\{\Upsilon_1, \ldots, \Upsilon_{m_1}, \Sigma_1, \ldots, \Sigma_l\} = \{\Upsilon_1', \ldots, \Upsilon_{m_1'}', \Sigma_1', \ldots, \Sigma_{l'}'\}.$$

Let us call A the set obtained from these segments. Then we can characterize the sets $\{\Sigma_1, \ldots, \Sigma_l\}$ and $\{\Sigma_1', \ldots, \Sigma_{l'}'\}$ as maximal subsets of A, such that the induced representations $\delta(\Sigma_i) \rtimes \mathscr{T}_0$ and $\delta(\Sigma_i') \rtimes \mathscr{T}_0$ are reducible for all i. Now, it is clear $\{\Sigma_1, \ldots, \Sigma_l\} = \{\Sigma_1', \ldots, \Sigma_{l'}'\}$. In particular, $l = l'$ and $\mathscr{T}' \cong \mathscr{T}$. ($\chi_\mu$–generic representations !). Furthermore, $m_1' = m_1$, and we have the equality of multisets $\{\Upsilon_1, \ldots, \Upsilon_{m_1}\} = \{\Upsilon_1', \ldots, \Upsilon_{m_1'}'\}$. This means that the sequence $\Upsilon_1, \ldots, \Upsilon_{m_1}$ is, up to a permutation, the sequence $\Upsilon_1', \ldots, \Upsilon_{m_1'}'$. The proof of (ii) is finished. □

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