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CERTAIN INTERESTING IMPLICATIONS OF T. J. RIVLIN'S RESULT ON MAXIMUM MODULUS OF A POLYNOMIAL

V. K. Jain, Kharagpur, India

Abstract. Let $f(z)$ be an arbitrary entire function and $M(f,r) = \max_{|z|=r} |f(z)|$. For a polynomial $p(z)$ of degree n, having no zeros in $|z| < 1$, Rivlin had obtained

$$M(p,r) \geqslant \left(\frac{r+1}{2}\right)^n M(p,1), \quad r \leqslant 1.$$

Using this and various associated results, we have obtained precise relationship between $M(p,\rho)$ and $M(p,1)$ for a polynomial $p(z)$ having all its zeros on $|z| = k$, $k > 0$. Some sort of a converse of Rivlin's result has also been obtained.

1. Introduction and statement of results

Let $f(z)$ be an entire function and $M(f,r) = \max_{|z|=r} |f(z)|$. If $p(z)$ is a polynomial of degree n, then as an easy consequence of maximum modulus principle, we have

THEOREM A. *For a polynomial $p(z)$ of degree n*

$$M(p,r) \geqslant r^n M(p,1), \quad r \leqslant 1, \tag{1.1}$$

with equality only for $p(z) = \lambda z^n$.

For polynomials not vanishing in $|z| < 1$, Rivlin [3] improved (1.1) and proved

THEOREM B. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then*

$$M(p,r) \geqslant \left(\frac{1+r}{2}\right)^n M(p,1), \quad r \leqslant 1,$$

with equality only for the polynomial $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, $|\alpha| = |\beta|$.

Govil [2] obtained the following generalization of Theorem B

THEOREM C. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for $0 \leqslant r \leqslant \rho \leqslant 1$,*

$$M(p,r) \geqslant \left(\frac{1+r}{1+\rho}\right)^n M(p,\rho).$$

The result is best possible and the equality holds for the polynomial $p(z) = (1+z)^n$.

Using Theorem C, Govil [2] obtained

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THEOREM D. *If $p(z)$ is a polynomial of degree n having all its zeros on $|z| = 1$, then for $0 \leqslant r \leqslant \rho \leqslant 1$ and for $1 \leqslant \rho \leqslant r$,*

$$M(p,r) \geqslant \left(\frac{1+r}{1+\rho}\right)^n M(p,\rho).$$

The result is best possible and equality holds for the polynomial $p(z) = (1+z)^n$.

In this paper, we have firstly considered polynomials having their zeros on $|z| = k$ and obtained precise relationship between $M(p,\rho)$ and $M(p,1)$ — something similar to the Theorem D. More precisely, we have proved

THEOREM 1. *Let $p(z)$ be a polynomial of degree n, having all its zeros on the circle $C: \ |z| = k,\ k > 0$. If $k \geqslant 1$, then*

$$M(p,\rho) \geqslant \left(\frac{\rho+k}{1+k}\right)^n M(p,1) \ \textit{for}\ 0 \leqslant \rho \leqslant 1 \ \textit{or}\ \rho \geqslant k^2, \tag{1.2}$$

$$\leqslant \left(\frac{\rho+k}{1+k}\right)^n M(p,1) \ \textit{for}\ 1 \leqslant \rho \leqslant k^2, \tag{1.3}$$

and if $k < 1$, then

$$M(p,\rho) \geqslant \left(\frac{\rho+k}{1+k}\right)^n M(p,1) \ \textit{for}\ 0 \leqslant \rho \leqslant k^2 \ \textit{or}\ \rho \geqslant 1, \tag{1.4}$$

$$\leqslant \left(\frac{\rho+k}{1+k}\right)^n M(p,1) \ \textit{for}\ k^2 \leqslant \rho \leqslant 1. \tag{1.5}$$

Equality holds in (1.2), (1.3), (1.4) and (1.5) for $p(z) = (z+k)^n$.

By taking $\rho = k^2$ in Theorem 1, we easily obtain

CORROLARY 1. *Let $p(z)$ be a polynomial of degree n, having all its zeros on $|z| = k,\ k > 0$. Then*

$$M(p,k^2) = k^n M(p,1).$$

Now we try to say something about converse of theorem B. The example $p(z) = \left(z+\frac{1}{2}\right)(z+3)$ shows that converse of Theorem B is false. However, we have the following result in the converse direction.

THEOREM 2. *If $p(z)$ is a polynomial of degree n such that*

$$M(p,r) \geqslant \left(\frac{1+r}{2}\right)^n M(p,1), \tag{1.6}$$

for all $r \in (r_0, 1)$, where $r_0 \in (0,1)$, then $p(z)$ can not have all its zeros in $|z| < 1$.

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

LEMMA 1. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k > 0$, then*

$$M(p,r) \geqslant \left(\frac{r+k}{1+k}\right)^n M(p,1), \quad r \leqslant \min(1,k^2). \tag{2.1}$$

Equality holds in (2.1) for $p(z) = (z+k)^n$.

This lemma is due to Govil [2].

LEMMA 2. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < k$, $k \geqslant 1$, then*

$$M(p,R) \leqslant \left(\frac{R+k}{1+k}\right)^n M(p,1), \quad 1 \leqslant R \leqslant k^2. \tag{2.2}$$

Equality holds in (2.2) for $p(z) = (z+k)^n$.

This lemma is due to Aziz and Mohammad [1].

LEMMA 3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leqslant k$, $k \leqslant 1$, then*

$$M(p,r) \leqslant \left(\frac{r+k}{1+k}\right)^n M(p,1), \quad k^2 \leqslant r \leqslant 1. \tag{2.3}$$

Equality holds in (2.3) for $p(z) = (z+k)^n$.

Proof. The polynomial

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}, \tag{2.4}$$

will have no zeros in $|z| < \frac{1}{k}$ and so, by Lemma 2

$$M(q,R) \leqslant \left(\frac{R+\frac{1}{k}}{1+\frac{1}{k}}\right)^n M(q,1), \quad 1 \leqslant R \leqslant \frac{1}{k^2},$$

i.e.

$$R^n M\left(p,\frac{1}{R}\right) \leqslant \left(\frac{R+\frac{1}{k}}{1+\frac{1}{k}}\right)^n M(p,1), \quad 1 \leqslant R \leqslant \frac{1}{k^2}.$$

On replacing R by $\frac{1}{r}$, Lemma 3 follows.

Remark. Arguments used in proof of Theorem B were used to obtain Lemma 1 and Lemma 2.

3. Proofs of the theorems

Proof of Theorem 1. The polynomial $q(z)$, given by (2.4), will have all its zeros on $|z| = \frac{1}{k}$, and so, by Lemma 1

$$M(q,r) \geqslant \left(\frac{r+\frac{1}{k}}{1+\frac{1}{k}}\right)^n M(q,1), \quad r \leqslant \min\left(1,\frac{1}{k^2}\right),$$

i.e.

$$r^n M\left(p, \frac{1}{r}\right) \geqslant \left(\frac{r+\frac{1}{k}}{1+\frac{1}{k}}\right)^n M(p,1), \quad r \leqslant \min\left(1, \frac{1}{k^2}\right).$$

On replacing r by $\frac{1}{R}$, we obtain

$$\frac{1}{R^n} M(p,R) \geqslant \left(\frac{\frac{1}{R}+\frac{1}{k}}{1+\frac{1}{k}}\right)^n M(p,1), \quad \frac{1}{R} \leqslant \min\left(1, \frac{1}{k^2}\right),$$

i.e.

$$M(p,R) \geqslant \left(\frac{R+k}{1+k}\right)^n M(p,1), \quad R \geqslant \max(1,k^2). \tag{3.1}$$

Now ineq. (1.2) and (1.4) follow from Lemma 1 and inequality (3.1). Ineq. (1.3) follows from Lemma 2. Ineq. (1.5) follows from Lemma 3.

This completes the proof of Theorem 1.

Proof of Theorem 2. On the contrary, let $p(z)$ have all its zeros in $|z| < 1$. Then $p(z)$ will have all its zeros in $|z| \leqslant k$, for some $k(< 1)$, also. And so, by Lemma 3

$$\begin{aligned} M(p,r) &\leqslant \left(\frac{r+k}{1+k}\right)^n M(p,1), \quad k^2 \leqslant r < 1 \\ &< \left(\frac{r+1}{2}\right)^n M(p,1), \quad k^2 \leqslant r < 1, \end{aligned}$$

which contradicts (1.6), thereby proving Theorem 2.

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Mathematics Department
I.I.T., Kharagpur - 721302
India