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ON A GENERAL ASYMPTOTIC PROBLEM ASSOCIATED WITH LERAY—LIONS OPERATORS

Dedicated to the memory
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Abstract. Firstly, we prove a pointwise comparison result for the suitable *symmetrized* problem that depends on a small positive parameter λ. Then, by these results and by the Schwarz symmetrization, we obtain some asymptotic relationship between the solutions u_ε of a general ε - problem and a sequence of real numbers λ_ε. Finally, it is shown an application the preceding results to getting a priori estimates in the homogenization theory.

1. Introduction and statement of problem

Let Ω be a bounded open set in $R^N (N \geqslant 1)$ and p and p' be two real numbers, $1 < p < +\infty$, $1/p + 1/p' = 1$. We consider the general ε - problem ($\varepsilon > 0$) for quasilinear elliptic equations of Leray - Lions type with p - growth in the gradient:

$$\begin{cases} A_\varepsilon u_\varepsilon + F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega), \end{cases} \tag{1}$$

where A_ε is a family of the operators of Leray - Lions type (*see* [10]) from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$, and F_ε is a family of the nonlinear operators of Nemytski type from $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ into $L^1(\Omega)$, which satisfy:

$$A_\varepsilon u = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_{\varepsilon,i}(x, u(x), \nabla u(x)) \quad \text{and} \quad F_\varepsilon(u, \nabla u)(x) = f_\varepsilon(x, u(x), \nabla u(x)). \tag{2}$$

The families of Caratheodory functions $A_{\varepsilon,i}$ and f_ε from $\Omega \times R \times R^N$ into R, for every $\varepsilon > 0$ satisfy the following properties:

$$\begin{cases} \exists \beta_{\varepsilon,i} > 0, \ \exists h_{\varepsilon,i} \in L^{p'}(\Omega), \ \forall \eta \in R, \ \forall \xi \in R^N \\ | A_{\varepsilon,i}(x, \eta, \xi) | \leqslant \beta_{\varepsilon,i}[h_{\varepsilon,i}(x) + | \eta |^{p-1} + | \xi |^{p-1}], \quad \text{a.e. in } \Omega, \end{cases} \tag{3}$$

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$$\begin{cases} \forall \eta \in R, \quad \forall \xi, \xi^* \in R^N, \; \xi \neq \xi^* \\ \sum_{i=1}^N [\, A_{\varepsilon,i}(x,\eta,\xi) - A_{\varepsilon,i}(x,\eta,\xi^*)\,](\xi_i - \xi_i^*) > 0, \quad \text{a.e. in } \Omega, \end{cases} \tag{4}$$

$$\exists \alpha_\varepsilon > 0,\ \forall \eta \in R,\ \forall \xi \in R^N,\ \sum_{i=1}^N A_{\varepsilon,i}(x,\eta,\xi)\xi_i \geqslant \alpha_\varepsilon \mid \xi \mid^p, \ \text{ a.e. in } \Omega, \tag{5}$$

$$\exists f_{\varepsilon,0} > 0,\ \forall \eta \in R,\ \forall \xi \in R^N,\ \mid f_\varepsilon(x,\eta,\xi) \mid \leqslant f_{\varepsilon,0}(1+ \mid \xi \mid^p), \ \text{ a.e. in } \Omega. \tag{6}$$

Under the assumptions $(2)-(6)$, for every $\varepsilon > 0$ we have the existence of a solution u_ε of the equation (1) (*see* for example [5], [6], [7] and [13]-[14]).

Let λ_ε be a sequence of real numbers defined by

$$\lambda_\varepsilon = \frac{f_{\varepsilon,0}}{\alpha_\varepsilon}, \quad \text{where } \alpha_\varepsilon \text{ and } f_{\varepsilon,0} \text{ are defined in (5) and (6).} \tag{7}$$

In this note we give all details and proofs of the theorems which were announced in the author's note [12]. Precisely, we shall investigate the asymptotic behaviour in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the solutions u_ε from (1) with respect to the asymptotic behaviour of λ_ε from (7). That is to say, we want to show that the precise boundeness in R and the convergence to 0 in R of a sequence λ_ε from (7), implies the precise boundeness in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and the convergence to 0 in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of a sequence u_ε from (1). We remark that the families of operators A_ε and F_ε from $(1)-(6)$ have only one asymptotic condition, that is, the asymptotic condition to (7) (see (9) and (12)).

About some investigations and applications of various ε - problems in the homogenization of partial differential equations *see* [1], [2], [3] and [4].

Next, we consider a *symmetrized* problem with a parameter $\lambda > 0$:

$$\begin{cases} -div(\mid \nabla v_\lambda \mid^{p-2} \nabla v_\lambda) - \lambda(1+ \mid \nabla v_\lambda \mid^p) = 0, \quad \text{in } \Omega^\#/\{0\}, \\ v_\lambda \in W_0^{1,p}(\Omega^\#) \bigcap L^\infty(\Omega^\#), \\ v_\lambda \text{ is positive, radially symmetric and decreasing function,} \end{cases} \tag{8}$$

where $\Omega^\#$ is N-dimensional ball centered at the origin 0 with $\mid \Omega^\# \mid = \mid \Omega \mid$ ($\mid A \mid$ denotes the Lebesgue measure of a measurable set A in R^N).

On the existence and the uniqueness results, and properties concerning the solution v_λ of the equation (8) *see* Lemma 4 below.

Firstly, we shall prove:

THEOREM 1. *Let u_ε be a solution of the equation* (1) *and let* $(2)-(6)$ *be satisfied. If for λ_ε from* (7) *exists a "small enough" constant $\beta > 0$ such that*

$$\lambda_\varepsilon \leqslant \beta, \quad \forall \varepsilon > 0, \tag{9}$$

then there exist two constants $c_1 = c_1(\beta) > 0$ and $c_2 = c_2(\beta) > 0$ such that the following a priori estimates hold true:

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leqslant c_1(\beta) \quad \text{and} \quad \| \nabla u_\varepsilon \|_{L^p(\Omega)} \leqslant c_2(\beta),\ \forall \varepsilon > 0. \tag{10}$$

Moreover, the constants c_1 and c_2 satisfy:

$$c_1(\beta) = \| v_\beta \|_{L^\infty(\Omega^\#)} \quad \text{and} \quad c_2(\beta) = \| \nabla v_\beta \|_{L^p(\Omega^\#)}, \tag{11}$$

where v_β is the unique solution of the equation (8) *for* $\lambda = \beta$.

The assumption, β is "small enough", is needed in the lemma 4 below.

Next, we shall obtain the limit behaviour of u_ε from (1) with respect to the limit behaviour of λ_ε from (7):

THEOREM 2. *Let u_ε be a solution of the equation* (1) *and let* (2) – (6) *be satisfied. If λ_ε from* (7) *satisfies:*

$$\lambda_\varepsilon \to 0, \quad as \ \varepsilon \to 0, \tag{12}$$

then the following estimate holds true:

$$u_\varepsilon \to 0 \qquad strongly\ in\ W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{13}$$

The proofs of the preceding theorems will be made in Section 3.

A GENERALIZATION. Now, we give some generalization of the results from Theorem 1 and Theorem 2 to a slightly general equation with p - growth in the gradient:

$$\begin{cases} A_\varepsilon u_\varepsilon + H_\varepsilon(u_\varepsilon) + F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \text{in } \Omega, \\ u_\varepsilon \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega), \end{cases} \tag{14}$$

where the families of operators A_ε and F_ε are as in the equation (1), and H_ε from $L^\infty(\Omega)$ into $L^1(\Omega)$ satisfies:

$$H_\varepsilon(u)(x) = h_\varepsilon(x, u(x)) \quad \text{a.e. in } \Omega, \tag{15}$$

where the family h_ε of Caratheodory function from $\Omega \times R$ into R, for every $\varepsilon > 0$ satisfies:

$$\begin{cases} \forall \eta \in R,\ h_\varepsilon(x,\eta) sgn(\eta) \geqslant 0 \text{ a.e. in } \Omega; \\ \exists h_{\varepsilon,0} \in L^1(\Omega),\ h_{\varepsilon,0} \geqslant 0,\ \forall M > 0,\ \exists C_\varepsilon = C_\varepsilon(M) > 0, \\ \forall \eta \in R,\ \mid \eta \mid \leqslant M, \quad \mid h_\varepsilon(x,\eta) \mid \leqslant C_\varepsilon(M) h_{\varepsilon,0}(x) \ \text{ a.e. in } \Omega. \end{cases} \tag{16}$$

THEOREM 3. *Let u_ε be a solution of the equation* (14) *and let λ_ε be defined by* (7). *Let* (2) – (6) *and* (15) – (16) *be satisfied. If λ_ε satisfies* (9) *then for u_ε we have* (10) – (11). *Also, if λ_ε satisfies* (12) *then for u_ε we have* (13).

Thanks to the proofs of Theorem 1 and Theorem 2, the proof of the preceding theorem we leave to reader.

AN APPLICATION TO THE HOMOGENIZATION THEORY. It is very well known that in many problems of the classical and abstract homogenization theory (*see* [1], [2], [3] and [4]), where α_ε and $f_{\varepsilon,0}$ from (5) – (6) are independent of ε (therefore (9) will be satisfied), it is very important to obtain a priori estimates which are independent of a small parameter ε (for $p = 2$ *see* [2]). These a priori estimates, in this note, we obtain particularly from the general result of Theorem 1. In this direction we will replace the assumptions (5)-(6) with the following:

$$\exists\alpha > 0,\ \forall\eta \in R,\ \forall\xi \in R^N,\ \sum_{i=1}^{N} A_{\varepsilon,i}(x,\eta,\xi)\xi_i \geqslant \alpha \mid \xi \mid^p, \text{ a.e. in } \Omega,\ \forall\varepsilon > 0, \tag{17}$$

$$\exists f_0 > 0,\ \forall\eta \in R,\ \forall\xi \in R^N,\ \mid f_\varepsilon(x,\eta,\xi) \mid \leqslant f_0(1+ \mid \xi \mid^p), \text{ a.e. in } \Omega,\ \forall\varepsilon > 0. \tag{18}$$

where the ratio f_0/α is "small enough" (see the proof of lemma 4 below).

According to the result of Theorem 1, it is easy to check that the following result holds true:

THEOREM 4. *Let u_ε be a solution of the equation* (1) *and let* (2) – (4) *and* (17) – (18) *be satisfied. Then there exists two positive constants c_1 and c_2 such that the following a priori estimates hold true:*

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leqslant c_1 \quad \text{and} \quad \| \nabla u_\varepsilon \|_{L^p(\Omega)} \leqslant c_2,\quad \forall\varepsilon > 0. \tag{19}$$

Moreover, there exists a small constant $\lambda > 0$ independent of ε such that the constants c_1 and c_2 satisfy:

$$c_1 = \| v_\lambda \|_{L^\infty(\Omega^*)} \quad \text{and} \quad c_2 = \| \nabla v_\lambda \|_{L^p(\Omega^*)}, \tag{20}$$

where v_λ is the unique solution of the equation (8).

2. Comparison results with a parameter

This section deal with the comparison results from Lemma 3 and Lemma 5 below. Before giving the proofs of these results, we first repeat a comparison principles of two Lemmas from [12].

For fixed $a, b \in R$, $a < b$ and for every $c \in (a, b]$, let $W_{a,c}$ denotes an arbitrary subset of $C([a, c])$ - denotes space of all continuous functions on $[a, c]$ and let K be an operator from $W_{a,c}$ into $C([a, c])$ which is independent of c and satisfies:

$$\begin{cases} \text{for every } c \in (a,b) \text{ there exists } \alpha = \alpha(c) \in (0,1) \text{ such that} \\ \| K\varphi - K\psi \|_{L^\infty(a,c)} \leqslant \alpha(c) \| \varphi - \psi \|_{L^\infty(a,c)} \quad \forall\varphi,\psi \in W_{a,c}. \end{cases} \tag{21}$$

We first give a *local* result in any $[a, b]$ for the sub { sup} -comparison:

LEMMA 1. *Assume that* (21) *holds and let $\varphi, \psi \in W_{a,b}$ be two arbitrary functions which satisfy: the restrictions $\varphi\mid_{[a,c]}$, $\psi\mid_{[a,c]} \in W_{a,c}$ for all $c \in (a, b)$ and*

$$\begin{cases} \varphi(a) = \psi(a), \\ \varphi(s) \leqslant (K\varphi)(s),\ \{\ \varphi(s) \geqslant (K\varphi)(s)\ \},\ \psi(s) = (K\psi)(s),\ \forall s \in [a,b]. \end{cases} \tag{22}$$

Then

$$\begin{cases} \textit{there exists no } c \in (a,b) \textit{ such that} \\ \varphi \neq \psi \textit{ in } (a,c] \textit{ and } \varphi(s) \geqslant \psi(s),\ \{\ \varphi(s) \leqslant \psi(s)\ \},\ \forall s \in [a,c]. \end{cases} \tag{23}$$

Let $a_0, b_0 \in R$, $a_0 < b_0$ be two fixed numbers. Now, we give a *global* result on $[a_0, b_0]$ for the sub { sup} -comparison:

LEMMA 2. *Let φ, $\psi \in C([a_0, b_0])$ be two arbitrary functions which satisfy:*

$$\begin{cases} \varphi(a_0) = \psi(a_0), \textit{ for all } a_1 \in [a_0, b_0) \textit{ for which } \varphi(a_1) = \psi(a_1) \\ \textit{there exists } b_1 = b_1(a_1) \in (a_1, b_0] \textit{ which satisfies:} \\ \textit{there exists no } c \in (a_1, b_1) \textit{ such that} \\ \varphi \neq \psi \textit{ in } (a_1, c] \textit{ and } \varphi(s) \geqslant \psi(s), \ \{ \varphi(s) \leqslant \psi(s) \}, \ \forall s \in [a_1, c]. \end{cases} \tag{24}$$

Then we have

$$\varphi(s) \leqslant \psi(s) \ , \ \{ \varphi(s) \geqslant \psi(s) \} \ , \ \forall s \in [a_0, b_0]. \tag{25}$$

For the proofs of Lemma 1 and Lemma 2 *see* in [12].

Now we consider a nonlinear ordinary differential equation with a parameter $\lambda > 0$:

$$\begin{cases} d\omega_\lambda / ds = 1 + C_0(\lambda) s^{p'(-1+1/N)} \omega_\lambda^{p'}(s) \ , \quad \forall s \in (0, |\Omega|], \\ \omega_\lambda(0) = 0 \ \text{ and } \ s^{p'(-1+1/N)} \omega_\lambda^{p'}(s) \to 0 \ \text{ as } \ s \to 0, \end{cases} \tag{26}$$

where $\omega_\lambda : [0, |\Omega|] \to [0, +\infty)$, and $C_0(\lambda)$ defined by

$$C_0(\lambda) = (\lambda / (N C_N^{1/N}))^{p'} . \tag{27}$$

Here C_N denotes the measure of N - dimensional unit ball in R^N. One can show that there exists a unique function $\omega_\lambda \in C^1([0, |\Omega|])$ which satisfies (26), where λ is "small enough" (see the proof of lemma 4 below).

We are now able to prove:

LEMMA 3. *Let λ and μ be two real numbers, $0 < \lambda < \mu$ and let ω_λ and ω_μ be two corresponding unique solutions of the equation* (26). *Then we have:*

$$\omega_\lambda(s) \leqslant \omega_\mu(s), \quad \forall s \in [0, |\Omega|]. \tag{28}$$

Before giving the proof of Lemma 3, we will need some technical results.

In this direction, we will use repeatedly the following notations: let the real numbers λ, a, b, m and M satisfy: $\lambda > 0$, $a \in [0, |\Omega|)$, $b \in (a, |\Omega|]$; for $a = 0$ let $m = 0$ and $M > 0$; for $a > 0$ let $m > 0$ and $M \geqslant m/a$. Next, let set $Z_{a,b} = Z_{a,b}(m, M)$ and operator $K_\lambda = K_\lambda(a, m)$ from $Z_{a,b}$ into $C([a, b])$, be given by:

$$Z_{a,b} = \{ \varphi : [a, b] \to [0, +\infty) : \varphi \in C([a, b]), \ \varphi(a) = m, \ \varphi(s) \leqslant Ms \ \text{ on } [a, b] \}, \tag{29}$$

$$(K_\lambda \varphi)(s) = m - a + s + C_0(\lambda) \int_a^s r^{p'(-1+1/N)} \varphi^{p'}(r) dr \ , \ \forall \varphi \in Z_{a,b}, \tag{30}$$

where $C_0(\lambda)$ was defined by (27).

With these notations we now can state the following technical results:

PROPOSITION 1. *For all $\lambda > 0$, $a \in [0, |\Omega|)$ and any m, M as in* (29) – (30), *we have:*

$$\begin{cases} \text{for every } c \in (a, |\Omega|] \text{ are } Z_{a,c} \subseteq C([a,c]),\ K_\lambda Z_{a,c} \subseteq C([a,c]) \text{ and} \\ \text{there exists } b = b(a, M, \lambda) \in (a, |\Omega|] \text{ such that} \\ \text{for every } c \in (a,b) \text{ there exists } \alpha = \alpha(a,c,M,\lambda) \in (0,1) \text{ such that} \\ \| K_\lambda \varphi - K_\lambda \psi \|_{L^\infty(a,c)} \leqslant \alpha(a,c,M,\lambda) \| \varphi - \psi \|_{L^\infty(a,c)},\ \forall \varphi, \psi \in Z_{a,c}. \end{cases} \tag{31}$$

Proof. On the similar way as in the proof of (19) from [12], we prove (31). □

PROPOSITION 2. *Let λ and μ be two real numbers, $0 < \lambda < \mu$. Then for any fixed a, b, m, M as in* (29), *we have:*

$$(K_\lambda \varphi)(s) \leqslant (K_\mu \varphi)(s), \quad \forall s \in [a,b],\ \forall \varphi \in Z_{a,b}. \tag{32}$$

Proof. From $\lambda < \mu$ follows $C_0(\lambda) < C_0(\mu)$ (*see* (27)) which together with (30) immediately implies (32) ($K_\lambda = K_\lambda(a,m)$, $K_\mu = K_\mu(a,m)$ and $Z_{a,b} = Z_{a,b}(m,M)$). □

PROPOSITION 3. *Let λ and μ be two real numbers, $0 < \lambda < \mu$ and let ω_λ and ω_μ be two corresponding unique solutions of the equation* (26). *Then for all $a \in [0, |\Omega|)$ for which $\omega_\lambda(a) = \omega_\mu(a)$ there exists M as in* (29) *such that for $m = \omega_\lambda(a) = \omega_\mu(a)$ and any $b = b(a) \in (a, |\Omega|]$ we have:*

$$\omega_\lambda,\ \omega_\mu \in Z_{a,b} \ \text{ and } \ \omega_\lambda \big|_{[a,c]},\ \omega_\mu \big|_{[a,c]} \in Z_{a,c},\ \ \forall c \in (a,b), \tag{33}$$

$$\omega_\lambda(s) \leqslant (K_\mu \omega_\lambda)(s) \ \text{ and } \ \omega_\mu(s) = (K_\mu \omega_\mu)(s), \quad \forall s \in [a,b], \tag{34}$$

$$\omega_\lambda(s) = (K_\lambda \omega_\lambda)(s) \ \text{ and } \ \omega_\mu(s) \geqslant (K_\lambda \omega_\mu)(s), \quad \forall s \in [a,b]. \tag{35}$$

Proof. Let $a \in [0, |\Omega|)$ and $\omega_\lambda(a) = \omega_\mu(a) = m$. Integrating (26) over $[a,s]$, from (30) we have that $\omega_\lambda = K_\lambda \omega_\lambda$ and $\omega_\mu = K_\mu \omega_\mu$. Now by these equalities and by (32) we obtain (34)-(35). Finally, we have (33) for $M = max\{M_\lambda; M_\mu\}$ where M_λ and M_μ we obtain by Remark 1 from [12]. □.

Proof of Lemma 3. With the help of the Proposition 1 and Proposition 3, we have in the particular case that (21)-(22) hold true on $[a,b]$ for all $a \in [0, |\Omega|)$ for which $\omega_\lambda(a) = \omega_\mu(a)$, $b = b(a, M, \mu)$ -from (31), M -from (33), $\varphi = \omega_\lambda$, $\psi = \omega_\mu$, $K = K_\mu(a,m)$, $m = \omega_\lambda(a) = \omega_\mu(a)$ and $W_{a,c} = Z_{a,c}(m,M)$ (*see* (29)-(30) for $\lambda = \mu$). Then from Lemma 1 and (23) in this particular case, we have (24) for $a_0 = 0, b_0 = |\Omega|$, $\varphi = \omega_\lambda$ and $\psi = \omega_\mu$. Then from Lemma 2 and (25) immediately follows (28). □

LEMMA 4. *There exists a unique solution $v_\lambda \in C(\overline{\Omega^\#}) \cap W^{1,\infty}(\Omega^\#) \cap C^2(\Omega^\#/\{0\})$ of the problem* (8), *where λ is "small enough". Moreover v_λ depends only on the constants N, p, λ, C_N, $|\Omega|$ and for all $x \in \overline{\Omega^\#}$ and for all $s \in (0, |\Omega|]$ we have:*

$$v_\lambda(x) = v_\lambda^*(C_N |x|^N) \ \text{ and } \ v_\lambda^*(s) = \frac{C_0(\lambda)}{\lambda} \int_s^{|\Omega|} r^{p'(-1+1/N)} \omega_\lambda^{p'/p}(r)dr, \tag{36}$$

where ω_λ *is the unique solution of the equation* (26) *and* $C_0(\lambda)$, λ, C_N *from* (27).

Remark. In the recently author's note [14] was remarked that the preceding existence result for the equation (8) and the existence result for the equation (26) are in fact only valid under the following supplementary hypothesis: λ is "small enough", that is to say if λ satisfies:

$$\lambda \leqslant \frac{NC_N^{1/N}}{2\mid\Omega^\#\mid^{1/N}}\left(\frac{p'}{N(p'+1)}\right)^{1/p'}.$$

This fact obviously follows from the classical Banach fixed - point theorem (on this classical technique, see for example the proof of Theorem 1, [12], pp. 364). But, in the case for λ "big enough" it seems it is not elementary to prove that there is no any bounded solution of (8) and (26). For more details, see [8]. □

Summarizing the above results of Lemma 3 and Lemma 4, we can prove:

LEMMA 5. *Let* λ *and* μ *be two real numbers,* $0 < \lambda < \mu$ *and let* v_λ *and* v_μ *be two corresponding unique solutions of the equation* (8). *Then the following comparisons hold true:*

$$v_\lambda(x) \leqslant v_\mu(x) \text{ in } \Omega^\# \quad and \quad \| v_\lambda \|_{L^\infty(\Omega^\#)} \leqslant \| v_\mu \|_{L^\infty(\Omega^\#)}, \tag{37}$$

$$\| \nabla v_\lambda \|_{L^p(\Omega^\#)} \leqslant \| \nabla v_\mu \|_{L^p(\Omega^\#)} . \tag{38}$$

Proof. Since $C_0(\lambda)/\lambda < C_0(\mu)/\mu$ (*see* (27)) then according to (28) and (36) obviously follows (37). Also we have:

$$\| \nabla v_\lambda \|^p_{L^p(\Omega^\#)} = (NC_N^{1/N})^p \int_0^{|\Omega|} s^{p(1-1/N)} \mid \frac{dv^*_\lambda}{ds} \mid^p ds =$$

$$= (NC_N^{1/N})^p\left(\frac{C_0(\lambda)}{\lambda}\right)^p \int_0^{|\Omega|} s^{p'(-1+1/N)} \omega_\lambda^{p'}(s)ds \leqslant$$

$$\leqslant (NC_N^{1/N})^p\left(\frac{C_0(\mu)}{\mu}\right)^p \int_0^{|\Omega|} s^{p'(-1+1/N)} \omega_\mu^{p'}(s)ds =$$

$$= (NC_N^{1/N})^p \int_0^{|\Omega|} s^{p(1-1/N)} \mid \frac{dv^*_\mu}{ds} \mid^p ds = \| \nabla v_\mu \|^p_{L^p(\Omega^\#)} .$$

3. Proofs of the main results

In this part, we shall give the proofs of Theorem 1 and Theorem 2. As the first, the relation between the general ε - problem (1) and the suitable *symmetrized* problem (8) for $\lambda = \lambda_\varepsilon$, where λ_ε is from (7), is given by the following result:

LEMMA 6. *Let u_ε be a solution of the equation* (1) *and let* (2) – (6) *be satisfied. Then a priori estimates hold true:*

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leqslant \| v_\varepsilon \|_{L^\infty(\Omega^\#)} \quad and \quad \| \nabla u_\varepsilon \|_{L^p(\Omega)} \leqslant \| \nabla v_\varepsilon \|_{L^p(\Omega^\#)}, \ \forall \varepsilon > 0, \qquad (39)$$

where v_ε is the unique solution of the equation (8) *for $\lambda = \lambda_\varepsilon$ and λ_ε is from* (7).

Proof. This proof immediately follows, using Theorem 3 from [12]. □

THE PROOF OF THEOREM 1. Thanks to (9) and (37)-(38) for $\lambda = \lambda_\varepsilon$ ($v_\lambda = v_\varepsilon$) and $\mu = \beta$ ($v_\mu = v_\beta$), from (39) we have:

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leqslant \| v_\varepsilon \|_{L^\infty(\Omega^\#)} \leqslant \| v_\beta \|_{L^\infty(\Omega^\#)} = c_1(\beta), \ \forall \varepsilon > 0, \qquad (40)$$

$$\| \nabla u_\varepsilon \|_{L^p(\Omega)} \leqslant \| \nabla v_\varepsilon \|_{L^p(\Omega^\#)} \leqslant \| \nabla v_\beta \|_{L^p(\Omega^\#)} = c_2(\beta), \ \forall \varepsilon > 0, \qquad (41)$$

which prove the desired results. □

THE PROOF OF THEOREM 2. According to (12) we have particularly that there exists a constant $\beta > 0$ such that $\lambda_\varepsilon \leqslant \beta, \forall \varepsilon > 0$. With the help of (39), (36) - for $\lambda = \lambda_\varepsilon$ (where λ_ε from (7)) and (28) - for $\lambda = \lambda_\varepsilon$, ($\omega_\lambda = \omega_\varepsilon$) and $\mu = \beta$, since $C_0(\lambda_\varepsilon)/\lambda_\varepsilon \to 0$ as $\lambda_\varepsilon \to 0$ (see (27)), we obtain:

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leqslant \| v_\varepsilon \|_{L^\infty(\Omega^\#)} = v_\varepsilon^*(0+) = \frac{C_0(\lambda_\varepsilon)}{\lambda_\varepsilon} \int_{0+}^{|\Omega|} r^{p'(-1+1/N)} \omega_\varepsilon^{p'/p}(r)dr \leqslant$$

$$\leqslant \frac{C_0(\lambda_\varepsilon)}{\lambda_\varepsilon} \int_{0+}^{|\Omega|} r^{p'(-1+1/N)} \omega_\beta^{p'/p}(r)dr \ \to 0 \ \text{ as } \ \lambda_\varepsilon \to 0,$$

Also we deduce:

$$\| \nabla u_\varepsilon \|^p_{L^p(\Omega)} \leqslant \| \nabla v_\varepsilon \|^p_{L^p(\Omega^\#)} = (NC_N^{1/N})^p \int_{0+}^{|\Omega|} s^{p(1-1/N)} \left| \frac{dv_\varepsilon^*}{ds} \right|^p ds =$$

$$= (NC_N^{1/N})^p (\frac{C_0(\lambda_\varepsilon)}{\lambda_\varepsilon})^p \int_{0+}^{|\Omega|} s^{p'(-1+1/N)} \omega_\varepsilon^{p'}(s)ds \leqslant$$

$$\leqslant (NC_N^{1/N})^p (\frac{C_0(\lambda_\varepsilon)}{\lambda_\varepsilon})^p \int_{0+}^{|\Omega|} s^{p'(-1+1/N)} \omega_\beta^{p'}(s)ds \ \to 0 \text{ as } \lambda_\varepsilon \to 0. \quad \square$$

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