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ONE–DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS MICROPOLAR FLUID: A LOCAL EXISTENCE THEOREM

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Abstract. An initial-boundary value problem for one-dimensional flow of a compressible viscous heat-conducting micropolar fluid is considered. It is assumed that the fluid is thermodinamicaly perfect and politropic. A local-in-time existence and uniquenes theorem is proved.

1. Introduction

Theory of a polar or Cosserat continuum ([4], [1], [5], [6]) is based on the assumption that an appropriate dynamical field in a medium is a torzor (e.g. [7]), the reduction elements of which are momentum and intrinsic spin. As a consequence, instead of the symmetry of the stress tenzor, a new conservation law (for the momentum moment) appears. Kinematical and contact fields corresponding to the spin are, respectively, microrotation velocity and couple stress tenzor. We consider here an isotropic, viscous and compressible fluid, that is (in a thermodinamical sense) perfect and politropic. In the setting of the field equations we use the Eulerian description.

Notation:

ρ – mass density
v – velocity
$D(v)$ – stretching, $D(v) = sym\nabla v$
p – pressure
T – stress tenzor
T_{ax} – an axial vector with the Cartesian components $(T_{ax})_i = e_{ijk}T_{kj}/2$, where e_{ijk} is the alternating tenzor
ω – microrotation velocity
ω_{skw} – a skew tenzor with the Cartesian components $(\omega_{skw})_{ij} = e_{ijk}\omega_k$
j – microinertia density (a positive scalar field)
M – couple stress tenzor
θ – absolute temperature
e – internal energy density
q – heat flux density vector
f – body force density

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m – body couple density

r – body heat density

Local forms of the conservation laws for the mass, momentum, momentum moment and energy are, respectively, as follows:

$$\dot{\rho} + \rho \operatorname{div} v = 0, \tag{1.1}$$

$$\rho \dot{v} = \operatorname{div} T + \rho f, \tag{1.2}$$

$$\rho j \dot{\omega} = \operatorname{div} M + T_{ax} + \rho m, \tag{1.3}$$

$$\rho \dot{e} = T \cdot \nabla v + M \cdot \nabla \omega - 2T_{ax} \cdot \omega + \operatorname{div} q + \rho r, \tag{1.4}$$

where $\dot{a}$ denotes material derivative of a field a:

$$\dot{a} = \frac{\partial a}{\partial t} + (\nabla a)v.$$

The linear constitutive equations for stress tenzor, couple stress tenzor and heat flux density vector are, respectively, of the forms:

$$T = -pI + \lambda(\operatorname{div} v)I + 2\mu D(v) + \chi(\nabla v + \omega_{skw}), \tag{1.5}$$

$$M = \alpha(\operatorname{div} \omega)I + \beta(\nabla \omega)^T + \gamma \nabla(\omega), \tag{1.6}$$

$$q = k\nabla\theta, \tag{1.7}$$

where λ, μ, χ, α, β, γ and k are scalar material coefficients, depending generaly on mass density and temperature and satisfying the conditions ([5], [6]):

$$3\lambda + 2\mu + \chi \geqslant 0, \qquad 2\mu + \chi \geqslant 0, \qquad \chi \geqslant 0, \tag{1.8}$$

$$3\alpha + \beta + \gamma \geqslant 0, \qquad |\beta| \leqslant 0, \qquad k \geqslant 0. \tag{1.9}$$

Assuming that the fluid is perfect and politropic, for pressure and internal energy we have the equations:

$$p = R\rho\theta, \tag{1.10}$$

$$e = c\theta, \tag{1.11}$$

where R and c are positive constants.

Initial-boundary value problems for the system (1.1)–(1.7), (1.10)–(1.11) so far were not considered (for incompressible flow see [10], [17], [18], [19], [21], [22], [23]).

It is well known that even for a classical fluid (when the coefficients j, χ, α, β and γ are equal zero) a few results are obtained for three-and-two-dimensional problems (see [2], [14] and [8] and references therein); a global existence theorems are proved for isentropic case ([15], [16]) and for one-dimensional flow ([11], [12], [2]; see also [9]).

2. Statement of the problem and the main results

In this paper we consider the system (1.1)–(1.7), (1.10)–(1.11) for one-dimensional flow, assuming that all material coeficients (including j) are constants.

Let (in a Cartesian coordinate frame) $v_2 = v_3 = \omega_2 = \omega_3 = 0$ and let the functions $\rho, v = v_1, \omega = \omega_1$ and θ depend on $x = x_1$ and t only. Inserting (1.5)–(1.7), (1.10)–(1.11) into (1.2)–(1.4) and taking $f = m = r = 0$, we obtain the system:

$$\dot{\rho} + \rho\frac{\partial v}{\partial x} = 0, \tag{2.1}$$

$$\rho\dot{v} = -\frac{\partial}{\partial x}(R\rho\theta) + \sigma_1\frac{\partial^2 v}{\partial x^2}, \tag{2.2}$$

$$j\rho\dot{\omega} = \sigma_2\frac{\partial^2\omega}{\partial x^2} - 2\chi\omega, \tag{2.3}$$

$$c\rho\dot{\theta} = -R\rho\theta\frac{\partial v}{\partial x} + \sigma_1\left(\frac{\partial v}{\partial x}\right)^2 + \sigma_2\left(\frac{\partial\omega}{\partial x}\right)^2 + 2\chi\omega^2 + k\frac{\partial^2\omega}{\partial x^2}, \tag{2.4}$$

where

$$\sigma_1 = \lambda + 2\mu + \chi, \qquad \sigma_2 = \alpha + \beta + \gamma.$$

Because of (1.8) and (1.9) it holds $\sigma_1 \geqslant 0$, $\sigma_2 \geqslant 0$; we assume

$$\sigma_1, \sigma_2, \chi, k \in R_+ =]0, +\infty[. \tag{2.5}$$

We shall consider the system (2.1)–(2.4) in the domain $]0, L[\times R_+$, $L \in R_+$, under the homogeneous boundary conditions:

$$v(0,t) = v(L,t) = 0, \tag{2.6}$$

$$\omega(0,t) = \omega(L,t) = 0, \tag{2.7}$$

$$\frac{\partial\theta}{\partial x}(0,t) = \frac{\partial\theta}{\partial x}(L,t) = 0 \tag{2.8}$$

for $t \succ 0$ and non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x), \tag{2.9}$$

$$v(x,0) = v_0(x), \tag{2.10}$$

$$\omega(x,0) = \omega_0(x), \tag{2.11}$$

$$\theta(x,0) = \theta_0(x) \tag{2.12}$$

for $x \in]0, L[$. Here ρ_0, v_0, ω_0 and θ_0 are given functions. We assume that the functions ρ_0 and θ_0 are strictly positive and bounded:

$$m \leqslant \rho_0 \leqslant M, \; m \leqslant \theta_0(x) \leqslant M \quad \text{for } x \in]0, L[, \tag{2.13}$$

where $m, M \in \mathbf{R}_+$.

It is convenient to transform our problem to the Lagrangian form. For $\xi \in]0, L[$ let $t \to \varphi_t(\xi)$ be a solution of the Cauchy problem

$$\frac{d\varphi_t}{dt} = v(\varphi_t, t), \quad \varphi_0(\xi) = \xi.$$

Because of (2.6) the mapping $\xi \to x = \varphi_t(\xi)$ is a diffeomorphism $]0, L[\to]0, L[$. To an Eulerian field $f(x,t)$ on $]0, L[\times\mathbf{R}_+$ it corresponds a Lagrangian field $\tilde{f}(\xi,t) = f(\varphi_t(\xi), t)$ on the same domain. Taking into account the equality

$$\dot{f} = \frac{\partial\tilde{f}}{\partial t} \circ \varphi_t^{-1},$$

one can easily obtain the system of equations for the functions $\widetilde{\rho}$, $\widetilde{v}$, $\widetilde{\omega}$ and $\widetilde{\theta}$. Let

$$\psi(\xi)=\int_0^\xi \rho_0(\xi)\,d\xi,\ \eta=\psi(L),\ \delta=\eta\sigma_1^{-1}(2\chi)^{-\frac{1}{2}}\sigma_2^{\frac{1}{2}},$$
$$\zeta_1=\eta^{-1}(2\chi)^{-\frac{1}{2}}\sigma_2^{\frac{1}{2}},$$
$$\zeta_2=\eta\sigma_1^{-1},$$
$$\zeta_3=\eta\sigma_1^{-\frac{3}{2}}\sigma_2^{\frac{1}{2}},$$
$$\zeta_4=c\eta^2\sigma_1^{-2}.$$

It is useful to introduce the new coordinates

$$x'=\eta^{-1}\psi(\xi),\ t'=\delta^{-1}t$$

and the new functions

$$\rho'(x',t')=\zeta_1\widetilde{\rho}(\psi^{-1}(\eta x'),\delta t'),$$
$$v'(x',t')=\zeta_2\widetilde{v}(\psi^{-1}(\eta x'),\delta t'),$$
$$\omega'(x',t')=\zeta_3\widetilde{\omega}(\psi^{-1}(\eta x'),\delta t'),$$
$$\theta'(x',t')=\zeta_4\widetilde{\theta}(\psi^{-1}(\eta x'),\delta t').$$

Let

$$K=Rc^{-1},\ A=j^{-1}\sigma_1^{-1}\sigma_2,\ D=kc^{-1}\sigma_1^{-1},$$
$$\rho_0'(x')=\zeta_1\rho_0(\psi^{-1}(\eta x')),$$
$$v_0'(x')=\zeta_2 v_0(\psi^{-1}(\eta x')),$$
$$\omega_0'(x')=\zeta_3\omega_0(\psi^{-1}(\eta x')),$$
$$\cdot\theta_0'(x')=\zeta_4\theta_0(\psi^{-1}(\eta x')).$$

Then the functions ρ', v', ω' and θ' satisfy the system that we write omiting for simplicity the primes:

$$\frac{\partial\rho}{\partial t}+\rho^2\frac{\partial v}{\partial x}=0, \tag{2.14}$$
$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\Big(\rho\frac{\partial v}{\partial x}\Big)-K\frac{\partial}{\partial x}(\rho\theta), \tag{2.15}$$
$$\rho\frac{\partial\omega}{\partial t}=A\Big[\rho\frac{\partial}{\partial x}\Big(\rho\frac{\partial\omega}{\partial x}\Big)-\omega\Big], \tag{2.16}$$
$$\rho\frac{\partial\theta}{\partial t}=-K\rho^2\theta\frac{\partial v}{\partial x}+\rho^2\Big(\frac{\partial v}{\partial x}\Big)^2+\rho^2\Big(\frac{\partial\omega}{\partial x}\Big)^2+\omega^2+D\rho\frac{\partial}{\partial x}\Big(\rho\frac{\partial\theta}{\partial x}\Big) \tag{2.17}$$

in $]0,1[\times\mathbf{R}^+$,

$$v(0,t)=v(1,t)=0, \tag{2.18}$$
$$\omega(0,t)=\omega(1,t)=0, \tag{2.19}$$
$$\frac{\partial\theta}{\partial x}(0,t)=\frac{\partial\theta}{\partial x}(1,t)=0 \tag{2.20}$$

for $t \in \mathbf{R}_+$,

$$\rho(x,0) = \rho_0(x), \tag{2.21}$$
$$v(x,0) = v_0(x), \tag{2.22,)}$$
$$\omega(x,0) = \omega_0(x), \tag{2.23,)}$$
$$\theta(x,0) = \theta_0(x), \tag{2.24}$$

for $x \in]0,1[$. The functions ρ_0 and θ_0 satisfy the conditions

$$m \leqslant \rho_0 \leqslant M, \quad m \leqslant \theta_0(x) \leqslant M \quad \text{for } x \in]0,1[, \tag{2.25}$$

where $m, M \in \mathbf{R}_+$. The problem (2.14)–(2.24) is equivalent to the problem (2.1)–(2.4), (2.6)–(2.12).

Definition 2.1. Let $T \in \mathbf{R}_+$; a generalised solution of the problem (2.14)–(2.24) in the domain $Q_T =]0,1[\times]0,T[$ is a function

$$(x,t) \to (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{2.26}$$

where

$$\rho \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T), \tag{2.27}$$
$$v, \omega, \theta \in L^\infty(0,T;H^1(]0,1[)) \cap H^1(Q_T) \cap L^2(0,T;H^2(]0,1[)), \tag{2.28}$$

that satisfies the equations (2.14)–(2.17) a.e. in Q_T, the conditions (2.18)–(2.24) in the sense of traces and the condition

$$\inf_{Q_T} \rho \succ 0. \tag{2.29}$$

Remark 2.1. From embedding and interpolation theorems ([13]) one can conclude that from (2.27) and (2.28) it follows:

$$\rho \in C([0,T], L^2(]0,1[)) \cap L^\infty(0,T;C([0,1])), \tag{2.30}$$
$$v, \omega, \theta \in L^2(0,T;C^{(1)}([0,1])) \cap C([0,T], H^1(]0,1[)), \tag{2.31}$$
$$v, \omega, \theta \in C(\overline{Q}_T). \tag{2.32}$$

Specially, the condition (2.29) has a sense.

The purpose of this paper is to prove the following results.

THEOREM 2.1. *For each $T \in \mathbf{R}_+$ the problem (2.14)–(2.24) has at most one generalised solution in Q_T.*

THEOREM 2.2. *Let the functions ρ_0, $\theta_0 \in H^1(]0,1[)$ satisfy the conditions (2.25) and let v_0, $\omega_0 \in H_0^1(]0,1[)$. Then there exists $T_0 \in \mathbf{R}_+$ such that the problem (2.14)–(2.24) has a generalised solution in $Q_0 = Q_{T_0}$, having the property*

$$\theta \succ 0 \quad in\ \overline{Q}_0. \tag{2.33}$$

The analogous theorems for the classical fluid were proved in [24], [25] and [2]. In our proof we use the Faedo–Galerkin method and follow ideas of the book [2].

3. The proof of Theorem 2.1.

Let $(\rho_i, v_i, \omega_i, \theta_i)$, $i = 1, 2$ be generalised solutions of the problem (2.14)–(2.24). Then the function $(\rho, v, \omega, \theta) = (\rho_1, v_1, \omega_1, \theta_1) - (\rho_2, v_2, \omega_2, \theta_2)$ satisfies the system:

$$\frac{\partial \rho}{\partial t} + \rho_1^2 \frac{\partial v}{\partial x} + \rho(\rho_1 + \rho_2)\frac{\partial v_2}{\partial x} = 0, \tag{3.1}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\Big(\rho_1 \frac{\partial v}{\partial x} + \rho \frac{\partial v_2}{\partial x}\Big) - K\frac{\partial}{\partial x}(\rho_1 \theta + \rho \theta_2), \tag{3.2}$$

$$\frac{\partial \omega}{\partial t} = A\Big[\frac{\partial}{\partial x}\Big(\rho_1 \frac{\partial \omega}{\partial x} + \rho \frac{\partial \omega_2}{\partial x}\Big) - \frac{\omega}{\rho_1} + \omega_2 \frac{\rho}{\rho_1 \rho_2}\Big], \tag{3.3}$$

$$\begin{aligned}\frac{\partial \theta}{\partial t} &= D\frac{\partial}{\partial x}\Big(\rho_1 \frac{\partial \theta}{\partial x} + \rho \frac{\partial \theta_2}{\partial x}\Big) - K\Big(\rho_1 \theta \frac{\partial v_1}{\partial x} + \theta_2 \rho \frac{\partial v_1}{\partial x} + \rho_2 \theta_2 \frac{\partial v}{\partial x}\Big)\\ &\quad + \rho_1 \frac{\partial v}{\partial x}\Big(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial x}\Big) + \rho\Big(\frac{\partial v_2}{\partial x}\Big)^2 + \frac{\omega}{\rho_1}(\omega_1 + \omega_2) - \omega_2^2 \frac{\rho}{\rho_1 \rho_2}\\ &\quad + \rho_1 \frac{\partial \omega}{\partial x}\Big(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial x}\Big) + \rho\Big(\frac{\partial \omega_2}{\partial x}\Big)^2,\end{aligned} \tag{3.4}$$

$$v(0,t) = v(1,t) = 0, \tag{3.5}$$

$$\omega(0,t) = \omega(1,t) = 0, \tag{3.6}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0, \tag{3.7}$$

$$\rho(x,0) = v(x,0) = \omega(x,0) = \theta(x,0) = 0. \tag{3.8}$$

In that what follows we denote by $C \succ 0$ a generic constant, not depending on $(\rho, v, \omega, \theta)$ and having possibly different values at different places. We also use the notation

$$\|f\| = \|f\|_{L^2(]0,1[)}.$$

Taking into account properties (2.30)–(2.32), from (3.1) and (3.8) we obtain

$$\|\rho(t)\|^2 \leqslant C\int_0^t \Big[\Big(1 + \max_{x\in[0,1]}\Big|\frac{\partial v_2}{\partial x}\Big|^2(\tau)\Big)\|\rho(\tau)\|^2 + \Big\|\frac{\partial v}{\partial x}(\tau)\|^2\Big] d\tau$$

or, because of the Gronwall's inequality,

$$\|\rho(t)\|^2 \leqslant C\int_0^t \Big\|\frac{\partial v}{\partial x}(\tau)\Big\|^2 d\tau. \tag{3.9}$$

From (3.2), (3.5) and (3.8) we get

$$\|v(t)\|^2 + \int_0^t \Big\|\frac{\partial v}{\partial x}(\tau)\Big\|^2 d\tau \leqslant C\int_0^t \Big[\Big(1 + \max_{x\in[0,1]}\Big|\frac{\partial v_2}{\partial x}\Big|(\tau)\Big)\|\rho(\tau)\|\,\Big\|\frac{\partial v}{\partial x}(\tau)\Big\| + \|\theta(\tau)\|\,\Big\|\frac{\partial v}{\partial x}(\tau)\Big\|\Big] d\tau$$

or applaying the Young's inequality and (3.9),

$$\|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau \leqslant C \int_0^t \left[\left(1 + \max_{x\in[0,1]}\left|\frac{\partial v_2}{\partial x}\right|(\tau)\right)^2 \left(\|v(\tau)\|^2 + \int_0^\tau \left\|\frac{\partial v}{\partial x}(\lambda)\right\|^2 d\lambda\right) + \|\theta(\tau)\|^2\right] d\tau.$$

Using now the Gronwall's inequality, we obtain

$$\|v(t)\|^2 + \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau \leqslant C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{3.10}$$

Analogously, from (3.3), (3.4), (3.6)–(3.10) there follow the inequalities

$$\|\omega(t)\|^2 + \int_0^t \left\|\frac{\partial \omega}{\partial x}(\tau)\right\|^2 d\tau \leqslant C \int_0^t \left\|\frac{\partial v}{\partial x}(\tau)\right\|^2 d\tau, \tag{3.11}$$

$$\|\theta(t)\|^2 + \int_0^t \left\|\frac{\partial \theta}{\partial x}(\tau)\right\|^2 d\tau \leqslant C \int_0^t \|\theta(\tau)\|^2 d\tau. \tag{3.12}$$

From (3.9)–(3.12) we conclude that $\rho = v = \omega = \theta = 0$.

4. Approximate solutions

A local generalised solution to the preoblem (2.14)–(2.24) we shall find as a limit of approximate solutions

$$(\rho^n, v^n, \omega^n, \theta^n), \quad n \in \mathbf{N}, \tag{4.1}$$

where

$$v^n(x,t) = \sum_{i=1}^n v_i^n(t) \sin(\pi i x), \tag{4.2}$$

$$\omega^n(x,t) = \sum_{j=1}^n \omega_j^n(t) \sin(\pi j x), \tag{4.3}$$

$$\theta^n(x,t) = \sum_{k=0}^n \theta_k^n(t) \cos(\pi k x); \tag{4.4}$$

here $v_i^n, \omega_j^n, \theta_k^n$ are unknown functions, defined and smooth on an interval $[0, T_n]$, $T_n \in \mathbf{R}_+$. Evidently, the boundary conditions

$$v^n(0,t) = v^n(1,t) = \omega^n(0,t) = \omega^n(1,t) = \frac{\partial \theta^n}{\partial x}(0,t) = \frac{\partial \theta^n}{\partial x}(1,t) = 0 \tag{4.5}$$

are satisfied. According to Feado–Galerkin method, we take the following approximation conditions:

$$\frac{\partial \rho^n}{\partial t} + (\rho^n)^2 \frac{\partial v^n}{\partial x} = 0, \quad \rho^n(x,0) = \rho_0(x), \tag{4.6}$$

$$\int_0^1 \left[\frac{\partial v^n}{\partial t} - \frac{\partial}{\partial x}\left(\rho^n \frac{\partial v^n}{\partial x}\right) + K\frac{\partial}{\partial x}(\rho^n \theta^n)\right] \sin(\pi i x)\, dx = 0, \quad i = 1,2,\dots,n, \tag{4.7}$$

$$\int_0^1 \left[\frac{\partial \omega^n}{\partial t} - A\frac{\partial}{\partial x}\left(\rho^n \frac{\partial \omega^n}{\partial x}\right) + A\frac{\omega^n}{\rho^n}\right] \sin(\pi j x)\, dx = 0, \quad j = 1,2,\dots,n, \tag{4.8}$$

$$\int_0^1 \left[\frac{\partial \theta^n}{\partial t} + K\rho^n\theta^n \frac{\partial v^n}{\partial x} - \rho^n\left(\frac{\partial v^n}{\partial x}\right)^2 - \rho^n\left(\frac{\partial \omega^n}{\partial x}\right)^2 - \frac{(\omega^n)^2}{\rho^n} - D\frac{\partial}{\partial x}\left(\rho^n \frac{\partial \theta^n}{\partial x}\right)\right] \cos(\pi k x)\, dx = 0, \quad k = 0,1,2,\dots,n. \tag{4.9}$$

From (4.6) and (4.2) it follows

$$\rho^n(x,t) = \rho_0(x)\Big(1 + \rho_0(x)\int_0^t \frac{\partial v^n}{\partial x}(x,\tau)\, d\tau\Big)^{-1} = \rho_0(x)\Big(1 + \rho_0(x)\sum_{i=1}^n (i\pi)\cos(\pi i x)\int_0^t v_i^n(\tau)\, d\tau\Big)^{-1}, \tag{4.10}$$

and because of (2.25), for sufficiently small T_n we have

$$\rho^n(x,t) \succ 0, \quad (x,t) \in [0,1] \times [0,T_n]. \tag{4.11}$$

Therefore the conditions (4.8) and (4.9) have a sense. Let v_{0i}, ω_{0j} $(i,j = 1,2,\dots)$ and θ_{0k} $(k = 0,1,2,\dots)$ be the Fourier coefficients of the functions v_0, ω_0 and θ_0, respectively:

$$v_{0i} = 2\int_0^1 v_0(x)\sin(\pi i x)\, dx, \quad i = 1,2,\dots,$$

$$\omega_{0j} = 2\int_0^1 \omega_0(x)\sin(\pi j x)\, dx, \quad j = 1,2,\dots,$$

$$\theta_{00} = \int_0^1 \theta_0(x)\, dx, \quad \theta_{0k} = 2\int_0^1 \theta_0(x)\cos(\pi k x)\, dx, \quad k = 1,2,\dots;$$

let

$$v_0^n(x) = \sum_{i=1}^{n} v_{0i} \sin(\pi i x), \tag{4.12}$$

$$\omega_0^n(x) = \sum_{j=1}^{n} \omega_{0j} \sin(\pi j x), \tag{4.13}$$

$$\theta_0^n(x) = \sum_{k=0}^{n} \theta_{0k} \cos(\pi k x). \tag{4.14}$$

The initial conditions for v^n, ω^n and θ^n we take in the form:

$$v^n(x,0) = v_0^n(x), \tag{4.15}$$

$$\omega^n(x,0) = \omega_0^n(x), \tag{4.16}$$

$$\theta^n(x,0) = \theta_0^n(x). \tag{4.17}$$

Let

$$z_r^n(t) = \int_0^t v_r^n(\tau)\, d\tau, \quad r = 1,2,\ldots,n. \tag{4.18}$$

Taking into account (4.2)–(4.4), (4.10) and (4.18), from (4.7)–(4.9) we obtain for $\left\{ \left(v_i^n, \omega_j^n, \theta_k^n, z_r^n\right) : i,j,r = 1,2,\ldots,n,\ k = 0,1,2,\ldots,n \right\}$ a Cauchy problem:

$$\dot{v}_i^n = \phi_i^n\left(v_1^n,\ldots,v_n^n,\omega_1^n,\ldots,\omega_n^n,\theta_0^n,\theta_1^n,\ldots,\theta_n^n,z_1^n,\ldots,z_n^n\right), \tag{4.19}$$

$$\dot{\omega}_j^n = \psi_j^n\left(v_1^n,\ldots,v_n^n,\omega_1^n,\ldots,\omega_n^n,\theta_0^n,\theta_1^n,\ldots,\theta_n^n,z_1^n,\ldots,z_n^n\right), \tag{4.20}$$

$$\dot{\theta}_k^n = \lambda_k \Pi_k^n\left(v_1^n,\ldots,v_n^n,\omega_1^n,\ldots,\omega_n^n,\theta_0^n,\theta_1^n,\ldots,\theta_n^n,z_1^n,\ldots,z_n^n\right), \tag{4.21}$$

$$\dot{z}_r^n = v_r^n, \tag{4.22}$$

$$v_i^n(0) = v_{0i}, \tag{4.23}$$

$$\omega_j^n(0) = \omega_{0j}, \tag{4.24}$$

$$\theta_k^n(0) = \theta_{0k}, \tag{4.25}$$

$$z_r^n(0) = 0, \tag{4.26}$$

where $\lambda_0 = 1$, $\lambda_k = 2$ for $k = 1,2,\ldots,n$, and

$$\phi_i^n = 2\int_0^1 \left[\frac{\partial}{\partial x}\left(\rho^n \frac{\partial v^n}{\partial x}\right) - K\frac{\partial}{\partial x}(\rho^n\theta^n)\right]\sin(\pi i x)\,dx. \tag{4.27}$$

$$\psi_j^n = 2\int_0^1 A\left[\frac{\partial}{\partial x}\left(\rho^n \frac{\partial \omega^n}{\partial x}\right) - \frac{\omega^n}{\rho^n}\right]\sin(\pi j x)\,dx, \tag{4.28}$$

$$\Pi_k^n = \int_0^1 \left[-K\rho^n\theta^n\frac{\partial v^n}{\partial x} + \rho^n\left(\frac{\partial v^n}{\partial x}\right)^2 + D\frac{\partial}{\partial x}\left(\rho^n\frac{\partial \theta^n}{\partial x}\right) + \frac{(\omega^n)^2}{\rho^n}\right.$$
$$\left. + \rho^n\left(\frac{\partial \omega^n}{\partial x}\right)^2\right]\cos(\pi k x)\,dx. \tag{4.29}$$

With the help of the Cauchy–Picard theorem (e.g. [20]) one can easily conclude that the following statements are valid.

LEMMA 4.1. *For each $n \in \mathbf{N}$ there exists $T_n \in \mathbf{R}_+$ such that the Cauchy problem (4.19)–(4.26) has a unique solution, defined on $[0, T_n]$; the functions v^n, ω^n and θ^n, defined by the formulas (4.2)–(4.4), belong to the class $C^\infty(\overline{Q}_n)$, $Q_n =]0,1[\times]0,T_n[$ and satisfy the conditions (4.15)–(4.17).*

LEMMA 4.2. *There exists $T_n \in \mathbf{R}_+$ such that function ρ^n, defined by (4.10), satisfies the condition*

$$\frac{m}{2} < \rho^n(x,t) < 2M \quad \text{in } \overline{Q}_n. \tag{4.30}$$

5. A priori estimates

Our purpose is to find out $T_0 \in \mathbf{R}_+$ such that for each $n \in \mathbf{N}$ there exists a solution of the problem (4.19)–(4.26), defined on $[0, T_0]$. It will be sufficient to find out uniform (in $n \in \mathbf{N}$) a priori estimates for a function $(\rho^n, v^n, \omega^n, \theta^n)$, defined through Lemmas 4.1. and 4.2. In that what follows, $C > 0$ denotes a generic constant, not depending on $n \in \mathbf{N}$.

LEMMA 5.1. *For $t \in [0, T_n]$ it holds the inequality*

$$\|\omega^n(t)\|^2 + \int_0^t \left(\left\|\frac{\partial \omega^n}{\partial x}(\tau)\right\|^2 + \|\omega^n(\tau)\|^2\right)d\tau \leqslant C. \tag{5.1}$$

Proof. Multiplying (4.8) by ω_j^n and summing over $j = 1, 2, \ldots, n$, after integration by parts we obtain

$$\frac{1}{2A}\frac{d}{dt}\|\omega^n(t)\|^2 + \int_0^1 \left[\rho^n(x,t)\left(\frac{\partial \omega^n}{\partial x}(x,t)\right)^2 + \frac{1}{\rho^n(x,t)}\left(\omega^n(x,t)\right)^2\right]dx = 0.$$

Integrating over $[0,t]$, $0 \prec t \leqslant T_n$, and taking into account (4.16), we have

$$\frac{1}{2A}\|\omega^n(t)\|^2 + \int_0^t\int_0^1 \left[\rho^n(x,t)\Big(\frac{\partial\omega^n}{\partial x}(x,t)\Big)^2 + \frac{1}{\rho^n(x,t)}\big(\omega^n(x,t)\big)^2\right] dx\,dt$$
$$= \frac{1}{2A}\|\omega_0^n\|^2 \leqslant \frac{1}{2A}\|\omega_0\|^2,$$

and using (4.30) we get (5.1). □

LEMMA 5.2. *For $t \in [0, T_n]$ it holds the inequality*

$$\Big|\int_0^1 \theta^n(x,t)\,dx\Big| \leqslant C\Big(1+\Big\|\frac{\partial v^n}{\partial x}(t)\Big\|^2\Big). \tag{5.2}$$

Proof. Multiplying (4.7) by v_i^n and summing over $i = 1, 2, \dots, n$, after integration by parts and using (4.9) for $k = 0$, we have

$$\frac{d}{dt}\Big(\frac{1}{2}\|v^n(t)\|^2 + \int_0^1 \theta^n(x,t)\,dx\Big) = \int_0^1 \frac{1}{\rho^n(x,t)}\big(\omega^n(x,t)\big)^2\,dx$$
$$+ \int_0^1 \rho^n(x,t)\Big(\frac{\partial\omega^n}{\partial x}(x,t)\Big)^2\,dx.$$

Taking into account (4.15), (4.17), (4.30), (5.1) and the inequality

$$\|v^n\| \leqslant 2^{-\frac{1}{2}}\Big\|\frac{\partial v^n}{\partial x}\Big\|,$$

we obtain (5.2). □

LEMMA 5.3. *For $(x,t) \in \overline{Q}_n$ it holds the inequality*

$$|\theta^n(x,t)| \leqslant C\Big(1+\Big\|\frac{\partial\theta^n}{\partial x}(t)\Big\| + \Big\|\frac{\partial v^n}{\partial x}(t)\Big\|^2\Big). \tag{5.3}$$

Proof. Let $t \in [0, T_n]$ and $x_1(t), x_2(t) \in [0,1]$, such that

$$m_n(t) = \min_{x\in[0,1]} \theta^n(x,t) = \theta^n(x_1(t),t),$$
$$M_n(t) = \max_{x\in[0,1]} \theta^n(x,t) = \theta^n(x_2(t),t).$$

For $x \in [0,1]$ it holds

$$\theta^n(x,t) - m_n(t) = \int_{x_1(t)}^x \frac{\partial\theta^n}{\partial x}(x,t)\,dx \leqslant \Big\|\frac{\partial\theta^n}{\partial x}(t)\Big\|,$$

and hence

$$\theta^n(x,t) \leqslant \left\|\frac{\partial\theta^n}{\partial x}(t)\right\| + m_n(t) \leqslant \left\|\frac{\partial\theta^n}{\partial x}(t)\right\| + \left|\int_0^1 \theta^n(x,t)\,dx\right|.$$

Analogously we have

$$\theta^n(x,t) - M_n(t) = \int_{x_2(t)}^{x} \frac{\partial\theta^n}{\partial x}(x,t)\,dx \geqslant -\left\|\frac{\partial\theta^n}{\partial x}(t)\right\|,$$

and

$$\theta^n(x,t) \geqslant -\left\|\frac{\partial\theta^n}{\partial x}(t)\right\| + M_n(t) \geqslant -\left\|\frac{\partial\theta^n}{\partial x}(t)\right\| - \left|\int_o^1 \theta^n(x,t)\,dx\right|.$$

So, it holds

$$\left|\theta^n(x,t)\right| \leqslant \left\|\frac{\partial\theta^n}{\partial x}(t)\right\| + \left|\int_0^1 \theta^n(x,t)\,dx\right|;$$

using (5.2) we get (5.3). □

LEMMA 5.4. *For $t \in [0, T_n]$ it holds the inequality*

$$\left\|\frac{\partial\rho^n}{\partial x}(t)\right\| \leqslant C\left(1 + \int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right). \tag{5.4}$$

Proof. The conclusion follows immediately from (4.10). □

LEMMA 5.5. *For $t \in [0, T_n]$ it holds*

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial\omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^2\right)\\ &\qquad + \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2\omega^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2\theta^n}{\partial x^2}(t)\right\|^2\\ &\leqslant C\left(1 + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^8 + \left\|\frac{\partial\omega^n}{\partial x}(t)\right\|^8 + \left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^8 + \left(\int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right).\end{aligned} \tag{5.5}$$

Proof. Multiplying (4.7), (4.8) and (4.9) respectively by $(\pi i)^2 v_i^n$, $(\pi j)^2\omega_j^n$ and $(\pi k)^2\theta_k^n$ and taking into account (4.2)–(4.4), after summation over $i,j,k = 1,2,\ldots,n$ and addition of the obtained equalities, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial\omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial\theta^n}{\partial x}(t)\right\|^2\right) + \int_0^1 \rho^n(x,t)\left[\left(\frac{\partial^2 v^n}{\partial x^2}(x,t)\right)^2\right.\\ &\qquad \left. + A\left(\frac{\partial^2\omega^n}{\partial x^2}(x,t)\right)^2 + D\left(\frac{\partial^2\theta^n}{\partial x^2}(x,t)\right)^2\right] dx = \sum_{r=1}^{10} I_r(t),\end{aligned} \tag{5.6}$$

where

$$I_1(t) = -\int_0^1 \frac{\partial \rho^n}{\partial x}\frac{\partial v^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}\,dx, \quad I_2(t) = K\int_0^1 \frac{\partial \rho^n}{\partial x}\theta^n\frac{\partial^2 v^n}{\partial x^2}\,dx,$$

$$I_3(t) = K\int_0^1 \rho^n\frac{\partial \theta^n}{\partial x}\frac{\partial^2 v^n}{\partial x^2}\,dx, \quad I_4(t) = A\int_0^1 \frac{1}{\rho^n}\omega^n\frac{\partial^2 \omega^n}{\partial x^2}\,dx,$$

$$I_5(t) = -A\int_0^1 \frac{\partial \rho^n}{\partial x}\frac{\partial \omega^n}{\partial x}\frac{\partial^2 \omega^n}{\partial x^2}\,dx, \quad I_6(t) = K\int_0^1 \rho^n\theta^n\frac{\partial v^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}\,dx,$$

$$I_7(t) = -\int_0^1 \rho^n\Big(\frac{\partial v^n}{\partial x}\Big)^2\frac{\partial^2 \theta^n}{\partial x^2}\,dx, \quad I_8(t) = -D\int_0^1 \frac{\partial \rho^n}{\partial x}\frac{\partial \theta^n}{\partial x}\frac{\partial^2 \theta^n}{\partial x^2}\,dx,$$

$$I_9(t) = -\int_0^1 \frac{1}{\rho^n}(\omega^n)^2\frac{\partial^2 \theta^n}{\partial x^2}\,dx, \quad I_{10}(t) = -\int_0^1 \rho^n\Big(\frac{\partial \omega^n}{\partial x}\Big)^2\frac{\partial^2 \theta^n}{\partial x^2}\,dx.$$

Taking into account (5.1)–(5.4) and the inequalities

$$|f|^2 \leqslant 2\|f\|\,\Big\|\frac{\partial f}{\partial x}\Big\|, \quad \Big|\frac{\partial f}{\partial x}\Big|^2 \leqslant 2\Big\|\frac{\partial f}{\partial x}\Big\|\,\Big\|\frac{\partial^2 f}{\partial x^2}\Big\|, \tag{5.7}$$

$$\|f\| \leqslant 2^{-\frac{1}{2}}\Big\|\frac{\partial f}{\partial x}\Big\|, \quad \Big\|\frac{\partial f}{\partial x}\Big\| \leqslant 2^{-\frac{1}{2}}\Big\|\frac{\partial^2 f}{\partial x^2}\Big\| \tag{5.8}$$

(for a function f vanishing at $x = 0$ and $x = 1$ or with the first derivative vanishing at the same points), one can estimate the functions $I_1(t) - I_{10}(t)$. For instance,

$$\begin{aligned} I_1(t) &\leqslant \max_{x\in[0,1]}\Big|\frac{\partial v^n}{\partial x}(x,t)\Big|\,\Big\|\frac{\partial \rho^n}{\partial x}(t)\Big\|\,\Big\|\frac{\partial^2 v^n}{\partial x^2}(t)\Big\| \\ &\leqslant 2^{\frac{1}{2}}\Big\|\frac{\partial v^n}{\partial x}(t)\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2 v^n}{\partial x^2}(t)\Big\|^{\frac{3}{2}}\Big\|\frac{\partial \rho^n}{\partial x}(t)\Big\| \\ &\leqslant C\Big\|\frac{\partial v^n}{\partial x}(t)\Big\|^{\frac{1}{2}}\Big\|\frac{\partial^2 v^n}{\partial x^2}(t)\Big\|^{\frac{3}{2}}\Big(1 + \Big(\int_0^t\Big\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\Big\|^2 d\tau\Big)^{\frac{1}{2}}\Big); \end{aligned}$$

applaying the Young inequality, we get

$$I_1(t) \leqslant \varepsilon\Big\|\frac{\partial^2 v^n}{\partial x^2}(t)\Big\|^2 + C\Big[1 + \Big\|\frac{\partial v^n}{\partial x}(t)\Big\|^4 + \Big(\int_0^t\Big\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\Big\|^2 d\tau\Big)^4\Big],$$

where $\varepsilon \succ 0$ is arbitrary. In an analogous way one obtains the inequalities:

$$I_2(t) \leqslant \varepsilon\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + C\left[1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^4 + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^8 + \left(\int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right],$$

$$I_3(t) \leqslant \varepsilon\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + C\left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2,$$

$$I_4(t) \leqslant \varepsilon\left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|^2 + C,$$

$$I_5(t) \leqslant \varepsilon\left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|^2 + C\left[1 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^4 + \left(\int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 d\tau\right)^4\right],$$

$$I_6(t) \leqslant \varepsilon\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^4 + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^8\right),$$

$$I_7(t) \leqslant \varepsilon\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + \varepsilon\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^8\right),$$

$$I_8(t) \leqslant \varepsilon\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left[1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^4 + \left(\int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^4\right],$$

$$I_9(t) \leqslant \varepsilon\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^2\right),$$

$$I_{10}(t) \leqslant \varepsilon\left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 + \varepsilon\left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|^2 + C\left(1 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^8\right).$$

Inequality (5.5) follows from (5.6) and (4.30). □

LEMMA 5.6. *There exists $T_0 \in \mathbf{R}_+$, such that for each $n \in \mathbf{N}$ the Cauchy problem (4.19)–(4.26) has a unique solution, defined on $[0, T_0]$. Moreover, the functions v^n, ω^n, θ^n and ρ^n satisfy the inequalities*

$$\max_{t\in[0,T_0]}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2\right) + \int_0^{T_0}\left(\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2\right) dt \leqslant C, \tag{5.9}$$

$$\max_{t\in[0,T_0]}\left\|\frac{\partial \rho^n}{\partial x}(t)\right\| \leqslant C, \tag{5.10}$$

$$\frac{m}{2} \leqslant \rho^n(x,t) \leqslant 2M, \quad (x,t) \in \overline{Q}_0, \quad Q_0 = Q_{T_0}. \tag{5.11}$$

Proof. Let

$$y_n(t) = \left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2 + \int_0^t \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau. \tag{5.12}$$

According to (5.5) it holds

$$\dot{y}_n \leqslant C(1 + y_n^4). \tag{5.13}$$

Because of (4.15)–(4.17) we have

$$y_n(0) = \left\|\frac{dv_0^n}{dx}\right\|^2 + \left\|\frac{d\omega_0^n}{dx}\right\|^2 + \left\|\frac{d\theta_0^n}{dx}\right\|^2 \leqslant \left\|\frac{dv_0}{dx}\right\|^2 + \left\|\frac{d\omega_0}{dx}\right\|^2 + \left\|\frac{d\theta_0}{dx}\right\|^2,$$

i.e.

$$y_n(0) \leqslant C. \tag{5.14}$$

Let $[0, T'[$, $T' \in \mathbf{R}_+$, be an existence interval of the Cauchy problem

$$\dot{y} = C(1 + y^4) \tag{5.15}$$

$$y(0) = C. \tag{5.16}$$

From (5.13)–(5.16) it follows

$$y_n(t) \leqslant y(t), \quad t \in [0, T'[. \tag{5.17}$$

Let $0 \prec T_0 \prec T'$. From (5.12) and (5.17) we obtain

$$\max_{t\in[0,T_0]}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2\right) + \int_0^{T_0} \left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau \leqslant C \tag{5.18}$$

and, using (5.5),

$$\frac{d}{dt}\left(\left\|\frac{\partial v^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \omega^n}{\partial x}(t)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|^2\right) + \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2 \omega^n}{\partial x^2}(t)\right\|^2 + \left\|\frac{\partial^2 \theta^n}{\partial x^2}(t)\right\|^2 dt \leqslant C;$$

taking into account (4.15)–(4.17) we obtain (5.9). From (5.9) and (5.4) it follows (5.10). According to (4.10) we have

$$\rho^n(x,t) \leqslant \frac{M}{1 - M\int_0^t \left|\frac{\partial v^n}{\partial x}(x,\tau)\right| d\tau}.$$

With the help of (5.7), (5.8) and (5.9) we find that

$$\int_0^t \left|\frac{\partial v^n}{\partial x}(x,\tau)\right| d\tau \leqslant \sqrt{2}\left(\max_{t\in[0,T_0]}\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|^2\right)^{\frac{1}{4}}\left(\int_0^{T_0}\left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^{\frac{1}{4}} T_0^{\frac{3}{4}} \leqslant CT_0^{\frac{3}{4}}.$$

Let $T_0 \prec \min\{T', (2M)^{-\frac{4}{3}}C^{-\frac{2}{3}}\}$; then for $(x,t) \in \overline{Q}_0$ we have

$$\rho^n(x,t) \leqslant 2M.$$

For such T_0 and $(x,t) \in \overline{Q}_0$, from (4.10) we obtain analogously

$$\rho^n(x,t) \geqslant \frac{m}{2}.$$

From (4.2)–(4.4) and (5.9) one can easy conclude that for $t \in [0, T_0]$ it holds

$$\sum_{i=1}^{n}\Big[|v_i^n(t)| + |\omega_i^n(t)| + |\theta_i^n(t)|\Big] \leqslant C. \tag{5.19}$$

From (4.21) and (4.29) we have

$$\theta_0^n(t) = \int_0^t\int_0^1 \left[-K\rho^n\theta^n\frac{\partial v^n}{\partial x} + \rho^n\left(\frac{\partial v^n}{\partial x}\right)^2 + \frac{(\omega^n)^2}{\rho^n} + \rho^n\left(\frac{\partial \omega^n}{\partial x}\right)^2\right] dx\,d\tau + \theta_{00}.$$

With the help of (5.3), (5.9), (5.11) and (5.7), (5.8), for $t \in [0, T_0]$ we obtain

$$|\theta_0^n(t)| \leqslant C. \tag{5.20}$$

From (5.19) and (5.20) we conclude that the solution of the problem (4.19)–(4.26) is defined on $[0, T_0]$. □

LEMMA 5.7. *Let T_0 be defined by Lemma 5.6. Then for each $n \in \mathbf{N}$ it holds*

$$\int_0^{T_0}\left(\left\|\frac{\partial v^n}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial \omega^n}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial \theta^n}{\partial t}(\tau)\right\|^2 + \left\|\frac{\partial \rho^n}{\partial t}(\tau)\right\|^2\right) d\tau \leqslant C. \tag{5.21}$$

Proof. Multiplying (4.7) by $\frac{dv_i^n}{dt}(t)$ and summing over $i = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}\left\|\frac{\partial v^n}{\partial t}(t)\right\|^2 &= \int_0^1\left(\frac{\partial \rho^n}{\partial x}\frac{\partial v^n}{\partial x}\frac{\partial v^n}{\partial t} + \rho^n\frac{\partial^2 v^n}{\partial x^2}\frac{\partial v^n}{\partial t} - K\frac{\partial \rho^n}{\partial x}\theta^n\frac{\partial v^n}{\partial t} - K\rho^n\frac{\partial \theta^n}{\partial x}\frac{\partial v^n}{\partial t}\right) dx\\ &\leqslant C\left(\max_{x\in[0,1]}\left|\frac{\partial v^n}{\partial x}(x,t)\right|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\|\right.\\ &\quad\left. + \max_{x\in[0,1]}|\theta^n(x,t)|\left\|\frac{\partial \rho^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\| + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial t}(t)\right\|\right).\end{aligned}$$

Applying (5.7), (5.8), (5.3) and (5.4) we find that

$$\begin{aligned}\left\|\frac{\partial v^n}{\partial t}(t)\right\|^2 &\leqslant C\left[\left\|\frac{\partial v^n}{\partial t}(t)\right\|\left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|\left(1 + \left(\int_0^t\left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^{\frac12}\right)\right.\\ &\quad + \left\|\frac{\partial^2 v^n}{\partial x^2}(t)\right\|\left\|\frac{\partial v^n}{\partial x}(t)\right\| + \left\|\frac{\partial v^n}{\partial t}(t)\right\|\left(1 + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|\right.\\ &\quad\left.\left. + \left\|\frac{\partial v^n}{\partial x}(t)\right\|^2\right)\left(1 + \left(\int_0^t\left\|\frac{\partial^2 v^n}{\partial x^2}(\tau)\right\|^2 d\tau\right)^{\frac12}\right) + \left\|\frac{\partial \theta^n}{\partial x}(t)\right\|\left\|\frac{\partial v^n}{\partial x}(t)\right\|\right].\end{aligned}$$

With the help of Young inequality and (5.9) one can easily conclude that

$$\int_0^{T_0} \left\| \frac{\partial v^n}{\partial t}(\tau) \right\|^2 d\tau \leqslant C.$$

In the same way from (4.8) and (4.9) we obtain the estimates for $\left\| \frac{\partial \omega^n}{\partial t} \right\|$ and $\left\| \frac{\partial \theta^n}{\partial t} \right\|$, respectively. The estimate for $\left\| \frac{\partial \rho^n}{\partial t} \right\|$ follows from (4.6) and (5.9). □

From Lemmas 5.6. and 5.7. we obtain immediately the next result.

PROPOSITION 5.1. *Let $T_0 \in \mathbf{R}_+$ be defined by Lemma 5.6. Then for the sequence $\{(\rho^n, v^n, \omega^n, \theta^n) : n \in \mathbf{N}\}$ the following statements hold true:*

(*i*) *$\{\rho^n\}$ is bounded in $L^\infty(Q_0)$, $L^\infty(0, T_0; H^1(]0,1[))$ and $H^1(Q_0)$;*

(*ii*) *$\{v^n\}$, $\{\omega^n\}$, $\{\theta^n\}$ are bounded in $L^\infty(0, T_0; H^1(]0,1[))$, $H^1(Q_0)$, and $L^2(0, T_0; H^2(]0,1[))$.*

6. The proof of Theorem 2.2.

In proofs that follow we use some well-known facts of Functions Analysis (e.g. [3]).

Let $T_0 \in \mathbf{R}_+$ be defined by Lemma 5.6. Theorem 2.2. is a consequence of the following lemmas.

LEMMA 6.1. *There exists a function*

$$\rho \in L^\infty(0, T_0; H^1(]0,1[)) \cap H^1(Q_0) \cap C(\overline{Q}_0)$$

and a subsequence of $\{\rho^n\}$ (for simplicity denoted again as $\{\rho^n\}$), such that

$$\rho^n \to \rho \text{ weakly-* in } L^\infty(0, T_0; H^1(]0,1[)), \tag{6.1}$$

$$\text{weakly in } H^1(Q_0), \tag{6.2}$$

$$\text{strongly in } C(\overline{Q_0}). \tag{6.3}$$

The function ρ satisfies the conditions

$$\frac{m}{2} \leqslant \rho \leqslant 2M \quad in \quad \overline{Q}_0, \tag{6.4}$$

$$\rho(x,0) = \rho_0(x), \quad x \in [0,1]. \tag{6.5}$$

Proof. The conclusions (6.1) and (6.2) follow immediately from Proposition 5.1. Let $(x,t), (x',t') \in \overline{Q}_0$. Then

$$|\rho^n(x,t) - \rho^n(x',t')| \leqslant |\rho^n(x,t) - \rho^n(x',t)| + |\rho^n(x',t) - \rho^n(x',t')|.$$

Using (4.6) and Proposition 5.1. we obtain

$$|\rho^n(x,t) - \rho^n(x',t)| \leqslant \int_{x'}^{x} \left|\frac{\partial \rho^n}{\partial x}(\xi,t)\right| d\xi \leqslant C|x-x'|^{\frac{1}{2}},$$

$$|\rho^n(x',t) - \rho^n(x',t')| \leqslant \int_{t'}^{t} \left|\frac{\partial \rho^n}{\partial t}(x',\tau)\right| d\tau \leqslant C \int_{t'}^{t} \left|\frac{\partial v^n}{\partial x}(x',\tau)\right| d\tau$$

$$\leqslant C \int_{t'}^{t} \|v^n(\tau)\|_{H^2(]0,1[)} d\tau \leqslant C|t-t'|^{\frac{1}{2}}.$$

The statement (6.3) follows now from the Arzela'–Ascoli theorem. The conditions (6.4) and (6.5) follow from (5.11) and (4.6), respectively. □

LEMMA 6.2. *There exist functions*

$$v, \omega, \theta \in L^\infty\left(0, T_0; H^1(]0,1[)\right) \cap H^1(Q_0) \cap L^2\left(0, T_0; H^2(]0,1[)\right)$$

and a subsequence of $\{v^n, \omega^n, \theta^n\}$ *(denoted again as* $\{v^n, \omega^n, \theta^n\}$*), such that*

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \quad \text{weakly-* in} \quad \left(L^\infty\left(0, T_0; H^1(]0,1[)\right)\right)^3, \tag{6.6}$$

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \quad \text{weakly in} \quad \left(H^1(Q_0)\right)^3, \tag{6.7}$$

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \quad \text{strongly in} \quad \left(L^2(Q_0)\right)^3, \tag{6.8}$$

$$(v^n, \omega^n, \theta^n) \to (v, \omega, \theta) \quad \text{weakly in} \quad \left(L^2(0, T_0; H^2]0,1[)\right)^3, \tag{6.9}$$

The functions v, ω *and* θ *satisfy the conditions*

$$v(0,t) = v(1,t) = \omega(0,t) = \omega(1,t) = 0, \qquad t \in [0, T_0], \tag{6.10}$$

$$\frac{\partial \theta}{\partial x}(0,t) = \frac{\partial \theta}{\partial x}(1,t) = 0 \quad \text{a.e. in} \quad]0, T_0[, \tag{6.11}$$

$$v(x,0) = v_0(x),\ \omega(x,0) = \omega_0(x),\ \theta(x,0) = \theta_0(x),\ x \in [0,1]. \tag{6.12}$$

Proof. The conclusions follow from Proposition 5.1. and embedding properties (see Remark 2.1.).

LEMMA 6.3. *The functions* ρ, v, ω *and* θ*, defined by Lemmas 6.1. and 6.2., satisfy the equations (2.14)–(2.17) a.e. in* Q_0.

Proof. Let $\{(\rho^n, v^n, \omega^n, \theta^n) : n \in \mathbf{N}\}$ be subsequence defined by Lemmas 6.1. and 6.2. The equation (2.14) follows then immediately from (4.6). Let us transform the equations (4.7)–(4.9) (integrating by parts) to slightly different forms:

$$\int_0^1 \left[\frac{\partial v^n}{\partial t}\sin(\pi i x) + \pi i \rho^n\Big(\frac{\partial v^n}{\partial x} - K\theta^n\Big)\cos(\pi i x)\right] dx = 0,$$

$$\int_0^1 \left[\Big(\frac{\partial \omega^n}{\partial t} + A\frac{\omega^n}{\rho^n}\Big)\sin(\pi j x) + A\pi j \rho^n \frac{\partial \omega^n}{\partial x}\cos(\pi j x)\right] dx = 0,$$

$$\int_0^1 \Bigg[\Big(\frac{\partial \theta^n}{\partial t} + K\rho^n\theta^n\frac{\partial v^n}{\partial x} - \frac{(\omega^n)^2}{\rho^n} - (\rho^n - \rho)\Big(\frac{\partial v^n}{\partial x}\Big)^2 - (\rho^n - \rho)\Big(\frac{\partial \omega^n}{\partial x}\Big)^2$$
$$+ \rho\omega^n\frac{\partial^2 \omega^n}{\partial x^2} + \frac{\partial \rho}{\partial x}\omega^n\frac{\partial \omega^n}{\partial x} + \frac{\partial \rho}{\partial x}v^n\frac{\partial v^n}{\partial x} + \rho v^n\frac{\partial^2 v^n}{\partial x^2}\Big)\cos(\pi k x)$$
$$- \pi k\Big(\rho\omega^n\frac{\partial \omega^n}{\partial x} + D\rho^n\frac{\partial \theta^n}{\partial x} + \rho v^n\frac{\partial v^n}{\partial x}\Big)\sin(\pi k x)\Bigg] dx = 0.$$

Taking limits (when $n \to \infty$), we obtain

$$\int_0^1 \left[\frac{\partial v}{\partial t}\sin(\pi i x) + \pi i \rho\Big(\frac{\partial v}{\partial x} - K\theta\Big)\cos(\pi i x)\right] dx = 0,$$

$$\int_0^1 \left[\Big(\frac{\partial \omega}{\partial t} + A\frac{\omega}{\rho}\Big)\sin(\pi j x) + A\pi j \rho \frac{\partial \omega}{\partial x}\cos(\pi j x)\right] dx = 0,$$

$$\int_0^1 \Bigg[\Big(\frac{\partial \theta}{\partial t} + K\rho\theta\frac{\partial v}{\partial x} - \frac{\omega^2}{\rho} + \rho\omega\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial \rho}{\partial x}\omega\frac{\partial \omega}{\partial x} + \frac{\partial \rho}{\partial x}v\frac{\partial v}{\partial x} + \rho v\frac{\partial^2 v}{\partial x^2}\Big)\cos(\pi k x)$$
$$- \pi k\Big(\rho v\frac{\partial v}{\partial x} + \rho\omega\frac{\partial \omega}{\partial x} + D\rho\frac{\partial \theta}{\partial x}\Big)\sin(\pi k x)\Bigg] dx = 0.$$

Now, integrating by parts and taking into account (6.10) and (6.11), we get the equations (2.15)–(2.17). □

LEMMA 6.4. *There exists $T_0 \in \mathbf{R}_+$ such that the function θ, defined by Lemma 6.2., satisfies the condition*

$$\theta \succ 0 \quad in \quad \overline{Q_0}. \tag{6.13}$$

Proof. Because of the inclusion $\theta \in C(\overline{Q}_0)$ (see Remark 2.1.), for each $\varepsilon \succ 0$ there exists $T_0 \in \mathbf{R}_+$, such that for $(x,t) \in \overline{Q}_0$ it holds

$$|\theta(x,t) - \theta(x,0)| = |\theta(x,t) - \theta_0(x)| \prec \varepsilon,$$

or

$$\theta(x,t) \succ \theta_0(x) - \varepsilon \geqslant m - \varepsilon.$$

□

Remark 6.1. In the second part of this work we intend to prove (with use of Theorem 2.2.) that a generalised solution of the problem (2.14)–(2.24) exists in Q_T for each $T \in \mathbf{R}_+$.

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