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ON A CHARACTERIZATION OF POLYNOMIALLY BARRELLED SPACES

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Abstract. A locally convex space E is polynomially barrelled if and only if, for every positive integer m and for every Banach space F, the space of all continuous m-homogeneous polynomials from E into F is quasi-complete for the topology of pointwise convergence.

Pfister [4] has proved that a locally convex space E is barrelled if and only if, for every Banach space F, the space of all continuous linear mappings from E into F is quasi-complete for the topology of pointwise convergence. The main purpose of the present note is to establish the corresponding result in the polynomial context. At the end of the note a discussion concerning the closed graph theorem for homogeneous polynomials is also presented.

In what follows we refer to [2] and [3] for the background on topological vector spaces. All vector spaces under consideration are vector spaces over a field $\mathbf{K}$ which is either $\mathbf{R}$ or $\mathbf{C}$. If E and F are locally convex spaces and m is a positive integer, $P(^mE;F)$ denotes the vector space of all continuous m-homogeneous polynomials from E into F, and τ_S represents the locally convex topology of pointwise convergence on $P(^mE;F)$. A locally convex space E is polynomially barrelled [5] if, for every positive integer m, each τ_S-bounded subset of $P(^mE) := P(^mE;\mathbf{K})$ is equicontinuous.

THEOREM. *Let E be a locally convex space. In order that E be a polynomially barrelled space it is necessary and sufficient that, for every positive integer m and for every Banach space F, the space $(P(^mE;F),\tau_S)$ be quasi-complete.*

Proof. Since the necessity is a particular case of Proposition 3.26 of [5], let us turn to the sufficiency.

Let m be a positive integer and let $X \subset P(^mE)$ be τ_S-bounded. Let $B(X)$ be the vector space of all $\mathbf{K}$-valued bounded mappings on X, endowed with the supremum norm: $\|h\| = \sup\{|h(f)|; f \in X\}$ for $h \in B(X)$. Then $(B(X), \|\cdot\|)$ is a Banach space. Define $P : E \to B(X)$ by $P(x)(f) = f(x)$ for $x \in E, f \in X$; note that $P(x) \in B(X)$ because X is τ_S-bounded. It is easily seen that P is an m-homogeneous polynomial from E into $B(X)$. We shall prove that P is continuous. In order to do so, we shall construct a set $\{P_{T,\epsilon}; T \in \Omega, \epsilon > 0\}$ of continuous m-homogeneous polynomials

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from E into $B(X)$ (where Ω denotes the set of all non-empty finite subsets of E) satisfying the following properties:

(1) For every $T \in \Omega$ and for every $\epsilon > 0$, $\|P_{T,\epsilon}(x) - P(x)\| \leqslant \epsilon$ if $x \in T$;
(2) For every $x \in E$, the set $\{P_{T,\epsilon}(x); T \in \Omega, \epsilon > 0\}$ is bounded in $B(X)$.

Let $T \in \Omega$ and $\epsilon > 0$ be given. Consider on X the pseudo-metric $d_T(f,g) = \max\{|f(x) - g(x)|; x \in T\}$ for $f, g \in X$. Then the uniformity determined by d_T is the smallest uniformity on X which makes the mappings $f \in X \mapsto f(x) \in \mathbf{K}$ $(x \in T)$ uniformly continuous. By Proposition 3.3, p.5 of [2], (X, d_T) is precompact because $X(x) = \{f(x); f \in X\}$ is a precompact subset of $\mathbf{K}$ for each $x \in T$ (recall that X is τ_S-bounded). Moreover, if τ_T denotes the topology induced by the pseudo-metric d_T, then (X, τ_T) is a normal space. For $f \in X$, put $B(f, \epsilon) = \{g \in X; d_T(f,g) < \epsilon\}$. By the precompactness of (X, d_T), there exist $f_1, \ldots, f_n \in X$ such that $X = B(f_1, \epsilon) \cup \ldots \cup B(f_n, \epsilon)$. By the Dieudonné–Bochner theorem, there exists a family $(\theta_i)_{1 \leqslant i \leqslant n}$ of real-valued τ_T-continuous mappings on X satisfying the following properties:

(3) $\theta_i \geqslant 0$ on X for $i = 1, \ldots, n$;
(4) $\sum_{i=1}^{n} \theta_i = 1$ on X;
(5) θ_i vanishes outside $B(f_i, \epsilon)$ for $i = 1, \ldots, n$.

Define $P_{T,\epsilon} : E \to B(X)$ by $P_{T,\epsilon}(x)(f) = \sum_{i=1}^{n} \theta_i(f) f_i(x)$ for $x \in E, f \in X$. It is easily seen that $P_{T,\epsilon} \in P(^mE; B(X))$.

Now, let us verify (1) and (2). Let $T \in \Omega$ and $\epsilon > 0$. If $x \in T$,

$$|P_{T,\epsilon}(x)(f) - P(x)(f)| = \left|\sum_{i=1}^{n} \theta_i(f) f_i(x) - \sum_{i=1}^{n} \theta_i(f) f(x)\right|$$
$$\leqslant \sum_{i=1}^{n} \theta_i(f)|f_i(x) - f(x)| \leqslant \epsilon\Big(\sum_{i=1}^{n} \theta_i(f)\Big) = \epsilon$$

for all $f \in X$ (by (3), (4) and (5)). Therefore (1) holds. If $x \in E$, (3) and (4) yield

$$\|P_{T,\epsilon}(x)\| = \sup\left\{\left|\sum_{i=1}^{n} \theta_i(f) f_i(x)\right|; f \in X\right\} \leqslant \sum_{i=1}^{n} |f_i(x)|$$

for all $T \in \Omega$ and for all $\epsilon > 0$. Therefore (2) holds.

For $T, T_1 \in \Omega$ and $\epsilon, \epsilon_1 > 0$, put $(T, \epsilon) \leqslant (T_1, \epsilon_1)$ if and only if $T \subset T_1$ and $\epsilon_1 \leqslant \epsilon$. In this way, $\Omega \times]0, +\infty[$ becomes a directed set and the set $\{P_{T,\epsilon}; T \in \Omega, \epsilon > 0\}$ may be regarded as a net in $P(^mE; B(X))$. By (2), the set $\{P_{T,\epsilon}; T \in \Omega, \epsilon > 0\}$ is τ_S-bounded in $P(^mE; B(X))$. Hence, by hypothesis, its closure in $(P(^mE; B(X)), \tau_S)$ is τ_S-complete. On the other hand, (1) ensures that $(P_{T,\epsilon})$ is a τ_S-Cauchy net. Consequently, there exists a $P_1 \in P(^mE; B(X))$ such that $(P_{T,\epsilon})$ converges to P_1 for τ_S. By (1), $P = P_1$ and the continuity of P is established. It then follows that X is uniformly bounded on a neighborhood of zero in E, and so X is equicontinuous by Theorem 1 of [6]. We have proved that every τ_S-bounded subset of $P(^mE)$ is equicontinuous, and hence E is polynomially barrelled. This completes the proof of the Theorem.

It is known [3] that a separated locally convex space E is barrelled if and only if, for every Banach space F, each linear mapping from E into F with a closed graph is continuous. In the same spirit, it is possible to establish a sufficient condition for a locally convex space to be polynomially barrelled (Proposition 1). We do not know if this condition is also necessary, although we have been able to obtain a partial result in this direction (Proposition 2).

PROPOSITION 1. *Let E be a locally convex space. In order that E be a polynomially barrelled space it is sufficient that, for every positive integer m and for every Banach space F, each m-homogeneous polynomial from E into F with a closed graph be continuous.*

Proof. Let m be a positive integer and let $X \subset P(^mE)$ be τ_S-bounded. Let $B(X)$ be the Banach space considered in the proof of the Theorem, and define $P : E \to B(X)$ by $P(x)(f) = f(x)$ for $x \in E, f \in X$. Then P is an m-homogeneous polynomial from E into $B(X)$ whose graph is obviously closed. By hypothesis, P is continuous, and the argument used in the proof of the Theorem ensures the equicontinuity of X. Therefore E is polynomially barrelled, as was to be shown.

PROPOSITION 2. *Let m be an integer, $m \geqslant 2$, $E_1, \dots, E_m$ separated locally convex (DF)-spaces, F a separated locally convex space, and $u : \prod_{k=1}^{m} E_k \to F$ a multilinear mapping with a closed graph. If*

(i) $E_1, \dots, E_m$ are barrelled spaces and F is an infra-(s)-space, or

(ii) $E_1, \dots, E_m$ are ultrabornological spaces and F is a webbed space, then u is continuous.

Proof. Let $k \in \{1, \dots, m\}$ and $x_j \in E_j$ $(1 \leqslant j \leqslant m, j \neq k)$ be fixed. Arguing as in the proof of the Theorem obtained in [1], we see that the linear mapping

$$t \in E_k \mapsto u(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_m) \in F$$

has a closed graph. By (4) a), p.45 of [3] (case(i)) or (2) , p.57 of [3] (case (ii)), this mapping is continuous. By Corollary 1, p.226 of [2] and Exercise 1, p.228 of [2], u is continuous, as was to be shown.

Remark. (i) Under the hypotheses of Proposition 2, u is an m-homogeneous polynomial from $\prod_{k=1}^{m} E_k$ into F by Proposition 1 of [6]. Moreover, $\prod_{k=1}^{m} E_k$ is polynomially barrelled. In fact, using the argument of the proof of Proposition 4.1 of [5], we see that every barrelled (DF)-space is polynomially barrelled.

(ii) In the non-linear case, the Theorem proved in [1] is a special case of both parts of Proposition 2; in the linear case, it is the classical closed graph theorem in the context of Banach spaces.

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